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A MULTI-PARAMETER SPLITTING EXTRAPOLATION AND A PARALLEL ALGORITHM FOR ELLIPTIC EIGENVALUE PROBLEM*

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Abstract

The finite element solutions of elliptic eigenvalue equations are shown to have a multi-parameter asymptotic error expansion. Based on this expansion and a splitting extrapolation technique, a parallel algorithm for solving multi-dimensional equations with high order accuracy is developed.

Key words: Finite element, multi-parameter error expansion, parallel algorithm, splitting extrapolation.

1. Introduction

The extrapolation method has become an important technique to obtain more accurate numerical solutions since it was first established by Richardson in 1926. The applications of extrapolation method in the finit difference can be found in [14]. In 1983, Q.Lin, T.Lü and S.Shen[8] introduced this technique into the finite element method, the development in this direction can be found in [5, 11, 12, 16]. However, this technique has a limitation when dealing with high dimentional problems, since the increasing of the dimension will cause an enormous amount of grid points which requires great computer power in case of the successive refinement. Recently, Zhou et al.[19,20] introduce a so called multi-parameter splitting extrapolation method. In this new method, the domain is divided into several subdomains based on the geometry of the domain, each of which is covered by different meshes so that the number of independent mesh parameters, say p, is as large as possible, and a higher order accuracy approximation is produced by $(p+1)$-processors in parallel. In general, p can be greater than the dimension of the problem. As a result, if the size of the original discrete problem is large, then the size of problems to be dealt with in each processor can be reduced significantly. In this paper, we adopt this method to the elliptic eigenvalue problem, a parallel algorithm for higher order approximations is also proposed.

2. Multi-Parameter Asymptotic Expansion for Eigenvalue

In this section, we only investigate simple eigenvalue problems for elliptic equations, so that we can concentrate on the main idea behind the construction without involving much effort in less important things, let us consider the Dirichlet problem

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$$\begin{cases} -\triangle u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega = (0,1)^n \subset R^n (n \geq 2)$.

Its weak form reads as follow: find $(\lambda, u) \in R \times (H_0^1(\Omega)\backslash\{0\})$ such that

$$a(u,v) = \lambda(u,v), \qquad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where $a(u,v) = \int_\Omega \nabla u \nabla v$, and $(f,v) = \int_\Omega fv$, $\int_\Omega \cdot = \int_\Omega \cdot dx_1 \cdots dx_n$.

Let Ω be divided into m rectangular subdomains $T = \{\Omega_j : j = 1, 2, \cdots, m\}$ so that the edges of each subdomain are parallel to the coordinate axe respectively and T is quasi-uniform. On the subdomain Ω_j, a rectangular mesh refinement with mesh parameters $\{h_{j,1}, \cdots, h_{j,n}\}$ is imposed, where $2h_{j,i}$ is the mesh size in the i^{th} coordinate direction. Assume that the union of all meshes form a quasi-uniform $n-$rectangular partition T^h of Ω with size h, then T^h is determined by some mesh parameters, say $h_1, \cdots, h_p$, with $h = \max\{h_i : i = 1, \cdots, p\}$ and $n \leq p \leq n + m - 1$. To minimize the sizes of the discrete subproblems, p may be chosen such that $p > n$.

Let $S^h(\Omega) = \{v \in C(\Omega)$: $v|_e$ is n-linear, $\forall e \in T^h\}$, $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$, and $i_h : C(\Omega) \longrightarrow S^h(\Omega)$ be the usual $n-$linear interpolation operator.

The finite element approximation of eigenvalue problem is determined by finding $(\lambda_h, u_h) \in R \times (S_0^h(\Omega)\backslash\{0\})$ satisfying

$$a(u_h, \varphi) = \lambda_h(u_h, \varphi), \quad \forall \varphi \in S_0^h(\Omega). \tag{2.3}$$

For continuous eigenvalue λ, there holds an orthonormal eigenfunction u and discrete solutions $(\lambda_h, u_h) \in R \times S_0^h(\Omega)$ such that

$$|\lambda - \lambda_h| + \|u - u_h\|_{0,2} \leq ch^2\|u\|_{2,2}, \tag{2.4}$$

where $\|\cdot\|_{0,2}$ denotes the usual Soblev space, we also denote it by $\|\cdot\|$ in the following. For simplicity, assume that $T^h|_{\Omega_i}$ is uniform and u is smooth enough. We denote $R_h u$ to be the Ritz projection of u which is determined by the equation

$$\int_\Omega \bigtriangledown(u - R_h u) \bigtriangledown v = 0, \quad \forall v \in S_0^h(\Omega).$$

For $e \in T^h$, denote the center of e by $x_e = (x_{e,1}, \cdots, x_{e,n})$ and $e = \prod_{j=1}^n [x_{e,j} - h_{e,j}, x_{e,j} + h_{e,j}]$. For $1 \leq j \leq n$, define

$$F_{e,j}(x_j) = \frac{1}{2}((x_j - x_{e,j})^2 - h_{e,j}^2).$$

From the definition of $F_{e,j}(x_j)$, we easily get the following useful identity

$$F_{e,j} = \frac{1}{6}(F_{e,j}^2)'' - \frac{1}{3}h_{e,j}^2. \tag{2.5}$$

We recall that there holds the following multi-parameter expantion (cf.[19,20]).

Lemma 2.1. *If u is smooth enough, then for any $e \in T^h$,*

$$\int_e \partial_j(u - i_h u) = \sum_{i \neq j} \int_e F_{e,i}\partial_i^2 \partial_j u - \sum_{\substack{\{n_1,\cdots,n_l\}\subset\{1,\cdots,n\}\backslash\{j\} \\ l\geq 2}} \int_e \Big(\prod_{s=1}^{l} F'_{e,n_s}\Big)\partial_{n_1}\cdots\partial_{n_l}\partial_j u \quad (2.6)$$

and

$$\int_e \partial_j(u - i_h u)\prod_{i=1}^{s}(x_{l_i} - x_{e,l_i}) = O(h^{3+n+s}), \quad (2.7)$$

where $s = 1, \cdots, n-1, \partial_j = \partial_{x_j}$, $l_i \neq j$. The last term in (2.6) disappears if $n = 2$.

Lemma 2.2. *If $u \in H_0^1(\Omega) \cap H^4(\Omega)$, then, there exists $\{w_1, \cdots, w_p\} \subset H_0^1(\Omega)$ such that*

$$R_h u(x) = i_h u(x) + \sum_{i=1}^{p} w_i h_i^2 + O(h^4) \quad (2.8)$$

holds in $L^2(\Omega)$.

The following identity for the interpolation operator will play an important role in deriving multi-parameter asymptotic error expansion.

Proposition 2.3. *For any $e \in T^h$, there holds*

$$\int_e (u - i_h u) = \sum_{i=1}^{n} \int_e F_{e,i}\partial_i^2 u - \sum_{\substack{\{n_1,\cdots,n_l\}\subset\{1,\cdots,n\} \\ l\geq 2}} \int_e \Big(\prod_{s=1}^{l} F'_{e,n_s}\Big)\partial_{n_1}\cdots\partial_{n_l} u \quad (2.9)$$

and

$$\int_e (u - i_h u)\prod_{i=1}^{s}(x_{l_i} - x_{e,l_i}) = O(h^{3+n+s}), \quad (2.10)$$

where $s = 1, \cdots, n-1$.

Proof. We give the proof for $n = 2$.

When $n = 2$,

$$\begin{aligned}\int_e (u - i_h u) &= \int_e F_1''(u - i_h u) = \Big(\int_{l_1} - \int_{l_3}\Big) F_1'(u - i_h u)F_2'' + \int_e F_1 \partial_1^2 u \\ &= \int_e F_1\partial_1^2 u + \int_e F_2 \partial_2^2 u - \int_e F_1' F_2' \partial_1 \partial_2 u, \end{aligned} \quad (2.11)$$

where l_1 and l_3 are edges parallel to x-axis.

For the general case, we can use induction method. We now turn to the proof for the second relation. Notice that $x_l - x_{e,l} = \frac{1}{6}(F_{e,l}^2)^{(3)}$.

Thus

$$\begin{aligned} I &\equiv \int_e (u - i_h u)(x_l - x_{e,l}) = \frac{1}{6}\int_e (u - i_h u)(F_{e,l}^2)^{(3)} \\ &= \frac{1}{6}\Big(\int_{x_l = x_{e,l}+h_{e,l}} - \int_{x_l = x_{e,l}-h_{e,l}}\Big)(F_{e,l}^2)^{(2)}(u - i_h u) + \frac{1}{6}\int_e (F_{e,l}^2)' \partial_l^2 u \\ &= \frac{h_{e,l}^2}{3}\int_e \partial_l (u - i_h u) - \frac{1}{6}\int_e F_{e,l}^2 \partial_l^3 u, \end{aligned} \quad (2.12)$$

which proves the result for $s = 1$ combining with Lemma 2.1. We finally get the result by induction.

We are now able to derive multi-parameter error expansion for the eigenvalues. For simplicity, we only consider the case of a simple eigenvalue λ. From (2.3) and the normalization condition for the eigenfunctions, we get

$$\|\nabla(u-u_h)\|^2 = \lambda + \lambda_h - 2\lambda(u,u_h).$$

The insertion of the identity $\|u-u_h\|^2 = 2-2(u,u_h)$ leads to the following equation:

$$\|\nabla(u-u_h)\|^2 = \lambda_h - \lambda + \lambda\|u-u_h\|^2.$$

Further, since the Ritz projection is the projection under the energy norm, we obtain

$$\begin{aligned}\|\nabla(u-u_h)\|^2 =&\|\nabla(u-R_hu)\|^2 + \|\nabla(R_hu-u_h)\|^2\\ =&\|\nabla(u-R_hu)\|^2 + \lambda(u-u_h,R_hu-u_h)+(\lambda-\lambda_h)(u_h,R_hu-u_h).\end{aligned} \tag{2.13}$$

Thus, from the above two identities together with the error estimates (2.4) we get

$$\lambda_h - \lambda = \|\nabla(u-R_hu)\|^2 + O(h^4)\|u\|_{2,2}^2. \tag{2.14}$$

Therefore, We only needs to expand the first term on the right hand side of (2.14), in order to get the multi-parameter expansions for the eigenvalue. Again, from the eigenequations (2.2) and (2.3), we have

$$\begin{aligned}\|\nabla(u-R_hu)\|^2 &= \lambda(u,u-R_hu)\\ &= \lambda(u,u-i_hu)+\lambda(u,i_hu-R_hu).\end{aligned} \tag{2.15}$$

For the first term on the right of (2.15), we apply proposition 2.1 and the identity (2.5) to see that

$$\begin{aligned}(u,u-i_hu) &= \sum_{e\in T^h}\int_e\sum_{i=1}^n F_{e,i}\partial_i^2(u(u-i_hu)) + O(h^4)\\ &= -\frac{1}{3}\sum_{e\in T^h}\int_e\sum_{i=1}^n h_{e,i}^2\partial_i^2((u-i_hu)u) + O(h^4)\\ &= -\frac{1}{3}\sum_{e\in T^h}\int_e\sum_{i=1}^n h_{e,i}^2 u\partial_i^2 u + O(h^4) = -\frac{1}{3}\sum_{i=1}^p h_i^2\int_\Omega N_iu\cdot u + O(h^4),\end{aligned} \tag{2.16}$$

where N_i is some (piecewise) differential operator of $3rd$ order. As for the second term on the right hand side of (2.15), we have, by applying lemma 2.2,

$$(u,i_hu-R_hu) = -\sum_{i=1}^p h_i^2(u,w_i) + O(h^4). \tag{2.17}$$

Combing (2.16) and (2.17), we finally obtain

Theorem 2.4. *Let λ be a simple eigenvalue, then, λ_h, its finite element approxiation admits the following muiti-parameter expansion*

$$\lambda_h - \lambda = \sum_{i=1}^p \xi_i h_i^2 + O(h^4) \tag{2.18}$$

for ξ_i independent of mesh parameter $h_i, i=1,\cdots,p$.

3. Multiparameter Expansions for Eigenfunction

In this section, we shall follow the line of Lin & Xie[9] and Blum[1], our aim is to get the multi-parameter expansion of the error $u_h - i_h u$ in the case of a simple eigenvalue λ. As in Blum[1], we denote $\prod$ and $\prod^{\perp}$ to be the orthogonal projections onto the eigenspace $E(\lambda)$ and its complement $E(\lambda)^{\perp}$ respectively, then

$$\begin{aligned} u_h - i_h u &= \prod(u_h - i_h u) + \prod^{\perp}(u_h - i_h u) \\ &= (u_h - u, u)u + (u - i_h u, u)u + \prod^{\perp}(u_h - i_h u). \end{aligned} \tag{3.1}$$

We shall now estimate the terms on the right hand side of (3.1) for $n = 2$. For the first term, we have $(u_h - u, u) = -\dfrac{1}{2}\|u - u_h\|^2 = O(h^4)$. For the second term, we apply (2.16) to get

$$(u - i_h u, u) = -\frac{1}{3}\sum_{i=1}^{p} h_i^2 u \int_\Omega N_i u \cdot u + O(h^4).$$

We now come to the estimation of the last term. Following Lin and Xie[9], we introduce the operators $K \equiv (-\Delta)^{-1}$ and $K_h = R_h K$, and obtain

$$\begin{aligned} u - u_h =& \lambda K u - \lambda_h K_h u_h = \lambda K(u - u_h) + \frac{1}{\lambda}(\lambda - \lambda_h)u + (u - R_h u) \\ &+ (\lambda - \lambda_h)(K_h - K)u + (\lambda - \lambda_h)K(u_h - u) + \lambda_h(K - K_h)(u_h - u). \end{aligned} \tag{3.2}$$

Note that $\|K - K_h\|_\infty = O(h^2|\ln h|^2)$, we finally see that

$$u - u_h = \lambda K(u - u_h) + \frac{1}{\lambda}(\lambda - \lambda_h)u + (u - R_h u) + O(h^4|\ln h|^2), \tag{3.3}$$

from which we obtain

$$i_h u - u_h = \lambda K(u - u_h) + \frac{1}{\lambda}(\lambda - \lambda_h)u + (i_h u - R_h u) + O(h^4|\ln h|^2), \tag{3.4}$$

or equivalently

$$(I - \lambda K)(i_h u - u_h) = \lambda K(u - i_h u) + \frac{1}{\lambda}(\lambda - \lambda_h)u + (i_h u - R_h u) + O(h^4|\ln h|^2), \tag{3.5}$$

since $v \equiv K(u - i_h u)$ satisfies the equation

$$(\nabla v, \nabla \varphi) = (u - i_h u, \varphi), \quad \forall \varphi \in H_0^1(\Omega). \tag{3.6}$$

For the right term of the equation (3.6), by using proposition (2.3), we can define the coefficients $e_\Omega^1, \cdots, e_\Omega^p$ such that

$$h_1^2(\nabla e_\Omega^1, \nabla\varphi) + \cdots + h_p^2(\nabla e_\Omega^p, \nabla\varphi) = -\frac{\lambda}{3}\sum_{i=1}^{p} h_i^2 \int_\Omega N_i u \cdot \varphi, \quad \forall \varphi \in H_0^1(\Omega). \tag{3.7}$$

Substituting (3.7), (2.8) into (3.5), we get

$$(I-\lambda K)(i_h u-u_h)=\sum_{i=1}^{p} h_i^2(e_\Omega^i-w_i)-\lambda^{-1}(\lambda-\lambda_h)u+O(h^4|\ln h|^2). \qquad (3.8)$$

Since $(I-\lambda K)$ is an ismorphism on the space $E(\lambda)^\perp$ and $\prod^\perp u=0$, thus

$$\prod^{\perp}(i_h u-u_h)=\sum_{i=1}^{p} h_i^2(I-\lambda K)^{-1}\prod^{\perp}(-e_\Omega^i+w_i)+O(h^4|\ln h|^2). \qquad (3.9)$$

In view of the above discussion, we finally arrive at the following result

Theorem 3.1. *Let the assumption of Theorem* 2.4 *be satisfied. Then, the eigenfunction corresponding to a simple eigenvalue* λ, *admits the expansion*

$$u_h-i_h u=\sum_{i}^{p} h_i^2\zeta_i+O(h^4|\ln h|^2), \qquad (3.10)$$

where ζ_i, $i=1,\cdots,p$ *are functions indenpent of mesh parameters* $h_1,\cdots,h_p$.

4. Partition for General Domain

In this section, we shall discuss the case when Ω is an polygonal convex domain in R^2 for sake of simplicity. We find that similar result is obtained.

First decompose Ω into several fixed convex quadrilaterals $T=\{\Omega_1,\cdots,\Omega_m\}$ such that T is quasi-uniform. Denote $(a_{i,j},b_{i,j})$ $(j=1,2,3,4)$ to be the 4 vertices of the Ω_i.

Let Φ_i:

$$x_1(\xi,\eta)=a_{i,1}(1-\xi)(1-\eta)+a_{i,2}\xi(1-\eta)+a_{i,3}\xi\eta+a_{i,4}(1-\xi)\eta,$$
$$x_2(\xi,\eta)=b_{i,1}(1-\xi)(1-\eta)+b_{i,2}\xi(1-\eta)+b_{i,3}\xi\eta+b_{i,4}(1-\xi)\eta$$

be the bilinear coordinate transformations from the unit square $[0,1]^2$ to $\Omega_i (i=1,\cdots,m)$.

Under such mapping, a line parallel to $\xi-$ or $\eta-$ axis in $[0,1]^2$ is transformed to the line linking the two equipartition points of a two opposite edges in Ω_i. For a function v defined on Ω_i, we define the function $\hat{v}$ on $[0,1]$ by

$$\hat{v}=v\circ\Phi_i. \qquad (4.1)$$

Conversely, a function $\hat{v}$ defined on $[0,1]^2$ determines a function v on Ω_i satisfying (4.1). Define

$$S_0^h(\Omega)=\{v\in H_0^1(\Omega): v\circ\Phi_i \text{is piecewise bilinear on } [0,1]^2,\ i=1,2,\cdots\}, \qquad (4.2)$$
$$u=\hat{u}\circ\Phi_i^{-1},\quad \text{on } \Omega_i,$$
$$i_h u=\hat{i_h}\hat{u}\circ\Phi_i^{-1},\quad \text{on } \Omega_i,$$

where $\hat{i_h}\hat{u}$ is the piecewise bilinear interpolant of $\hat{u}$ on $[0,1]^2$. $i_h u(x)=u(x)$ holds for x being the nodal points in Ω and $S_0^h(\Omega)$ is determined by some parameters, say $h_1,\cdots,h_p$. By induction, it can be proved that for any polygonal domain with a proper choice of $\{\Omega_1,\cdots,\Omega_m\}$, p satisfies $p\geq 2$.

Then there exist an interpolation operator I_h (cf. Lin (1990) and Lin Yan and Zhou (1991)) constants ξ_i and functions w_i such that

$$\lambda_h - \lambda = \sum_{i=1}^{p} \xi_i h_i^2 + O(h^4) \tag{4.3}$$

$$I_h u_h = u + \sum_{i=1}^{p} w_i h_i^2 + O(h^4 |\ln h|^2) \tag{4.4}$$

hold for any $x \in \Omega$ having positive distance from the vertices of T.

Remark 4.1. It should point out that other kinds of partitions can also be employed and the corresponding multi-parameter asymptotic error expansions are also exist (cf. Zhou, Liem and Shih (1994)).

5. A Parallel Algorithm

Consider an 2−dimensional problem. Suppose that the domain Ω is divided into m nonoverlapping subdomains $\{\Omega_j : j = 1, 2, \cdots, m\}$, on which meshes are imposed and $h_1, h_2 \cdots$ are the mesh parameters. Among them, p parameters are independent, we denoted it by $h_1, \cdots, h_p$ without loss of generality. Let $h = \max\{h_i : i = 1, \cdots, p\}$ and denote the numerical solution by $u(h_1, \cdots, h_p)$. In many cases, there exists a multi-parameter expansion

$$\lambda(h_1, \cdots, h_p) = \lambda + \sum_{i=1}^{p} \xi_i h_i^2 + O(h^4), \tag{5.1}$$

$$u(h_1, \cdots, h_p) = u + \sum_{i=1}^{p} \eta_i h_i^2 + O(h^4 |\ln h|^2), \tag{5.2}$$

where u is the exact solution and ξ, η_i $(i = 1, \cdots, p)$ are independent of $(h_1, \cdots, h_p)$.

It is obvious that a careful choice of mesh parameters will save the computational work and computer storage and yields a higher accuracy approximation, i.e., there holds

$$\lambda^c \equiv \Big(4 \sum_{i=1}^{p} \lambda_i - (4p-3)\lambda_0\Big)/3 = \lambda + O(h^4), \tag{5.3}$$

$$u^c \equiv \Big(4 \sum_{i=1}^{p} u_i - (4p-3)u_0\Big)/3 = u + O(h^4 |\ln h|^2), \tag{5.4}$$

where $\lambda_i = \lambda(h_1, \cdots, h_{i-1}, h_i/2, h_{i+1}, \cdots, h_p)$ and $u_i = u(h_1, \cdots, h_{i-1}, h_i/2, h_{i+1}, \cdots, h_p)$. Thus, a parallel algorithm for higher accuracy approximations follows:

Algorithm.

Step 1. Compute $\lambda_i, u_i (0 \le i \le p)$ in parallel.

Step 2. Set $\lambda^c = \Big(4 \sum_{i=1}^{p} \lambda_i - (4p-3)\lambda_0\Big)/3$, and $u^c = \Big(4 \sum_{i=1}^{p} u_i - (4p-3)u_0\Big)/3$.

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