Journal of Computational Mathematics, Vol.16, No.3, 1998, 275–288.

THE GLOBAL DUFORT-FRANKEL DIFFERENCE APPROXIMATION FOR NONLINEAR REACTION-DIFFUSION EQUATIONS*1)

Bai-nian Lu
(*Institute of Computational and Applied Mathematics, Xiangtan University, Xiangtan, 411105, China; Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China*)
Guo-hua Wan
(*Department of Mathematics, Shaanxi Normal University, Xi'an, 710062, China*)
Bo-lin Guo
(*Center for Nonlinear Studies, Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China*)

Abstract

In this paper, the initial value problem of nonlinear reaction-diffusion equation is considered. The Dufort-Frankel finite difference approximation for the long time scheme is given for the d-dimensional reaction-diffusion equation with the two different cases. The global solution and global attractor are discussed for the Dufort-Frankel scheme. Moreover properties of the solution are studied. The error estimate is presented in a finite time region and in the global time region for some special cases. Finally the numerical results for the equation are investigated for Allen-Cahn equation and some other equations and the homoclinic orbit is simulated numerically.

Keywords: globel Dufort-Frankel method, reaction-diffusion eqution, global attractor, error estimate, numerical experiments

1. Introduction

In this paper we consider the following initial-value problem of nonlinear reaction-diffusion equation:

$$\begin{cases} u_t = \gamma \Delta u - f(u); & (x,t) \in \Omega \times \mathbf{R}^+ & (1.1a) \\ u = 0, & x \in \partial\Omega & (1.1b) \\ u(x,0) = u_0(x), & x \in \Omega & (1.1c) \end{cases}$$

Here Ω is a bounded domain in $\mathbf{R}^d$ $(d \le 3)$ with a Lipschitz boundary $\partial\Omega$ and γ is a positive constant.

Let the set $\{|u^\star| : f(u^\star) = 0\}$ be not empty and $\bar{u} = \max\{|u^\star| : f(u^\star) = 0\}$.

* Received April 30, 1996.
1) This project is partly supported by the Natural Science Foundation of China and partly by State Education Committee.

Assumption on the nonlinear function $f(u)$ is either (i). $rf(r) > 0$ for $|r| \geq \bar{u}$; or (ii). $f(-\bar{u}) = f(\bar{u})$.

Remark 1.1. Indeed, the assumption (ii) can be reduced as: there exist at least two different real roots for the nonlinear term $f(u)$ because we only need introduce a transform $v = u - (u_{\max} + u_{\min})/2$. Under this transform, the case (ii) holds for the new equation in which $v(x, t)$ is a new unknown function. Where $u_{\max}$ and $u_{\min}$ are the maximum and minimum roots of the nonlinear function $f(u)$.

The assumption (i) can be found in Hale[1]. Temam[2] who studied the global attractor for the equation (1.1) with the nonlinear term

$$f(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_{2p-1} > 0.$$

In particular, the equation (1.1) which satisfies the condition (i) contains the Allen-Cahn equation[3] provided that $f(s) = \beta s(s^2 - 1)$. It is clear that the Allen-Cahn equation satisfies both assumptions (i) and (ii).

We also shall discuss the attractor for the case (ii) which the nonlinear function f satisfies (ii) but (i). For example, $f(s) = \beta(s^2 - 1)$, or $f(s) = \beta(|s| - 1)$. The nonlinear function $f(\cdot)$ which satisfy the assumptions(i) and (ii) are sketched on the Figures 1 and 2 respectively.

Fig.1 Case (i). $rf(r) > 0$ for $|r| \geq \bar{u}$ Fig.2 Case (ii). $f(\bar{u}) = f(-\bar{u})$

Throughout this paper, similar to discuss in Lu Bainian and Wan Guihua[4], we shall discuss the absorbing set and attractor of the discrete Dufort-Frankel finite difference equation of the equation (1.1) under condition (i) for any initial data and under (ii) for small initial data respectively. Moreover in section 2, we shall give some notations and some lemmas. In section 3, we shall study the global attractor for the finite difference equation and some properties of the difference equation. In section 4, we give the convergent theorem. Finally in section 5, we shall give several numerical examples to check our theoretical results which discussed in section 3. Furthermore, we study some properties for equation (1.1) with nonlinear term in the case (ii).

Elliott and Stuart[5] studied the Euler scheme for the equation (1.1). We know if we simulate the global solution or global attractor numerically to discuss the properties of the solution of (1.1), it will take a lots of CPU time on computer. So it is necessary to find a numerical scheme with large discrete step-lengthen. Indeed, Dufort-Frankel finite difference scheme is unconditionally stable for linear reaction-diffusion equation, but the Euler scheme is conditionally stable[6]. Therefore, in this paper we shall use Dufort-Frankel finite difference scheme to discrete the nonlinear reaction-diffusion equation and to discuss its properties. In order to discrete the nonlinear function $f(u)$, we apply the mean value around the center lattice but not including the center lattice. Taking the 2-d as an example, the nonlinear term is taken by $f((u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1})/4)$. It is very interesting if we take the nonlinear term as $f(u_{i,j})$ the numerical experiments show that it is not stable. Moreover, we have not got the mathematical analysis in this case. We know in linear case the Dufort-Frankel scheme is unconditionally stable, unfortunately, in view of mathematical analysis we have not got unconditional stability for the Dufort-Frankel scheme. But in view of numerical experiments, we shall see that the time-step length of the Dufort-Frankel scheme is much larger than the one of Euler scheme.

2. Dufort-Frankel Finite Difference Approximation and Some Lemmas

In this section we consider the Dufort-Frankel discretisation of (1.1), for simplicity sake, we only discuss in the case two-dimension but in order to get general d-dimensional results, in many places we still write d instead of 2 in supplied conditions and some notations.

Let $\Omega = [0,l] \times [0,l]$. The solution domain in $(x,y) - t$ space is covered by a rectangular grid with grid spacings of h and τ in the x, y and t directions respectively. Where $h = l/J$, J is a positive integer.

We introduce the following notations: $t_n := n\tau$, $x_i = ih$, $y_j = jh$, $u^n_{i,j} = u(x_i, y_j, t_n)$ and $u^n = (u^n_{i,j})_{1\le i,j\le J-1}$. $u^n_{\hat{t}} = \dfrac{u^{n+1} - u^{n-1}}{2\tau}$, $u^n_t = \dfrac{u^{n+1} - u^n}{\tau}$ and $u^n_{\bar{t}} = \dfrac{u^n - u^{n-1}}{\tau}$. $u^n_{i,j\,x} = \dfrac{u^n_{i+1,j} - u^n_{i,j}}{h}$, $u^n_{i,j\,\bar{x}} = \dfrac{u^n_{i,j} - u^n_{i-1,j}}{h}$ and $u^n_{i,j\,x\bar{x}} = (u^n_{i,j\,x})_{\bar{x}}$.

We may similarly define: u^n_y, $u^n_{\bar{y}}$ and $u^n_{y\bar{y}}$. The notations $u^n_x = (u^n_{i,j\,x})_{1\le i,j\le J-1}$ and $u^n_y = (u^n_{i,j\,y})_{1\le i,j\le J-1}$ and so on. The discrete Laplace operator Δ_h is defined by $\Delta_h u^n = u^n_{x\bar{x}} + u^n_{y\bar{y}}$.

$\nabla_h u^n$ is the discrete gradient which is defined by $(u^n_x,\, u^n_y)$.

We consider the Dufort-Frankel finite difference approximation:

$$\begin{cases} u^n_{\hat{t}} = \gamma\Delta_h u^n - d\gamma\Big(\dfrac{\tau}{h}\Big)^2\Delta_\tau u^n - f(\hat{u}^n); & \text{(2.1a)} \\ u^0_t = \gamma\Delta_h u^0 - f(u^0); & \text{(2.1b)} \\ u^0 = u_0. & \text{(2.2c)} \end{cases}$$

Where $\hat{u}^n_{i,j} = 2^{-d} \displaystyle\sum_{|k-i|+|m-j|=1} u^n_{k,m}$ and $f(\hat{u}^n) = (f((\hat{u}^n_{i,j})))_{1\le i,j\le J-1}$.

We introduce the discrete inner product $(\cdot,\cdot)$ as following

$$(u^n,\, v^n) = h^d \sum_{j=1}^{J-1} u^n_{i,j}\, v^n_{i,j}$$

so the norm is $\|u^n\|^2 = (u^n,\ u^n)$.

Lemma 2.1. *For any discrete function u^n, there is*
(i). $(u_t^n, u^{n+1}) = \frac{1}{2}\|u^n\|_t^2 + \frac{\tau}{2}\|u_t^n\|^2$. (ii). $(u_t^n, u^n) = \frac{1}{2}\|u^n\|_t^2 - \frac{\tau}{2}\|u_t^n\|^2$.

Lemma 2.2. *For any discrete function u^n, there is*

$$(u_{\hat{t}}^n, u^{n+1}) = \frac{1}{2}\|u^n\|_{\hat{t}}^2 + \frac{\tau}{4}(\|u_t^n\|^2 + \|u_t^{n-1}\|^2) + \frac{\tau}{2}(u_t^{n-1}, u_t^n)$$

Proof. By the Lemma 2.1 (i), implies

$$\begin{aligned}(u_{\hat{t}}^n, u^{n+1}) &= \frac{1}{2}[(u_t^n, u^{n+1}) + (u_t^{n-1}, u^{n+1})]\\ &= \frac{1}{4}[\|u^n\|_t^2 + \tau\|u_t^n\|^2] + \frac{1}{2}[(u_t^{n-1}, u^n) + \tau(u_t^{n-1}, u_t^n)]\\ &= \frac{1}{4}[(\|u^n\|_t^2 + \|u^{n-1}\|_t^2) + \tau(\|u_t^n\|^2 + \|u_t^{n-1}\|^2)] + \frac{\tau}{2}(u_t^{n-1}, u_t^n)\\ &= \frac{1}{2}\|u^n\|_{\hat{t}}^2 + \frac{\tau}{4}(\|u_t^n\|^2 + \|u_t^{n-1}\|^2) + \frac{\tau}{2}(u_t^{n-1}, u_t^n).\end{aligned}$$

Lemma 2.3. *For any discrete function u^n, there is*

$$-(\Delta_h u^n, u^{n+1}) = \frac{1}{2}(\|\nabla_h u^{n+1}\|^2 + \|\nabla_h u^n\|^2) - \frac{\tau^2}{2}\|\nabla_h u_t^n\|^2.$$

Proof. By the Lemma 2.1(ii), implies

$$\begin{aligned}-(\Delta_h u^n, u^{n+1}) &= -(\Delta_h u^n, u^n) - \tau(\Delta_h u^n, u_t^n) = \|\nabla_h u^n\|^2 + \tau(\nabla_h u^n, \nabla_h u_t^n)\\ &= \|\nabla_h u^n\|^2 + \frac{\tau}{2}\|\nabla_h u^n\|_t^2 - \frac{\tau^2}{2}\|\nabla_h u_t^n\|^2\\ &= \frac{1}{2}(\|\nabla_h u^{n+1}\|^2 + \|\nabla_h u^n\|^2) - \frac{\tau^2}{2}\|\nabla_h u_t^n\|^2.\end{aligned}$$

Lemma 2.4. *For any discrete function u^n, there is*

$$(\Delta_\tau u^n, u^{n+1}) = -(u_t^{n-1}, u_t^n) + (u_t^n, u^{n+1})_{\bar{t}}.$$

Proof. It is clear that

$$\begin{aligned}(\Delta_\tau u^n, u^{n+1}) &= \frac{1}{\tau}[(u_t^n, u^{n+1}) - (u_t^{n-1}, u^{n+1})]\\ &= \frac{1}{\tau}[(u_t^n, u^{n+1}) - (u_t^{n-1}, u^{n+1} - u^n) - (u_t^{n-1}, u^n)]\\ &= -(u_t^{n-1}, u_t^n) + (u_t^n, u^{n+1})_{\bar{t}}.\end{aligned}$$

Lemma 2.5. *For any discrete function u^n, there is*

$$(u^n, u^{n+1}) = \frac{1}{2}(\|u^{n+1}\|^2 + \|u^n\|^2) - \frac{\tau^2}{2}\|u_t^n\|^2.$$

Proof. By the Lemma 2.1, implies

$$(u^n, u^{n+1}) = \|u^n\|^2 + \tau(u^n, u^n_t) = \|u^n\|^2 + \tau\Big[\frac{1}{2}\|u^n\|^2_t - \frac{\tau}{2}\|u^n_t\|^2\Big].$$

Lemma 2.6. *If the discrete function u^n satisfies $u^n|_{\partial\Omega_h} = 0$, then*

$$\|\nabla_h u^n\|^2 \geq \frac{2d}{l^2}\|u^n\|^2.$$

Proof. When $d = 1$, Zhou[7] gave proof of the Lemma. In the following, we discuss for $d-$dimensional case:

$$\begin{aligned}
u^n_{i,j} =& h\sum_{k=0}^{i-1} u^n_{k,j\ x}, \qquad |u^n_{i,j}|^2 \leq h^2 i \sum_{k=0}^{i-1} |u^n_{k,j\ x}|^2 \\
\|u^n\|^2 =& h^d \sum_{i=1}^{J-1}\sum_{j=1}^{J-1} |u^n_{i,j}|^2 \leq h^{2+d}\sum_{i=1}^{J-1}\sum_{j=1}^{J-1} i \sum_{k=0}^{J-1} |u^n_{k,j\ x}|^2 \\
=& h^2 \sum_{i=1}^{J-1} i\Big(h^d \sum_{j=1}^{J-1}\sum_{k=0}^{J-1} |u_{k,j\ x}|^2\Big) \leq h^2 \frac{J^2}{2}\|u^n_x\|^2 = \frac{l^2}{2}\|u^n_x\|^2.
\end{aligned}$$

Similar to prove that $\|u^n\|^2 \leq \dfrac{l^2}{2}\|u^n_y\|^2$.

So by two inequalities above, we have

$$d\|u^n\|^2 \leq \frac{l^2}{2}(\|u^n_x\|^2 + \|u^n_y\|^2) = \frac{l^2}{2}\|\nabla_h u^n\|^2.$$

Lemma 2.7. *Assume that $\{f^n\}_{n\geq 0}$ is a non-negative series, α and β are positive constants. If $\{f^n\}$ satisfies*

$$f^{n+1} \leq \alpha f^n + \beta f^{n-1} \tag{2.2}$$

Then

$$f^{n+1} \leq \frac{1}{\sqrt{\alpha^2 + 4\beta}}[(\lambda_1^{n+1} - \lambda_2^{n+1})f^1 + \beta(\lambda_1^n - \lambda_2^n)f^0]$$

Where $\lambda_1 = \dfrac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2}$ *and* $\lambda_2 = \dfrac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2}$.

Proof. Let the matrix $\mathbf{A}$ and the 2-dimensional vector $\mathbf{x}$ be defined by

$$\mathbf{A} = \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix}, \quad \mathbf{x}^n = \begin{bmatrix} f^{n+1} \\ f^n \end{bmatrix}.$$

It is easy to check that λ_1 and λ_2 are two eigenvalues of the matrix $\mathbf{A}$, and $(\lambda_1, 1)^T$ and $(\lambda_2, 1)^T$ are the eigenvectors respectively associated to λ_1 and λ_2. Where $(\cdot, \cdot)^T$ denotes transformation of the vector.

Set $a_1 = \dfrac{f^1 - \lambda_2 f^0}{\sqrt{\alpha^2 + 4\beta}}$ and $a_2 = \dfrac{\lambda_1 f^0 - f^1}{\sqrt{\alpha^2 + 4\beta}}$. Then $\mathbf{x}^0 = a_1(\lambda_1, 1)^T + a_2(\lambda_2, 1)^T$. Because the matrix $\mathbf{A}$ and the vector $\mathbf{x}^n$ are positive, by (2.2) we have

$$\mathbf{x}^n \leq \mathbf{A}\mathbf{x}^{n-1} \tag{2.3}$$

then (2.3) implies $\mathbf{x}^n \leq \mathbf{A}^n\mathbf{x}^0 = a_1\lambda_1^n(\lambda_1, 1)^T + a_2\lambda_2^n(\lambda_2, 1)^T$. So $f^{n+1} \leq a_1\lambda_1^{n+1} + a_2\lambda_2^{n+1}$. This completes the proof.

3. Existence of Absorbing Sets and Attractors

In the first part of this section the assumption (i) to the nonlinear term f is made. In the last part of this section we shall give the remarks to illustrate the same results hold to the assumption (ii) provided that the initial data is small.

Lemma 3.1. *If the mesh parameters and the first level data satisfy*

$$\frac{\tau}{h^2} < (2d\gamma)^{-1}, \quad \text{and} \quad Kh^2 < 2^d\gamma, \qquad (3.1)$$
$$\|u^1\|_\infty \le \max\{\bar{u}, \|u_0\|_\infty\}$$

then $\sup_{0\le n<\infty} \|u^n\|_\infty \le \max\{\bar{u}, \|u_0\|_\infty\}$. *Where* $K = \max_{|s|\le\max\{\bar{u},\|u_0\|_\infty\}} |f'(s)|$.

Proof. Say that $a = \max\{\bar{u}, \|u_0\|_\infty\}$. We write (2.1a) componentwise as

$$u_{i,j}^{n+1} - a = u_{i,j}^{n-1} - a + 2\tau\gamma\Delta_h(u_{i,j}^n - a) - 2d\tau\gamma\Big(\frac{\tau}{h}\Big)^2\Delta_\tau(u_{i,j}^n - a) - 2\tau f(\hat{u}_{i,j}^n) \qquad (3.2)$$

Rewrite (3.2) as the following

$$\begin{aligned}\Big(1+\frac{2d\tau\gamma}{h^2}\Big)(u_{i,j}^{n+1}-a) =&\Big(1-\frac{2d\tau\gamma}{h^2}\Big)(u_{i,j}^{n-1}-a)\\ &+\frac{2\tau\gamma}{h^2}\sum_{|k-i|+|m-j|=1}(u_{k,m}^n-a) - 2\tau(f(\hat{u}_{i,j}^n)-f(a)) - 2\tau f(a).\end{aligned}$$

By the Talyor's formula, it implies

$$f(\hat{u}_{i,j}^n) - f(a) = f'(\xi_{i,j}^n)2^{-d}\sum_{|k-i|+|m-j|=1}(u_{k,m}^n - a).$$

Assume that $\sup_{0\le k\le n}\|u^k\|_\infty \le a$, by the definition of $\hat{u}^n$, then $|\xi_{i,j}^n| \le a$. So

$$\begin{aligned}\Big(1+\frac{2d\tau\gamma}{h^2}\Big)(u_{i,j}^{n+1}-a) \le&\Big(1-\frac{2d\tau\gamma}{h^2}\Big)(u_{i,j}^{n-1}-a)\\ &+\Big(\frac{2\tau\gamma}{h^2} - 2^{1-d}\tau K\Big)\sum_{|k-i|+|m-j|=1}(u_{k,m}^n-a) - 2\tau f(a)\end{aligned}$$

Since $a \ge \bar{u}$, so $f(a) \ge 0$, and by the assumption (3.1), imply, $u_{i,j}^{n+1} \le a$. Similarly the proof prove above, write (2.1a) componentwise as

$$\begin{aligned}\Big(1+\frac{2d\tau\gamma}{h^2}\Big)(u_{i,j}^{n+1}+a) =&\Big(1-\frac{2d\tau\gamma}{h^2}\Big)(u_{i,j}^{n-1}+a)\\ &+\frac{2\tau\gamma}{h^2}\sum_{|k-i|+|m-j|=1}(u_{k,m}^n+a) - 2\tau(f(\hat{u}_{i,j}^n)-f(-a)) - 2\tau f(-a).\end{aligned}$$

By the Talyor's formula, it implies

$$f(\hat{u}_{i,j}^n) - f(-a) = f'(\bar{\xi}_{i,j}^n)2^{-d}\sum_{|k-i|+|m-j|=1}(u_{k,m}^n + a).$$

Assume that $\sup\limits_{0\le k\le n}\|u^k\|_\infty \le a$, by the definition of $\hat{u}^n$, then $|\bar{\xi}^n_{i,j}| \le a$. So

$$\Big(1+\frac{2d\tau\gamma}{h^2}\Big)(u^{n+1}_{i,j}+a) \le \Big(1-\frac{2d\tau\gamma}{h^2}\Big)(u^{n-1}_{i,j}+a) + \Big(\frac{2\tau\gamma}{h^2}-2^{1-d}\tau K\Big)\sum_{|k-i|+|m-j|=1}(u^n_{k,m}+a) - 2\tau f(-a)$$

Since $-a \le -\bar{u}$, so $f(-a) \le 0$, and by the assumption (3.1), imply, $u^{n+1}_{i,j} \ge -a$. Therefore, the lemma holds.

Lemma 3.2. *If the assumption* (3.1) *holds, then for any* $\epsilon > 0$ *there exists an* $N > 0$ *when* $n > N$*,*

$$\sup_{1\le n<\infty}\|u^n\|_\infty \le \bar{u}+\epsilon$$

Proof. Similar to prove in the Lemma 3.1,

$$\Big(1+\frac{2d\tau\gamma}{h^2}\Big)(u^{n+1}_{i,j}-\bar{u})_+ \le \Big(1-\frac{2d\tau\gamma}{h^2}\Big)(u^{n-1}_{i,j}-\bar{u})_+ + \Big(\frac{2\tau\gamma}{h^2}-2^{1-d}\tau K\Big)\sum_{|k-i|+|m-j|=1}(u^n_{k,m}-\bar{u})_+ - 2\tau f(\bar{u}).$$

Where the positive part of $u^n_{i,j}-\bar{u}$ is defined by

$$(u^n_{i,j}-\bar{u})_+ = \begin{cases} 0; & \text{If } u^n_{i,j}-\bar{u}<0; \\ u^n_{i,j}-\bar{u}; & \text{If } u^n_{i,j}-\bar{u}\ge 0 \end{cases}$$

It is clear that

$$\Big(1+\frac{2d\tau\gamma}{h^2}\Big)\|(u^{n+1}-\bar{u})_+\|_\infty \le \Big(1-\frac{2d\tau\gamma}{h^2}\Big)\|(u^{n-1}_{i,j}-\bar{u})_+\|_\infty + 2d\Big(\frac{2\tau\gamma}{h^2}-2^{1-d}\tau K\Big)\|(u^n-\bar{u})_+\|_\infty$$

Set $\delta = \dfrac{2d\tau\gamma}{h^2}$, $\alpha = \dfrac{1-\delta}{1+\delta}$, $\beta = \dfrac{2(\delta-2^{1-d}d\tau K)}{1+\delta}$ and $f^n = \|(u^n-\bar{u})_+\|_\infty$, then we have that (2.2) holds. By the lemma 2.7, implies

$$\|(u^{n+1}-\bar{u})_+\|_\infty \le \frac{1}{\sqrt{\alpha^2+4\beta}}[(\lambda_1^{n+1}-\lambda_2^{n+1})\|(u^1-\bar{u})_+\|_\infty + \beta(\lambda_1^n-\lambda_2^n)\|(u^0-\bar{u})_+\|_\infty.$$

By the assumption (3.1), it is not difficult to get $\max\{|\lambda_1|,|\lambda_2|\}<1$, implies

$$\lim_{n\to\infty}\sup\|(u^n-\bar{u})_+\|_\infty = 0. \tag{3.3}$$

Similarly we define the notation $(u^n_{i,j}+\bar{u})_-$ as the negative part by

$$(u^n_{i,j}+\bar{u})_- = \begin{cases} 0; & \text{If } u^n_{i,j}+\bar{u}\ge 0; \\ -(u^n_{i,j}+\bar{u}); & \text{If } u^n_{i,j}+\bar{u}<0 \end{cases}$$

and similar to the proof above we have

$$\Big(1+\frac{2d\tau\gamma}{h^2}\Big)\|(u^{n+1}+\bar{u})_-\|_\infty \leq \Big(1-\frac{2d\tau\gamma}{h^2}\Big)\|(u_{i,j}^{n-1}-\bar{u})_-\|_\infty + 2d\Big(\frac{2\tau\gamma}{h^2}-2^{1-d}\tau K\Big)\|(u^n-\bar{u})_-\|_\infty.$$

Thus,

$$\lim_{n\to\infty}\sup\|(u^n+\bar{u})_-\|_\infty = 0. \tag{3.4}$$

By (3.3) and (3.4) the lemma is proved.

Remark 3.1. By the Lemmas 3.1 and 3.2, we know if the initial data satisfies $\|u_0\|_\infty \leq \bar{u}$, then $\|u^n\|_\infty \leq \bar{u}$ for all integer $n > 0$; If the initial data does not satisfy $\|u_0\|_\infty \leq \bar{u}$, then $\lim\limits_{n\to\infty}\|u^n\|_\infty \leq \bar{u}$. So when we compute the global attractor, we only take the initial data satisfies $\|u_0\|_\infty \leq \bar{u}$.

Lemma 3.3. *Suppose that the mesh parameters satisfy* (3.1) *and* $h^2 \leq 2^d\gamma$, *then*

$$\sup_{1\leq n<\infty}\|u^{n+1}\| \leq [C/D+(F^0-C/D)(1+D\tau)^{-(n+1)}]/\Big(\frac{h^2}{2}+d\gamma\tau\Big).$$

Where $C = 2h^2a^2l^d(\max\limits_{|s|\leq a}|f(s)-s|+2)$ *and* $D = h^2/\max\Big(\frac{h^2}{2}+d\gamma\tau,\ 2^d\tau\gamma\Big)$.

Proof. We rewrite (2.1a) as the following

$$u_{\hat{t}}^n - \gamma\Delta_h u^n + d\gamma\Big(\frac{\tau}{h}\Big)^2\Delta_\tau u^n + \hat{u}^n = g(\hat{u}^n). \tag{3.5}$$

where $g(s) = -f(s)+s$. It is clear that $\hat{u}^n = h^2 2^{-d}\Delta_h u^n + u^n$ so the (3.5) is equivalent to

$$u_{\hat{t}}^n - (\gamma-2^{-d}h^2)\Delta_h u^n + d\gamma\Big(\frac{\tau}{h}\Big)^2\Delta_\tau u^n + u^n = g(\hat{u}^n). \tag{3.6}$$

Taking the discrete inner product between (3.6) and u^{n+1}, by the lemmas 2.2–2.6, follows

$$\begin{aligned}&\frac{1}{2}\|u^n\|_{\hat{t}}^2 + \frac{\tau}{4}(\|u_t^n\|^2+\|u_t^{n-1}\|^2) + \frac{\tau}{2}(u_t^{n-1},\ u_t^n)\\ &\quad+\frac{\gamma-2^{-d}h^2}{2}(\|\nabla_h u^{n+1}\|^2+\|\nabla_h u^n\|^2) - \frac{(\gamma-2^{-d}h^2)\tau^2}{2}\|\nabla_h u_t^n\|^2\\ &\quad+-d\gamma\Big(\frac{\tau}{h}\Big)^2(u_t^{n-1},u_t^n) + d\gamma\Big(\frac{\tau}{h}\Big)^2(u_t^n,u^{n+1})_{\bar{t}}\\ &\quad+\frac{1}{2}(\|u^{n+1}\|^2+\|u^n\|^2) - \frac{\tau^2}{2}\|u_t^n\|^2 = (g(\hat{u}^n),u^{n+1}).\end{aligned} \tag{3.7}$$

By the assumption of the Lemma, implies

$$\Big(d\gamma\Big(\frac{\tau}{h}\Big)^2-\frac{\tau}{2}\Big)(u_t^{n-1},u_t^n) \leq \Big[\frac{\tau}{4}-\frac{d\gamma}{2}\Big(\frac{\tau}{h}\Big)^2\Big](\|u_t^{n-1}\|^2+\|u_t^n\|^2) \tag{3.8}$$

$$\frac{\tau^2}{4}(\gamma-2^{-d}h^2)\|\nabla_h u_t^n\|^2 \leq \frac{1}{2}(\gamma-2^{-d}h^2)(\|\nabla_h u^{n+1}\|^2+\|\nabla_h u^n\|^2) \tag{3.9}$$

$$\frac{\tau^2}{4}(\gamma - 2^{-d}h^2)\|\nabla_h u_t^n\|^2 \le \frac{\tau^2 d}{2h^2}(\gamma - 2^{-d}h^2)\|u_t^n\|^2 \tag{3.10}$$

substitution (3.8), (3.9) and (3.10) into (3.7), we get

$$\begin{aligned}\frac{1}{2}\|u^n\|_{\hat{t}}^2 + 2^{-(1+d)}d\tau^2\|u_t^n\|^2 + \frac{d\gamma}{2}\Big(\frac{\tau}{h}\Big)^2\|u_t^{n-1}\|^2 + d\gamma\Big(\frac{\tau}{h}\Big)^2(u_t^n, u^{n+1})_{\bar{t}} \\ + \frac{1}{2}(\|u^{n+1}\|^2 + \|u^n\|^2) \le (g(\hat{u}^n), u^{n+1}) + \frac{\tau^2}{2}\|u_t^n\|^2.\end{aligned} \tag{3.11}$$

By the Lemma 3.1, we get

$$\frac{\tau^2}{2}\|u_t^n\|^2 \le \frac{1}{2}\|u^{n+1} - u^n\|^2 \le \|u^{n+1}\|^2 + \|u^n\|^2 \le 2(Jh)^d \sup_{j\in Z}\|u^j\|_\infty \le 2a^2l^d. \tag{3.12}$$

Where $a = \max\{\bar{u}, \|u_0\|_\infty\}$.

By the definition of $\|u^n\|_{\hat{t}}^2$, we have

$$\frac{1}{2}\|u^n\|_{\hat{t}}^2 = \frac{1}{4}(\|u^{n+1}\|^2 + \|u^n\|^2)_{\bar{t}} \tag{3.13}$$

Substitute lemma 2.1(i), (3.12) and (3.13) into (3.11), follows

$$\begin{aligned}\Big[\frac{1}{4}(\|u^{n+1}\|^2 + \|u^n\|^2) + \frac{d\gamma}{2}\Big(\frac{\tau}{h}\Big)^2(\|u^n\|_t^2 + \tau\|u_t^n\|^2)\Big]_{\bar{t}} + 2^{-(1+d)}d\tau^2\|u_t^n\|^2 \\ + \frac{d\gamma}{2}\Big(\frac{\tau}{h}\Big)^2\|u_t^{n-1}\|^2 + \frac{1}{2}(\|u^{n+1}\|^2 + \|u^n\|^2) \le a^2l^d(M+2)\end{aligned} \tag{3.14}$$

where $M = \max_{|s|\le a}|g(s)|$.

Substitute $\|u^n\|_t^2 = (\|u^{n+1}\|^2 - \|u^n\|^2)/\tau$ into (3.14), multiple $2h^2$ and omit the nonnegative term $d\gamma\tau^2\|u_t^{n-1}\|^2$, implies

$$\begin{aligned}\Big[\Big(\frac{h^2}{2} + d\gamma\tau\Big)\|u^{n+1}\|^2 + \Big(\frac{h^2}{2} - d\gamma\tau\Big)\|u^n\|^2 + \tau^3 d\gamma\|u_t^n\|^2\Big]_{\bar{t}} \\ + 2^{-d}h^2 d\tau^2\|u_t^n\|^2 + h^2(\|u^{n+1}\|^2 + \|u^n\|^2) \le 2h^2a^2l^d(M+2)\end{aligned} \tag{3.15}$$

For the sake of convenience, we introduce the following symbols

$$\begin{aligned}F^n &= \Big(\frac{h^2}{2} + d\gamma\tau\Big)\|u^{n+1}\|^2 + \Big(\frac{h^2}{2} - d\gamma\tau\Big)\|u^n\|^2 + \tau^3 d\gamma\|u_t^n\|^2, \\ D &= \min\Big\{\frac{h^2}{h^2/2 - d\gamma\tau}, \ \frac{h^2}{h^2/2 + d\gamma\tau}, \ \frac{2^{-d}h^2}{\tau\gamma}\Big\}.\end{aligned}$$

Then (3.15) may be write: $F_t^n \le C - DF^{n+1}$.

In fact $D = h^2/\max\Big(\frac{h^2}{2} + d\gamma\tau, \ 2^d\tau\gamma\Big)$. By the Lemma[4,8], we get $F^n \le C/D + (F^0 - C/D)(1 + D\tau)^{-(n+1)}$. Therefore, by the definition of F^n, this completes the lemma.

Remark 3.2. We may check the proofs of the Lemmas 3.1-3.3, it is still hold for the assumption (ii) when the initial data $\|u_0\|_\infty \le \bar{u}$. Moreover, when the nonlinear function $f(s)$ satisfies (ii) but (i), if $f(s) > 0$ for $|s| > \bar{u}$, then the solution u^n of (2.1) is up-bounded and satisfies

$$\lim_{n\to\infty} \sup u^n \le \max\{\|u^0\|_\infty, \bar{u}\}.$$

On the other hand if $f(s) < 0$ for $|s| > \bar{u}$, the u^n is blow-bounded and satisfies

$$\lim_{n\to\infty} \inf u^n \ge -\max\{\|u^0\|_\infty, \bar{u}\}.$$

Theorem 3.1. *If the mesh parameters satisfy*

$$\frac{\tau}{h^2} < (2d\gamma)^{-1}, \quad \text{and} \quad Kh^2 < 2^d\gamma \min(1, K) \tag{3.16}$$

then

(i). For any discrete function $u^0 \in \mathbf{R}^{dJ}$, there exists a unique solution of (2.1) *u^n for all $n > 0$. The mapping $u_0 \to u^n$ is continuous in L_2 and L_∞ for each $n > 0$ and hence the family of solution operators $\{S^n\}_{n>0}$, defined by $S^n u_0 = u^n$, form a continuous semigroup on L_2 and L_∞.*

(ii). There exist constants $\{\bar{\rho}_i\}_{i=1}^2$ independent of h, τ and J such that the balls

$$\mathcal{B}_1 = \{u \in L_\infty : \|u\|_\infty \le \bar{\rho}_1\}, \quad \mathcal{B}_2 = \{u \in L_2 : \|u\| \le \bar{\rho}_2\}$$

are absorbing sets for the semigroup $\{S^n\}_{n>0}$, that is, for each $u_0 \in L_\infty$ there exists $\{N_i\}_{i=1}^2$ (depending on $\{u_0,\ \rho_i\}$) such that $S^n u_0 \in \mathcal{B}_i$, $\forall n > N_i$ $(i = 1, 2)$.

(iii). There exists a global attractor $\mathcal{A} \subset L_\infty$(or L_2) for the semigroup $\{S^n\}_{n>0}$. Furthermore $\mathcal{A}$ is connected and there exists a constant ρ independent of h, τ and J such that $\max\{\|u\|_\infty, \|u\|\} \le \rho$, *$\forall u \in \mathcal{A}$.*

Proof. It is clear that (2.1) is an explicit difference scheme, then follows the existence and uniqueness. Because L_∞ and L_2 are finite dimensional vector space, by Lemmas 3.1 and $\|u^n\| \le l^{d/2}$ we have (i) holds.

(ii). By the lemma 3.2, taking $\rho_1 = \bar{u}$ and $\bar{\rho}_1 = \rho_1 + \epsilon$, we have $\mathcal{B}_1$ is a absorbing set.

By Lemma 3.3, we have

$$\lim_{n\to\infty} \sup \|u^n\| \le \frac{2C}{D(h^2 + 2\tau d\gamma)}. \tag{3.17}$$

By (3.16), when $d \le 2$, we have $D = 2h^2/(h^2 + 2\tau d\gamma)$; when $d \ge 3$, we have $D = h^2/2^d\tau\gamma$. By (3.17), implies,

$$\lim_{n\to\infty} \sup \|u^n\| \le \begin{cases} a^2 l^d (\max\limits_{|s|\le\bar{u}} |f(s) - s| + 2); & \text{for} \quad d \le 2, \\ 2^{d-1} d^{-1} a^2 l^d (\max\limits_{|s|\le\bar{u}} |f(s) - s| + 2); & \text{for} \quad d \ge 3. \end{cases}$$

So we take

$$\rho_2=\begin{cases} a^2l^d(\max\limits_{|s|\le\bar u}|f(s)-s|+2) & \text{for} \quad d\le 2\\ 2^{d-1}d^{-1}a^2l^d(\max\limits_{|s|\le\bar u}|f(s)-s|+2) & \text{for} \quad d\ge 3\end{cases}$$

and $\bar\rho_2=\rho_2+\epsilon$, then $\mathcal{B}_2$ is a absorbing set.

(iii). It is clear $\{S^n\}_{n>0}$ is a uniform compact semigroup in L_∞ and L_2, by the results of Temma[2], we have this result holds.

4. Error Estimate of the Dufort-Frankel Scheme

In this section we shall give the convergent result:

Theorem 4.1. *Assume that u^n and $U(x,y,t)$ are the solutions of* (2.1) *and* (1.1) *respectively and $U\in C^3(\mathbf{R}^+;\ H_0^1(\Omega)\cap H^4(\Omega))$. Suppose that the mesh parameters satisfy*

$$\frac{\tau}{h^2}<(2d\gamma)^{-1},$$

then for any integer M, there exists a constant $C(T)$ independent of h and τ

$$\sup_{0\le n\le M}\|u^n-U(t_n)\|\le C(h^2+\tau)$$

where $T\ge M\tau$.

Proof. Set $e^n=U(t_n)-u^n$, here $U(t_n)=(U(x_i,y_j,t_n))_{1\le i,j\le J-1}$. Substitute U^n into (2.1a), implies

$$U^n_{\hat t}=\gamma\Delta_hU^n-d\gamma\Big(\frac{\tau}{h}\Big)^2\Delta_\tau U^n-f(\hat U^n)+r^n. \tag{4.1a}$$

Where r^n is the truncation error, it is clear that $r^n=\mathcal{O}(h^2+(\tau/h)^2)$. By the assumption of the theorem, $r^n=\mathcal{O}(h^2+\tau)$.

(4.1a)−(2.1a), yields

$$e^n_{\hat t}-\gamma\Delta_he^n+d\gamma\Big(\frac{\tau}{h}\Big)^2\Delta_\tau\hat e^n=f'(\hat\eta^n)e^n+\mathcal{O}(r^n). \tag{4.2}$$

Taking the discrete inner product between (4.2) and e^{n+1}, by the lemmas 2.2–2.6, follows

$$\begin{aligned}&\frac12\|e^n\|^2_{\hat t}+\frac{\tau}{4}(\|e^n_t\|^2+\|e^{n-1}_t\|^2)+\frac{\tau}{2}(e^{n-1}_t,e^n_t)+\frac{\gamma}{2}(\|\nabla_he^{n+1}\|^2+\|\nabla_he^n\|^2)\\ &\qquad-\frac{\gamma\tau^2}{2}\|\nabla_he^n_t\|^2-d\gamma\Big(\frac{\tau}{h}\Big)^2(e^{n-1}_t,e^n_t)+d\gamma\Big(\frac{\tau}{h}\Big)^2(e^n_t,e^{n+1})_{\bar t}\\ &=(f'(\hat\eta^n)e^n+r^n,\ e^{n+1}).\end{aligned} \tag{4.3}$$

By the assumption of the Theorem, implies

$$\Big(d\gamma\Big(\frac{\tau}{h}\Big)^2-\frac{\tau}{2}\Big)(e^{n-1}_t,e^n_t)\le\Big[\frac{\tau}{4}-\frac{d\gamma}{2}\Big(\frac{\tau}{h}\Big)^2\Big](\|e^{n-1}_t\|^2+\|e^n_t\|^2) \tag{4.4}$$

$$\frac{\gamma\tau^2}{2}\|\nabla_h e_t^n\|^2 \leq \frac{\gamma\tau^2 d}{h^2}\|e_t^n\|^2, \tag{4.5}$$

substitution (4.4)–(4.5) into (4.3), we get

$$\begin{aligned}\frac{1}{2}\|e^n\|_{\hat{t}}^2 - \frac{d\gamma\tau}{2}\Big(\frac{\tau}{h}\Big)^2\|e_t^n\|_{\hat{t}}^2 + \frac{\gamma}{2}(\|\nabla_h e^{n+1}\|^2 + \|\nabla_h e^n\|^2) + d\gamma\Big(\frac{\tau}{h}\Big)^2(e_t^n, e^{n+1})_{\bar{t}}\\ \leq (f'(\hat{\eta}^n)\hat{e}^n + r^n,\ e^{n+1}).\end{aligned} \tag{4.6}$$

Substituting lemma 2.1(i) into (4.6), follows

$$\frac{1}{2}\|e^n\|_{\hat{t}}^2 + \frac{\gamma}{2}(\|\nabla_h e^{n+1}\|^2 + \|\nabla_h e^n\|^2) + \frac{d\gamma}{2}\Big(\frac{\tau}{h}\Big)^2(\|e^n\|_t^2)_{\bar{t}} \leq (f'(\hat{\eta}^n)\hat{e}^n + r^n,\ e^{n+1}). \tag{4.7}$$

By the following identifies

$$\frac{1}{2}\|e^n\|_{\hat{t}}^2 = \frac{1}{4}(\|e^{n+1}\|^2 + \|e^n\|^2)_{\bar{t}} \tag{4.8}$$

and

$$\frac{d\gamma}{2}\Big(\frac{\tau}{h}\Big)^2(\|e^n\|_t^2)_{\bar{t}} = \frac{d\gamma}{2\tau}\Big(\frac{\tau}{h}\Big)^2(\|e^{n+1}\|^2 - \|e^n\|^2)_{\bar{t}}. \tag{4.9}$$

Substituting (4.8) and (4.9) and ε inequality into (4.7) yields

$$\begin{aligned}\Big[\Big(\frac{1}{4} + \frac{d\gamma\tau}{2h^2}\Big)\|e^{n+1}\|^2 + \Big(\frac{1}{4} - \frac{d\gamma\tau}{2h^2}\Big)\|e^n\|^2\Big]_{\bar{t}} + \frac{\gamma}{2}(\|\nabla_h e^{n+1}\|^2 + \|\nabla_h e^n\|^2)\\ \leq \Big(\frac{L}{2} + \varepsilon\Big)(\|e^n\|^2 + \|e^{n+1}\|^2) + C(\varepsilon)(h^4 + \tau^2)\end{aligned} \tag{4.10}$$

where $L = \max\limits_{|s|\leq a}|f'(s)|$.

By the Lemma 2.6, (4.10) can be rewritten

$$\begin{aligned}\Big[\Big(\frac{1}{4} + \frac{d\gamma\tau}{2h^2}\Big)\|e^{n+1}\|^2 + \Big(\frac{1}{4} - \frac{d\gamma\tau}{2h^2}\Big)\|e^n\|^2\Big]_{\bar{t}} + \frac{\gamma d}{l^2}(\|e^{n+1}\|^2 + \|e^n\|^2)\\ \leq \Big(\frac{L}{2} + \varepsilon\Big)(\|e^n\|^2 + \|e^{n+1}\|^2) + C(\varepsilon)(h^4 + \tau^2).\end{aligned} \tag{4.11}$$

For the sake of convenience, we introduce the following symbol

$$F^n = \Big(\frac{1}{4} + \frac{d\gamma\tau}{2h^2}\Big)\|e^{n+1}\|^2 + \Big(\frac{1}{4} - \frac{d\gamma\tau}{2h^2}\Big)\|e^n\|^2.$$

By Gronwall's inequality, implies

$$F^n \leq C(h^2 + \tau + \|e^1\|) \tag{4.12}$$

Similarly, we can prove

$$\|e^1\| \leq C(h^2 + \tau) \tag{4.13}$$

Therefore by (4.12) and (4.13), the Theorem holds.

Remark 4.1. We prove the error estimate Theorem 4.1. Unfortunately, the error is only estimated in a finite time region $[0,\ T]$. In some special cases, such as for any $\delta > 0$ with $\frac{\gamma d}{l^2} - \frac{L}{2} - \varepsilon > \delta$, then error estimate theorem 4.1 holds in global time region. For the generalized case we have not got the estimate theorem.

5. Numerical Results

In this section we shall present some numerical results which illustrate the material in section 3.

The energy function, *i.e.* Lyapunov function[1], is defined by

$$I(u^n) = \frac{\gamma}{2}\|\nabla_h u^n\|^2 + (F(u^n), 1)$$

where $F(u^n)$ is defined by $\int f(s)ds$.

Experiment 1. In order to simulate Allen-Cahn equation, we take $f(s) = \beta s(s^2 - 1)$. Here set $\beta = 1$, $\gamma = 0.0025$, $\tau = 0.473$, $l = 1$ and $J = 100$ with the initial data

$$u_i(0) = \begin{cases} \sin(7\pi i h) + 0.002 * random & \text{for } 1 \le i \le J-1 \\ 0.0 & \text{for } i = 0,\ J \end{cases} \tag{5.1}$$

Fig.3a $u(x, t_n)$ for $t_n = 0, 16, 30$ Fig.3b Evolution of $u(x,t)$ Fig.3c Energy $I(u^n)$

The Figure 3 shows the numerical results that are simulated by Dufort-Frankel scheme and Euler explicit scheme[5] respectively. For the Euler scheme, when we take the time step-length $\tau = 0.007$, i.e $\dfrac{\tau}{h^2} = 70$, the solution is overflow for 8'th level; but for Dufort-Frankel scheme, when $\tau = 1.473$, i.e. $\dfrac{\tau}{h^2} = 1473$, the solution is still stable. Therefore we take that $\tau = 0.0004736$ to calculate the Allen-Cahn equation with the initial data (5.1) by Euler and $\tau = 0.473$ by Dufort-Frankel schemes respectively. In Figure 3a, we draw the pictures of $t = 0$, $t = 16$ and $t = 30$ by the Dufort-frankel scheme. In the Figure 3b we draw the picture by the Dufort-frankel scheme with the evolution as the time increasing. In the Figure 3c we draw the picture of energy $I(u^n)$ from $t = 0$ to $t = 100$ by the Dufort-frankel scheme. It is clear that the energy function is decreasing from $t = 0$ to $t = 16$. The energy is a constant between $t = 1.7$ to $t = 16$. Therefore, the solution is an state steady solution[4,5]. Moreover the energy is a constant again between $t = 24$ to $t = 100$, so the solution is another state steady solution (see figure 3b). By the figures 3a–3c, we know that the numerical solution is a homoclinic orbit.

From the pictures 3a and 3b, we know that the phase change is very slowly, this phenomena has been analyzed in Carr[9] for the small γ.

For the Euler scheme, the time length is much smaller than one of Dufort-Frankel scheme provided the same space length. the numerical is almost same. So the Dufort-frankel scheme is a large time step length algorithm.

Experiment 2. In this example we take that the parameters as same as the experiment 1 but the initial data as

$$u_i(0) = \begin{cases} \sin(7\pi ih) - 0.002 * random & \text{for } 1 \le i \le J-1 \\ 0.0 & \text{for } i = 0,\ J \end{cases}$$

the numerical results can be seen in figure 4. We may compare our results with Eillott and Stuart[5].

Experiment 3. In this example we take that $f(s) = 0.1(s^2 - 1)$, initial data and the parameters as same as the experiment 1 but the $\tau = 0.002$. The numerical results can be seen in figure 5 at the time $t = 3000$.

Fig.4 Evolution of $u(x,t)$

Fig.5 Solution at $t = 2000$

Acknowledgment. The idea of Remark 1.1 is suggested by Professor Fu Hongyuan, the first author would like to thank Professor Fu Hongyuan for his helpful suggestion.

References

[1] J.K. Hale, Asymptotic Behaviour of Dissipative Systems, Math. Surveys and Monographs No. 25, American Mathematical Society, 1988

[2] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York (1988).

[3] N. Chafee, E. Infante, A Bifurcation problem for a nonlinear parabolic problem, *J. Applic. Anal.,* **4** (1974), 17–37.

[4] B.N. Lu, G.H. Wan, The numerical analysis of a Mean-field equation for the Ising model with Glauber dynamics, *J. Comp. Math.,* **15** : 3 (1997), 203–218.

[5] C.M. Elliott & A.M. Stuart, The global dynamics of discrete semilinear parabolic equations, *SIAM J. Numer Anal.,* **30** (1993), 1622–1663.

[6] J. Noye, Finite Difference Techniques for Partial Differential Equations, Computational Techniques for Differential Equations, Elsevier Science Publishing Company, Inc, 1984.

[7] Y.L. Zhou, Applications of Discrete Functional Analysis to the Finite Difference Method, International Academic Publishers, 1990.

[8] C.M. Elliott, A.R. Gardiner, I. Kostin, B.N. Lu, Mathematical and Numerical Analysis of a Mean-Field Equation for the Ising Model with Glauber Dynamics, *Cont. Math.,* **172** (1994), 217–241.

[9] J. Carr, R.L. Pego, Metastable patterns of $u_t = \epsilon^2 u_{xx} - f(u)$, *Comm. Pure Appl. Math.,* **42** (1989), 523–576.