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A SPLITTING ITERATION METHOD FOR DOUBLE X_0-BREAKING BIFURCATION POINTS*

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Abstract

A splitting iteration method is proposed to compute double X_0-breaking bifurcation points. The method will reduce the computational work and storage, it converges linearly with an adjustable speed. Numerical computation shows the effectiveness of splitting iteration method.

Key words: Double X_0-breaking bifurcation point, splitting iteration method, extended system

1. Introduction

Consider the following two-parameter dependent nonlinear problem

$$f(x, \lambda, \mu) = 0,\ \ f : X \times R^2 \to X, \tag{1.1}$$

where $X = R^n$, λ, μ are real parameters, $f \in C^r(r \geq 3)$, $D_x f_0 (\equiv D_x f(x_0, \lambda_0, \mu_0))$ is a Fredholm map with index zero. One of our main assumptions, which arise in many applications[1,2,5−7], is that f satisfies Z_2-symmetry: there exists a linear operator $S : X \to X$ such that

$$S \neq I, S^2 = I, Sf(x, \lambda, \mu) = f(Sx, \lambda, \mu), \quad \forall (x, \lambda, \mu) \in X \times R^2 \tag{1.2}$$

It is well-known that X has the following natural decomposition:

$$X = X_s \oplus X_a,$$

where

$$X_s = \{x \in X : Sx = x\}, \quad X_a = \{x \in X : Sx = -x\}$$

are the set of symmetric elements and the set of anti-symmetric elements respectively[7].

We also assume that there is an invariant subspace $X_0 \subset X_s$ such that

$$f(x, \lambda, \mu) \in X_0, \quad \forall (x, \lambda, \mu) \in X_0 \times R^2. \tag{1.3}$$

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The usual and best case is $X_0 = \{0\}$. We call (x_0, λ_0, μ_0) a singular point of (1.1) if $f(x_0, \lambda_0, \mu_0) = 0$ and $\dim(\text{Null}(f_x(x_0, \lambda_0, \mu_0)) \geq 1$. In this paper, we are concerned with double X_0-breaking bifurcation points (x_0, λ_0, μ_0) in the sense that[6]

$$f(x_0, \lambda_0, \mu_0) = 0, x_0 \in X_0, \tag{1.4a}$$

$$\text{Null } (D_x f_0) = \text{span } \{\phi_s, \phi_a\}, \phi_s \in X_s, \phi_s \notin X_0, \phi_a \in X_a, \tag{1.4b}$$

$$\text{Range } (D_x f_0) = \{y \in X : \langle \psi_s, y \rangle = \langle \psi_a, y \rangle = 0\},\ \psi_s \in X_s, \psi_a \in X_a, \tag{1.4c}$$

$$\langle \psi_s, D_\lambda f_0 \rangle = \langle \psi_s, D_\mu f_0 \rangle = 0. \tag{1.4d}$$

In addition, as is common, we assume that $\langle \psi_r, \phi_r \rangle = \langle \psi_r, \psi_r \rangle = \langle \phi_r, \phi_r \rangle = 1$, $r = s, a$, $\langle \psi_r, \phi_\delta \rangle = 0$, $(r, \delta) = (s, a)$ or (a, s). X_0-breaking bifurcation point is one of the three most important kinds of bifurcation points (the others are turning points and pitchfork points[2]). For the computation of double X_0-breaking bifurcation points of (1.1), Werner[6] proposed a regular extended system which is a direct method and is at least three times larger than the original equation (1.1). Here we will propose a splitting iteration method. The method produces smaller systems and it could simultaneously compute the point (x_0, λ_0, μ_0), the null vectors of $D_x f_0$, $D_x f_0{}^*$ in a coupled way. This method converges linearly with an adjustable speed and its computational cost at each iteration step remains the same level as that for the regular solution of (1.1)[3,4].

We will construct small extended systems in section 2, then propose the splitting iteration method in section 3. Numerical examples are given in section 4 to show the effectiveness of the method.

2. Extended Systems

First, we introduce the following lemma, which could be proved directly by differentiating (1.2).

Lemma 1. *$\forall x \in X_0$, $\lambda \in R, \mu \in R$,*

(i) $f(x, \lambda, \mu), D_\lambda f(x, \lambda, \mu), D_\mu f(x, \lambda, \mu) \in X_0$;

(ii) X_0, X_s and X_a are invariant subspaces of $D_x f(x, \lambda, \mu), D_{x\lambda} f(x, \lambda, \mu)$, $D_{x\mu} f(x, \lambda, \mu)$;

(iii) $\forall v, w \in X_0$ $D_{xx} f(x, \lambda, \mu) vw \in X_0$;

(iv) $\forall v \in X_s, w \in X_s$, or $\forall v \in X_a, w \in X_a$, $D_{xx} f(x, \lambda, \mu) vw \in X_s$;

(v) $\forall v \in X_s, w \in X_a$, $D_{xx} f(x, \lambda, \mu) vw \in X_a$.

It follows from Lemma 1 and (1.4d) that there exist $v_0, u_0 \in X_0$ such that

$$D_x f_0 v_0 + D_\lambda f_0 = 0, \tag{2.1a}$$

$$D_x f_0 u_0 + D_\mu f_0 = 0, \tag{2.1b}$$

and hence we could introduce the following notations

$$A_r := \langle \psi_r, (D_{xx} f_0 v_0 + D_{x\lambda} f_0) \phi_r \rangle, \tag{2.2a}$$

$$B_r := \langle \psi_r, (D_{xx} f_0 u_0 + D_{x\mu} f_0) \phi_r \rangle, r = s, a. \tag{2.2b}$$

Werner[6] first introduced the following extended systems to determine double X_0-breaking bifurcation points.

$$F(y) = \begin{bmatrix} f(x,\lambda,\mu) \\ D_x f\phi_1 \\ D_x f\phi_2 \\ l_1\phi_1 - 1 \\ l_2\phi_2 - 1 \end{bmatrix} = 0 \tag{2.3}$$

$$y = (x, \phi_1, \phi_2, \lambda, \mu), \ y_0 = (x_0, \phi_s, \phi_a, \lambda_0, \mu_0),$$

$$F : Y \to Y := X_0 \times X_s \times X_a \times R^2$$

where $l_1 \in X'$, $l_2 \in X'$ such that $l_1\phi_s = 1$, $l_2\phi_a = 1$. It is obvious that extended system (2.3) is at least three times larger than (1.1). In the sequel we will propose some small extended systems instead of (2.3).

We shall assume the following nondegeneracy condition

$$\det K \neq 0 \tag{2.4}$$

where

$$K = \begin{bmatrix} A_s & B_s \\ A_a & B_a \end{bmatrix}.$$

Set

$$u_1 = (x, \lambda, \mu) \in X_0 \times R^2, \quad u_2 = (x_1, c_1) \in X_s \times R,$$
$$u_3 = (x_2, c_2) \in X_a \times R, \quad u_4 = (x_3, c_3) \in X_s \times R, \quad u_5 = (x_4, c_4) \in X_a \times R,$$
$$u = (u_1, u_2, u_3, u_3, u_4, u_5) \in V := (X_0 \times R^2) \times (X_s \times R \times X_a \times R)^2,$$
$$\|u\|_V := \|x\| + |\lambda| + |\mu| + \sum_{j=1}^{4} (\|x_j\| + |c_j|).$$

We define the following extended systems:

$$F_1(u) = \begin{bmatrix} f(x,\lambda,\mu) \\ \langle x_3, D_x f(x,\lambda,\mu)x_1\rangle / m^5 \\ \langle x_4, D_x f(x,\lambda,\mu)x_2\rangle / m^5 \end{bmatrix}, \quad F_1 : V \to X_0 \times R^2, \tag{2.5a}$$

$$F_2(u) = \begin{bmatrix} D_x f(x,\lambda,\mu)x_1 + c_1 x_3/m^2 \\ (\langle x_3, x_1\rangle - m^5)/m^2 \end{bmatrix}, \quad F_2 : V \to X_s \times R, \tag{2.5b}$$

$$F_3(u) = \begin{bmatrix} D_x f(x,\lambda,\mu)x_2 + c_2 x_4/m^2 \\ (\langle x_4, x_2\rangle - m^5)/m^2 \end{bmatrix}, \quad F_3 : V \to X_a \times R, \tag{2.5c}$$

$$F_4(u) = \begin{bmatrix} D_x f(x,\lambda,\mu)^* x_3 + c_3 x_3/m^2 \\ (\langle x_3, x_3\rangle - m^4)/2m^2 \end{bmatrix}, \quad F_4 : V \to X_s \times R, \tag{2.5d}$$

$$F_5(u) = \begin{bmatrix} D_x f(x,\lambda,\mu)^* x_4 + c_4 x_4/m^2 \\ (\langle x_4, x_4\rangle - m^4)/2m^2 \end{bmatrix}, \quad F_5 : V \to X_a \times R. \tag{2.5e}$$

Here $m > 0$ is a normalization parameter, and A^* denotes the conjugate operator of a linear operator A.

Let

$$u_1^* = (x_0, \lambda_0, \mu_0), \quad u_2^* = (m^3\phi_s, 0), \quad u_3^* = (m^3\psi_a, 0),$$
$$u_4^* = (m^2\psi_s, 0), \quad u_5^* = (m^2\psi_a, 0), \quad u^* = (u_1^*, u_2^*, \cdots, u_5^*),$$
$$F(u) = (F_1(u), F_2(u), \cdots, F_5(u)).$$

It is easy to check that

$$F(u^*) = 0.$$

By a direct calculation, we obtain

$$D_{u_1}F_1(u^*) = \begin{pmatrix} D_x f_0 & D_\lambda f_0 & D_\mu f_0 \\ \langle \psi_s, D_{xx} f_0 \phi_s \cdot \rangle & \langle \psi_s, D_{x\lambda} f_0 \phi_s \rangle & \langle \psi_s, D_{x\mu} f_0 \phi_s \rangle \\ \langle \psi_a, D_{xx} f_0 \phi_a \cdot \rangle & \langle \psi_a, D_{x\lambda} f_0 \phi_a \rangle & \langle \psi_a, D_{x\mu} f_0 \phi_a \rangle \end{pmatrix},$$
$$D_{u_2}F_2(u^*) = \begin{pmatrix} D_x f_0 & \psi_s \\ \langle \psi_s, \cdot \rangle & 0 \end{pmatrix}, \quad D_{u_3}F_3(u^*) = \begin{pmatrix} D_x f_0 & \psi_a \\ \langle \psi_a, \cdot \rangle & 0 \end{pmatrix},$$
$$D_{u_4}F_4(u^*) = (D_{u_2}F_2(u^*))^*, \quad D_{u_5}F_5(u^*) = (D_{u_3}F_3(u^*))^*.$$

We could obtain the following theorem.

Theorem 1. *Let* (x_0, λ_0, μ_0) *be a double* X_0*-breaking bifurcation point of* $f(x, \lambda, \mu) = 0$*. We assume that* (2.4) *holds. Then* $D_{u_i}F_i(u^*)$ $(i = 1, 2, \cdots, 5)$ *is regular.*

Proof. It is sufficient to show that there exists only the trivial solution $\theta = 0$ for the following linear system

$$D_{u_1}F_1(u^*)\theta = 0, \quad \theta = (x, \lambda, \mu) \in X_0 \times R^2. \tag{2.6}$$

Expanding (2.6), we obtain

$$D_x f_0 x + \lambda D_\lambda f_0 + \mu D_\mu f_0 = 0, \tag{2.7a}$$
$$\langle \psi_s, D_{xx} f_0 \phi_s x \rangle + \lambda \langle \psi_s, D_{x\lambda} f_0 \phi_s \rangle + \mu \langle \psi_s, D_{x\mu} f_0 \phi_s \rangle = 0, \tag{2.7b}$$
$$\langle \psi_a, D_{xx} f_0 \phi_a x \rangle + \lambda \langle \psi_a, D_{x\lambda} f_0 \phi_a \rangle + \mu \langle \psi_a, D_{x\mu} f_0 \phi_a \rangle = 0. \tag{2.7c}$$

It follows from (2.7a) that

$$x = \lambda v_0 + \mu u_0. \tag{2.8}$$

Substituting (2.8) into (2.7b) and (2.7c), we

$$\langle \psi_s, D_{xx} f_0 (\lambda v_0 + \mu u_0) \phi_s \rangle + \lambda \langle \psi_s, D_{x\lambda} f_0 \phi_s \rangle + \mu \langle \psi_s, D_{x\mu} f_0 \phi_s \rangle = 0, \tag{2.9a}$$
$$\langle \psi_a, D_{xx} f_0 (\lambda v_0 + \mu u_0) \phi_a \rangle + \lambda \langle \psi_a, D_{x\lambda} f_0 \phi_a \rangle + \mu \langle \psi_a, D_{x\mu} f_0 \phi_a \rangle = 0, \tag{2.9b}$$

or equivalently,

$$\begin{bmatrix} A_s & B_s \\ A_a & B_a \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.10}$$

According to the condition (2.4), it follows from (2.10) and (2.8) that

$$x = 0, \quad \lambda = 0, \quad \mu = 0,$$

so $\theta = 0$ and we have proved that $D_{u_1}F_1(u^*)$ is regular.

Next we show that $D_{u_2}F_2(u^*)$ is regular. It is sufficient to show that there exists only the trivial solution for the linear system

$$D_{u_2}F_2(u^*)\theta_1 = 0, \quad \theta_1 = (x_1, c_1) \in X_s \times R. \tag{2.11}$$

Expanding (2.11), we obtain

$$D_x f x_1 + c_1\psi_s = 0, \tag{2.12a}$$

$$\langle \psi_s, x_1\rangle = 0. \tag{2.12b}$$

Multiplying (2.12a) by ψ_s, we could get $c_1 = 0$, and hence $x_1 = \alpha\phi_s$ for some constant α. Substituting x_1 into (2.12b) we could get $\alpha = 0$, therefore $x_1 = 0, c_1 = 0$.

Similarly, we could prove the regularities of $D_{u_i}F_i(u^*)$, $i = 3, 4, 5$.

3. Splitting Iteration Method

To compute double X_0-breaking bifurcation point, we introduce the following iteration process:

$$u^{k+1} = H(u^k), \quad k = 0, 1, 2, \cdots, \tag{3.1}$$

where

$$H(u) = (H_1(u), H_2(u), \cdots, H_5(u)). \tag{3.2}$$

$$H_i(u) = u_i - [D_{u_i}F_i(u)]^{-1}F_i(u), \quad i = 1, 2, \cdots, 5. \tag{3.3}$$

Therefore

$$D_{u_i}H_i(u^*) = I - [D_{u_i}F_i(u^*)]^{-1}D_{u_i}F_i(u^*) = 0, \tag{3.4a}$$

$$D_{u_j}H_i(u^*) = -[D_{u_i}F_i(u^*)]^{-1}D_{u_j}F_i(u^*), \quad i, j = 1, 2, \cdots, 5,\ i \neq j. \tag{3.4b}$$

It follows from (3.4) that

$$\begin{aligned}
&\|DH_1(u^*)\| = 0,\\
&\|DH_2(u^*)\| \leq m\|A_1^{-1}\|(m^2(\|D_{xx}f_0\phi_s\| + \|D_{x\lambda}f_0\phi_s\| + \|D_{x\mu}f_0\phi_s\|) + 1),\\
&\|DH_3(u^*)\| \leq m\|A_2^{-1}\|(m^2(\|D_{xx}f_0\phi_a\| + \|D_{x\lambda}f_0\phi_a\| + \|D_{x\mu}f_0\phi_a\|) + 1),\\
&\|DH_4(u^*)\| \leq m^2\|A_1^{-1}\|(\|D_{xx}f_0^*\psi_s\| + \|D_{x\lambda}f_0^*\psi_s\| + \|D_{x\mu}f_0^*\psi_s\|),\\
&\|DH_5(u^*)\| \leq m^2\|A_2^{-1}\|(\|D_{xx}f_0^*\psi_a\| + \|D_{x\lambda}f_0^*\psi_a\| + \|D_{x\mu}f_0^*\psi_a\|),
\end{aligned}$$

where

$$A_1 = \begin{pmatrix} D_x f_0, & \psi_s \\ \langle\psi_s, \cdot\rangle & 0\end{pmatrix}, \quad A_2 = \begin{pmatrix} D_x f_0, & \psi_a \\ \langle\psi_a, \cdot\rangle & 0\end{pmatrix}.$$

We note that in the above equalities we have used $\|A_i^{*-1}\| = \|A_i^{-1}\|$, $i = 1, 2$. It follows from Theorem 1 that A_1 and A_2 have bounded inverse. Since A_1, A_2 are independent of m, we could chose $m > 0$ small enough such that

$$\|DH_i(u^*)\| \leq 1/20, \quad i = 1, 2, \cdots, 5. \tag{3.5}$$

Theorem 2. *Let (x_0, λ_0, μ_0) be a double X_0-breaking bifurcation point of $f(x, \lambda, \mu) = 0$, Chose $m > 0$ small enough so that (3.5) holds. Then there exists a $\delta_0 > 0$ so that for all $u \in B(u^*, \delta_0) = \{u \in V : \|u_i - u_i^*\| \leq \delta_0,\ i = 1, 2, \cdots, 5\}$, we have:*

$$\|DH_i(u)\| \leq 1/10. \tag{3.6}$$

Furthermore, for all $u^0 \in B(u^*, \delta_0)$, the iteration (3.1) is defined and

$$\|u_i^k - u_i^*\| \leq (1/2)^k \delta_0, \quad i = 1, 2, \cdots, 5. \tag{3.7}$$

Proof. According to the definition of F_i and $f \in C^3$, we know that $D_{u_i}F_i(u)$ is C^1 continuous map. Since $D_{u_i}F_i(u^*)$ has bounded inverse, we could find $\delta_1 > 0$ so that $\forall u \in B(u^*, \delta_1)$, $D_{u_i}F_i(u)$ is also bounded invertible and C^1 continuous. Therefore $D_{u_i}H_i(u)$ is also continuous. (3.5) implies there exists $\delta_2 > 0$ such that

$$\|DH_i(u)\| \leq 1/10, \quad \forall u \in B(u^*, \delta_2),\ i = 1, 2, \cdots, 5.$$

Let $\delta_0 = \min(\delta_1, \delta_2)$, then (3.6) holds. Next, we want to prove (3.7) by induction. First, (3.7) is obviously holds for $k = 0$. Next we assume that (3.7) holds for all $k \leq n$, then

$$\begin{aligned}\|u_i^{n+1} - u_i^*\| = \|H_i(u^n) - H_i(u^*)\| &\leq \int_0^1 \|DH_i(u^* + t(u^n - u^*))\| \cdot \|u^n - u^*\| dt \\ &\leq 1/10 \|u^n - u^*\| \leq (1/2)^{n+1}\delta_0\end{aligned}$$

Hence, (3.7) holds for all natural number k and we complete the proof.

4. Numerical Examples

Example 4.1. Consider the following nonlinear problem in R^4

$$f(x, \lambda, \mu) = \begin{pmatrix} 2x_1 - \lambda e^{x_1} + \epsilon x_2 - \mu {x_3}^2 + x_4 \\ 2x_2 - \lambda e^{x_2} + \epsilon x_1 - \mu {x_4}^2 + x_3 \\ 2x_3 - \lambda e^{x_3} + \epsilon x_4 - \mu {x_1}^2 + x_2 \\ 2x_4 - \lambda e^{x_4} + \epsilon x_3 - \mu {x_2}^2 + x_1 \end{pmatrix} = 0 \tag{4.1}$$

where $x = (x_1, x_2, x_3, x_4)$. It is easy to check that $f(x, \lambda, \mu) = 0$ satisfies Z_2-symmetry with

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and we could find that

$$\begin{aligned}&X_0 = \{(v, v, v, v) : v \in R\}, \quad X_s = \{(v_1, v_1, v_2, v_2) : v_i \in R,\ i = 1, 2\}, \\ &X_a = \{(v_3, -v_3, v_4, -v_4) : v_i \in R, i = 3, 4\}.\end{aligned}$$

Chose $\epsilon = 2.0, m = 1.0$, using the splitting iteration method, we could find one double X_0-breaking bifurcation point $(x_0, \lambda_0, \mu_0) = (x_1, x_1, x_1, x_1, \lambda_0, \mu_0) = (0.166667, 0.166667, 0.166667, 0.166667, 0.846482, -6.0)$. The numerical results are show in Table 1.

Table 1

iteration	x_1	λ	μ
0	0.5	0.5	-4.0
1	0.214286	0.749388	-4.285714
2	0.166010	0.843295	-5.632184
3	0.166674	0.846478	-6.001211
4	0.166667	0.846482	-6.000000
5	0.166667	0.846482	-6.000000

Example 4.2. Consider the following Brusselator model[5]:

$$\lambda D_1 \frac{d^2 u}{d\xi^2} + (\mu - 1)u + A^2 v + N(u, v) = 0, \tag{4.2a}$$

$$\lambda D_2 \frac{d^2 v}{d\xi^2} - \mu u - A^2 v - N(u, v) = 0, \tag{4.2b}$$

$$u(0) = u(3) = 0, \quad v(0) = v(3) = 0, \tag{4.2c}$$

where

$$N(u, v) = \frac{\mu}{A} u^2 + 2Auv + u^2 v.$$

We could observe that (4.2) satisfies Z_2-symmetry with

$$S(u(\xi), v(\xi)) = (u(3 - \xi), v(3 - \xi)).$$

Discretize (4.2) by the central difference with step $h = 1.0$, we could obtain

$$f(x, \lambda, \mu) = \begin{bmatrix} \lambda D_1(-2x_1 + x_2) + (\mu - 1)x_1 + A^2 x_3 + N(x_1, x_3) \\ \lambda D_1(x_1 - 2x_2) + (\mu - 1)x_2 + A^2 x_4 + N(x_2, x_4) \\ \lambda D_2(-2x_3 + x_4) - \mu x_1 - A^2 x_3 - N(x_1, x_3) \\ \lambda D_2(x_3 - 2x_4) - \mu x_2 - A^2 x_4 - N(x_2, x_4) \end{bmatrix} = 0, \tag{4.3}$$

where $x = (x_1, x_2, x_3, x_4) := (u_1, u_2, v_1, v_2)$. It is easy to check that $f(x, \lambda, \mu) = 0$ also satisfies Z_2-symmetry with

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$X_0 = \{(0, 0, 0, 0)\}, \ X_s = \{(v_1, v_1, v_2, v_2) : v_i \in R, \ i = 1, 2\},$$
$$X_a = \{(v_3, -v_3, v_4, -v_4) : v_i \in R, \ i = 3, 4\}.$$

Set $m = 1.0$, chose $D_1 = 0.0016, D_2 = 0.008, A = 2.0$, we could get a double X_0-breaking bifurcation point $(x_0, \lambda_0, \mu_0) = (0, 322.748626, 3.865591)$. The numerical results are in Table 2.

Table 2

iteration	λ	μ	iteration	λ	μ
0	310.712324	3.775435			
1	323.802114	3.881061	5	322.750715	3.865619
2	322.996705	3.869072	6	322.749006	3.865596
3	322.804518	3.866339	7	322.748685	3.865592
4	322.759678	3.865737	8	322.748626	3.865591

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