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THE LARGE TIME CONVERGENCE OF SPECTRAL METHOD FOR GENERALIZED KURAMOTO-SIVASHINSKY EQUATION (II)*1)

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Abstract

In this paper, we study fully discrete spectral method and long time behavior of solution of generalized Kuramoto-Sivashinsky equation with periodic initial condition. We prove that the large time error estimation for fully discrete solution of spectral method. We prove the existence of approximate attractors $\mathcal{A}_N$, $\mathcal{A}_N^k$ respectively and $d(\mathcal{A}_N, \mathcal{A}) \to 0$, $d(\mathcal{A}_N^k, \mathcal{A}) \to 0$.

Key words: Kuramoto-Sivashinsky equation, large time convergence, Approximate attractor, Upper semicontinuity of attractors.

1. Introduction

In the paper [1], we studed the generalized Kuramoto-Sivashinsky equation

$$u_t + \gamma u_{xxxx} + \beta u_{xxx} + \alpha u_{xx} + f(u)_x + \phi(u)_{xx} = g(u) + h(x,t). \tag{1.1}$$

We proved the existence and uniqueness of global solution for periodic initial problem and gave the large time error estimation for the solution of continuous spectral method.

The aim of this paper is to study fully discrete spectral method and the long time behavior of the solution of this system. In §1 we given the large time error estimation for fully discrete solution of spectral method. In §2 we prove the existence of approximate attractors $\mathcal{A}_N$, $\mathcal{A}_N^k$ and in §3 we prove the convergence of approximate attractors $d(\mathcal{A}_N, \mathcal{A})$, $d(\mathcal{A}_N^k, \mathcal{A}) \to 0$.

2. The Large Time Error Estimation of Fully Discrete Approximate Solution

For the problem (1.1), we construct the following fully discrete approximate spectral scheme

$$\Big(\frac{1}{k}(u_N^n - u_N^{n-1}) + \alpha u_{Nxx}^n + \beta u_{Nxxx}^n + \gamma u_{Nxxxx}^n + f(u_N^n)_x + \phi(u_N^n)_{xx}$$

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$$- g(u_N^n) - h(x,t_n), \chi\Big) = 0, \quad \forall \chi \in S_N, \tag{2.1}$$

$$u_N^0 = u_{0N} = F_N u_0, \tag{2.2}$$

where the k is step size of time, F_N is the orthogonal projective operator from $L^2(\Omega)$ to S_N.

Lemma 1. *If $f(t)$, $f'(t) \in L^2(R^+)$, let $f_n = f(nk)$, $k > 0$, then*

$$k\sum_{n=1}^{\infty} f_n^2 \le (1+k)\int_0^{+\infty} |f|^2 dt + k\int_0^{+\infty} |f'|^2 dt. \tag{2.3}$$

Proof. Using the integration by parts

$$\begin{aligned} kf_n^2 &= \int_{t_{n-1}}^{t_n} (t-t_{n-1})\frac{d}{dt}f^2 dt + \int_{t_{n-1}}^{t_n} f^2 dt = 2\int_{t_{n-1}}^{t_n} (t-t_{n-1})ff'dt + \int_{t_{n-1}}^{t_n} f^2 dt \\ &\le \int_{t_{n-1}}^{t_n} (t-t_{n-1})(f^2+f'^2)dt + \int_{t_{n-1}}^{t_n} f^2 dt \\ &\le (1+k)\int_{t_{n-1}}^{t_n} f^2 dt + k\int_{t_{n-1}}^{t_n} f'^2 dt. \end{aligned} \tag{2.4}$$

Summing up for n in both sides of (2.4), we obtain the conclusion of Lemma.

Now we make priori estimates for the solution of (2.1)–(2.2).

Lemma 2. *Under the conditions of Lemma 1 of* [1] *and assume $h_t \in L^2(Q_\infty)$, then we have the estimates for the solution of* (2.1)–(2.2)

$$\|u_N^n\|^2 \le \frac{1}{(1+\lambda k)^n}\|u_0\|^2 + (1+k)\int_0^{+\infty} \|h(t)\|^2 dt + k\int_0^{+\infty} \|h_t(t)\|^2 dt \le C^*, \tag{2.5}$$

$$k\sum_{n=1}^{\infty} \|D_x^j u_N^n\|^2 \le \tilde{C}_j\Big(\|u_0\|^2 + (1+k)\int_0^{+\infty} \|h(t)\|^2 dt + k\int_0^{+\infty} \|h_t(t)\|^2 dt\Big) = C_j^*, \quad 0 \le j \le 2, \tag{2.6}$$

where $\lambda = -2\Big[g_0 + \dfrac{1}{2}(\alpha + \phi_0 + 1)\Big] > 0$, C^*, C_j^*, $0 \le j \le 2$ are constants independent of N.

Lemma 3. *If the conditions of Lemma* 3 *of* [1] *and Lemma* 2 *are satisfied and assume $h_{xt} \in L^2(Q_\infty)$, then we have*

$$\begin{aligned} \|u_{Nx}^n\|^2 &\le \frac{1}{(1-2kg_0)^n}\|u_{0x}\|^2 + C\Big[(1+k)\int_0^{+\infty} \|h_x(t)\|^2 + k\int_0^{+\infty} \|h_{xt}(t)\|^2 dt\Big] \\ &\quad + Ck\sum_{j=1}^{\infty} \|u_N^j\|^2 \le E^*, \quad \forall n \ge 0, \end{aligned} \tag{2.7}$$

$$k\sum_{n=1}^{\infty} \|D_x^3 u_N^n\|^2 \le C_3^*, \tag{2.8}$$

where the same as Lemma 2*, the constants E^* and C_3^* are all independent of N.*

The proof of Lemmas 2,3 is similar to the Lemmas 1, 3 of [1].

Now we estimate the error of fully discrete solution.

Let $u^n = u(x, t_n)$, by (1.1) at $t = t_n$, we have

$$\begin{aligned}&\Big(\frac{1}{k}(u^n - u^{n-1}) + \alpha u^n_{xx} + \beta u^n_{xxx} + \gamma u^n_{xxxx} + f(u^n)_x + \phi(u^n)_{xx} - g(u^n) - h(x, t_n), \chi\Big)\\ &\qquad = \Big(\frac{1}{k}(u^n - u^{n-1}) - u^n_t, \chi\Big), \quad \forall \chi \in S_N. \qquad (2.9)\end{aligned}$$

Putting $u^n - u^n_N = u^n - F_N u^n - (u^n_N - F_N u^n) = \xi^n - \zeta^n$, then ζ^n satisfy

$$\begin{aligned}&\Big(\frac{1}{k}(\zeta^n - \zeta^{n-1}) + \alpha \zeta^n_{xx} + \beta \zeta^n_{xxx} + \gamma \zeta^n_{xxxx} + f(u^n_N)_x - f(u^n)_x\\ &+ \phi(u^n_N)_{xx} - \phi(u^n)_{xx} - g(u^n_N) + g(F_N u^n), \chi\Big)\\ =&\Big(\frac{1}{k}(\xi^n - \xi^{n-1}) + \alpha \xi^n_{xx} + \beta \xi^n_{xxx} + \gamma \xi^n_{xxxx} + g(F_N u^n) - g(u^n), \chi\Big)\\ &+ \Big(u^n_t - \frac{1}{k}(u^n - u^{n-1}), \chi\Big), \quad \forall \chi \in S_N. \qquad (2.10)\end{aligned}$$

Let $\chi = \zeta^n$, similar to the estimate of [1] and noting

$$\Big|\Big(u^n_t - \frac{1}{k}(u^n - u^{n-1}), \zeta^n\Big)\Big| \le \begin{cases} \|\zeta^n\|\Big\|\frac{1}{k}\int_{t_{n-1}}^{t_n}(s - t_{n-1})u_{tt}(s)ds\Big\| \le \frac{1}{8}\|\zeta^n\|^2\\ \qquad +Ck\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2 ds, & (2.11)\\ \frac{1}{8}\|\zeta^n\|^2 + \frac{C}{kn^2}\int_{t_{n-1}}^{t_n} s^2\|u_{tt}(s)\|^2 ds, & (2.12)\end{cases}$$

$$(g(u^n_N) - g(F_N u^n), \zeta^n) = (g'(\rho^n_3)\zeta^n, \zeta^n) \le g_0\|\zeta^n\|^2,$$

but

$$\begin{aligned}g(F_N u^n) - g(u^n) &= -g'(u^n)\xi^n + \frac{1}{2}g''(\eta^n_3)(\xi^n)^2\\ &= -g''(\rho^n_4)u^n\xi^n - g'(0)\xi^n + \frac{1}{2}g''(\eta^n_3)(\xi^n)^2.\end{aligned}$$

By the orthogonal of ζ^n with ξ^n, $(g'(0)\xi^n, \zeta^n) = 0$, hence we have

$$\begin{aligned}|(g(F_N u^n) - g(u^n), \zeta^n)| &\le C\|g''_n\|_{L^\infty}(\|u^n\|_1 + \|u^n_N\|_1)\|\xi^n\|\|\zeta^n\|\\ &\le \frac{1}{8}\|\zeta^n\|^2 + C\|g''_n\|^2_{L^\infty}(\|u^n\|^2_1 + \|u^n_N\|^2_1)\|\xi^n\|^2.\end{aligned}$$

Substituting the above estimates into (2.10) (let $\chi = \zeta^n$), we obtain

$$\begin{aligned}\frac{1}{2k}(\|\zeta^n\|^2 - \|\zeta^{n-1}\|^2) + \Big(\gamma - \frac{\alpha}{2} - 2\eta\Big)\|\zeta^n_{xx}\|^2 &\le \Big(g_0 + \frac{\alpha}{2} + \frac{1}{4}\Big)\|\zeta^n\|^2\\ + C_1(\|u^n\|^2_1 + \|u^n_N\|^2_1)\|\zeta^n\|^2 &+ C_2(\|u^n\|^2_1 + \|u^n_N\|^2_1)\|\xi^n\|^2\end{aligned}$$

$$+ C_3k\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2ds. \tag{2.13}$$

Noting $\gamma-\frac{\alpha}{2}-2\eta=\frac{1}{2}\gamma^*>0$, $g_0+\frac{\alpha}{2}+\frac{1}{4}=-\frac{\delta}{2}<0$,

$$R_1^n=2C_1(\|u^n\|_1^2+\|u_N^n\|_1^2),\quad R_2^n=2C_2(\|u^n\|_1^2+\|u_N^n\|_1^2),$$

then (2.14) can be written

$$\|\zeta^n\|^2\le\|\zeta^{n-1}\|^2+kR_1^n\|\zeta^n\|^2+kR_2^n\|\xi^n\|^2+C_4k^2\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2ds \tag{2.14}$$

i.e.

$$\|\zeta^n\|^2\le\frac{1}{1-kR_1^n}\Big[\|\zeta^{n-1}\|^2+kR_2^n\|\xi^n\|^2+C_5k^2\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2ds\Big]. \tag{2.15}$$

According to Lemma 3 and Theorem 1 of [1] and we take k small enough such that $kR_1^n=2kC_1(\|u^n\|_1^2+\|u_N^n\|_1^2)<\frac{1}{2}$. When $0<x<\frac{1}{2}$, $\frac{1}{1-x}<1+2x$ and $\zeta(0)=0$, thus from (2.15) we have

$$\begin{aligned}\|\zeta^n\|^2&\le(1+2kR_1^n)\Big[\|\zeta^{n-1}\|^2+kR_2^n\|\xi^n\|^2+C_5k^2\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2ds\Big]\le\cdots\\&\le\prod_{j=1}^n(1+2kR_1^j)\sum_{p=1}^n\Big[kR_2^p\|\xi^p\|^2+C_5k^2\int_{t_{p-1}}^{t_p}\|u_{tt}(s)\|^2ds\Big].\end{aligned}$$

Thanks to $\ln(1+x)\le x$, for $x\ge0$, thus

$$\ln\prod_{j=1}^n(1+2kR_1^j)=\sum_{j=1}^n\ln(1+2kR_1^j)\le2k\sum_{j=1}^nR_1^j,$$

from this we obtain

$$\prod_{j=1}^n(1+2kR_1^j)\le e^{2k\sum_{j=1}^nR_1^j}\le\exp\Big\{2kC\sum_{j=1}^\infty(\|u^j\|_1^2+\|u_N^j\|_1^2)\Big\}\le C.$$

Hence

$$\begin{aligned}\|\zeta^n\|^2&\le Ck\sum_{j=1}^\infty R_2^j\|\xi^j\|^2+Ck^2\int_0^{+\infty}\|u_{tt}(s)\|^2ds\\&\le CN^{-2m}\sup_{0<j<+\infty}\|u^j\|_m^2k\sum_{j=1}^\infty(\|u^j\|_1^2+\|u_N^j\|_1^2)+Ck^2\int_0^{+\infty}\|u_{tt}(t)\|^2dt\\&\le CN^{-2m}\sup_{0<j<+\infty}\|u^j\|_m^2\Big[(1+k)\int_0^{+\infty}\|u(t)\|_1^2dt+k\int_0^{+\infty}\|u_t(t)\|_1^2dt\\&\quad+k\sum_{j=1}^\infty\|u_N^j\|^2\Big]+Ck^2\int_0^{+\infty}\|u_{tt}(t)\|^2dt\le C(N^{-2m}+k^2),\quad\forall n\ge0.\end{aligned} \tag{2.16}$$

Using the triangle inequality, we obtain

$$\|u^n - u_N^n\| \le \|u^n - F_N u^n\| + \|u_N^n - F_N u^n\| \le C(N^{-m} + k), \quad \forall n \ge 0.$$

Summing up for n from 1 to Q on both sides of (2.13)

$$\gamma^* k \sum_{n=1}^{Q} \|\zeta_{xx}^n\|^2 + \delta k \sum_{n=1}^{Q} \|\zeta^n\|^2 \le 2C_6 k \sum_{n=1}^{Q} (\|u^n\|_1^2 + \|u_N^n\|_1^2)(\|\zeta^n\|^2 + \|\xi^n\|^2)$$
$$+ 2C_7 k^2 \int_0^{t_Q} \|u_{tt}(s)\|^2 ds \le C_8(N^{-2m} + k^2) k \sum_{n=1}^{\infty} (\|u^n\|_1^2 + \|u_N^n\|_1^2)$$
$$+ 2C_7 k^2 \int_0^{+\infty} \|u_{tt}(s)\|^2 ds$$

where C_6, C_7 and C_8 are constants independent of Q. Thus we have

$$k \sum_{n=1}^{\infty} (\|\zeta_{xx}^n\|^2 + \|\zeta^n\|^2) \le C(N^{-2m} + k^2).$$

By the triangle inequality and interpolating inequality

$$k \sum_{n=1}^{\infty} \|D_x^j(u^n - u_N^n)\|^2 \le CN^{-2m} k \sum_{n=1}^{\infty} \|u^n\|_{m+j}^2 + C(N^{-2m} + k^2), \quad 0 \le j \le 2.$$

Summing up, we obtain

Theorem 1. *Suppose that the conditions of Lemma* 3 *is satisfied and* f, ϕ, $g \in C^2$, *the solution of* (1.1) $u \in L^\infty(R^+; H_p^m(\Omega)) \cap L^2(R^+; H_p^1(\Omega))$, $u_t \in L^2(R^+; H_p^1(\Omega))$, $u_{tt} \in L^2(Q_\infty)$ $(m \ge 1)$, *then for the solution of fully discrete problem* (2.1)–(2.2) *we have the following large time error estimation*

$$\|u^n - u_N^n\| \le C(N^{-m} + k), \quad \forall n \ge 0.$$

If further suppose that $u, u_t \in L^2(R^+; H_p^{m+j}(\Omega))$, $0 \le j \le 2$, *we have yet*

$$k \sum_{n=1}^{\infty} \|D_x^j(u^n - u_N^n)\|^2 \le C(N^{-2m} + k^2).$$

3. The Existence of Global Approximate Attractors $\mathcal{A}_N$, $\mathcal{A}_N^k$

For the problem (1.1), the existence of global attractor has been proved in [5]. Similar proof can be obtained for semidiscrete spectral approximate. Now we prove semigroup operator $S_N^k(n)$ of discrete problem (2.1) has a global attractor.

Lemma 4. *Under the conditions of Lemma* 3, *suppose that* $f \in C^1$, $\phi \in C^2$, $g \in C^1$, $u_0 \in H_p^2(\Omega)$, *then we have*

$$\|u_{Nxx}^n\|^2 \le \frac{1}{(1 - 2g_0 k)^n} \|u_{0xx}\|^2 + Ck \sum_{n=0}^{\infty} (\|u_N^n\|^2 + \|u_{Nx}^n\|^2)$$

$$+C\int_0^\infty (\|h(t)\|^2+\|h_t(t)\ |^2)dt \le E, \quad \forall n\ge 0,$$

where the constant E is independent of N.

Combining the Lemmas 2-4, we can obtain that there exist constants $A_i (i = 0, 1, 2)$ independent of N, such that the solution of (2.1)-(2.2) belong to the set $B_0 \triangleq \{u_N^n \in H_p^2(\Omega) \cap S_N | \|D_x^i u_N^n\| \le A_i,\ i = 0, 1, 2\}$ for enough large n. Hence the set B_0 is a bounded absorbing set of $\{S_N^k(n)\}$. Moreover, the semigroup operator $S_N^k(n)$ maps $H_p^2 \cap S_N$ to $H_p^2 \cap S_N$ and S_N is a space of finite dimension for every N, hence $S_N^k(n)$ is completely continuous operator for $n \ge 1$. By the result of [2], we have

Theorem 2. *The semigroup operator $S_N(t)$ and $S_N^k(n)$ have a compact global attractors $\mathcal{A}_N$ and $\mathcal{A}_N^k$ respectively.*

4. The Convergence of the Approximate Attractors $\mathcal{A}_N$ and $\mathcal{A}_N^k$

In order to prove the convergence of the approximate attractors, we need the following theorem [2].

Let H_η is a family closed subspace of H, $0 < \eta \le \eta_0$ and $\cup_{0<\eta\le\eta_0} H_\eta$ is dense in H . For every $\eta > 0$, the semigroup operator $S_\eta(t)$ maps H_η into itself and every operator $S_\eta(t)$ is continuous for $t \ge 0$. Suppose that for every compact interval $I^* \subset (0, +\infty)$

$$\delta_\eta(I^*) = \sup_{\substack{u_0\in H_\eta \\ \|u_0\|_H\le R}} \sup_{t\in I^*} d(S_\eta(t)u_0,\ S(t)u_0) \to 0, \quad \eta \to 0.$$

We also assume that for every $\eta > 0$, $S_\eta(t)$ possesses an attractor $\mathcal{A}_\eta$ that attracts any bounded open neigbourhood which include $\mathcal{A}_\eta \cup \mathcal{A}$.

Theorem 3. *Under the above assumptions, we have*

$$d(\mathcal{A}_\eta,\ \mathcal{A}) \to 0, \quad \eta \to 0,$$

where $d(X, Y)$ is semidistance between X and Y, i.e. $d(X, Y) = \sup_{x\in X}\inf_{y\in Y} \|x - y\|_H$.

At first we discuss $d(\mathcal{A}_N, \mathcal{A}) \to 0$, if $N \to +\infty$.

Lemma 5. *Under the conditions of Lemma 3, we suppose that $f \in C^2$, $\phi \in C^3$ $g \in C^1$, then we have*

$$\begin{aligned}\|u_{xxx}(t)\|^2 \le& \frac{2}{t^2}\int_0^t s\|u_{xxx}(s)\|^2 ds \\ &+ \frac{C}{t^2}\int_0^t [\|su_x(s)\|^2 + \|su_{xx}(s)\|^2 + \|sh_x(s)\|^2]ds, \quad \text{for } t > 0.\end{aligned}$$

Proof. On both sides of (1.1) taking the inner product of with $t^2u_{x^6}$, similar to Lemma 4, we can obtain

$$\frac{1}{2}\frac{d}{dt}\|tu_{xxx}\|^2 + \frac{\gamma}{2}\|tu_{x^5}\|^2 \le t\|u_{xxx}\|^2 + C(\|tu_x\|^2 + \|tu_{xx}\|^2 + \|th_x\|^2). \tag{4.1}$$

Integrating (4.1) on both sides, we can obtain the result of Lemma immeditely.

According to (3.6) of [1]

$$\|u_N(t) - F_N u(t)\|^2 = \|\zeta(t)\|^2 \le \int_0^t R_2(s)\|\xi(s)\|^2 \exp\Big(\int_s^t R_1(\tau)d\tau\Big)ds\Big),$$

where $R_1(t) = 2(C_1 + C_3)(\|u(t)\|_1^2 + \|u_N(t)\|_1^2)$, $R_2(t) = 2[2C_2\|u(t)\|^2 + C_4(\|u(t)\|_1^2 + \|u_N(t)\|_1^2 + C_5\|g'\|_{L^\infty}^2]$ and C_i $(i = 1, \cdots, 5)$ are constants independent of t and N. By using Lemma 1 and 3 of [1],

$$\sup_{t\in R^+} \|R_2(t)\| \le C, \ \exp\Big(\int_0^{+\infty} R_1(t)dt\Big) \le C,$$

we have

$$\begin{aligned}\|u_N(t) - F_N u(t)\|^2 &\le C\int_0^{+\infty} \|\xi(s)\|^2 ds = C\int_0^{+\infty} \|u(s) - F_N u(s)\|^2 ds \\ &\le CN^{-6}\int_0^{+\infty} \|u_{xxx}(s)\|^2 ds.\end{aligned}$$

Using the inverse property

$$\|u_N(t) - F_N u(t)\|_2^2 \le CN^4\|u_N(t) - F_N u(t)\|^2 \le CN^{-2}\int_0^{+\infty} \|u_{xxx}(s)\|^2 ds$$

and by Lemma 5, we have

$$\begin{aligned}\|u(t) - u_N(t)\|_2^2 &\le 2[\|u(t) - F_N u(t)\|_2^2 + \|u_N(t) - F_N u(t)\|_2^2] \\ &\le CN^{-2}\Big(\|u_{xxx}(t)\|^2 + \int_0^{+\infty} \|u_{xxx}(s)\|^2 ds\Big) \\ &\le CN^{-2}\Big(\frac{1}{t}\int_0^{+\infty} \|u_{xxx}(s)\|^2 ds + \int_0^{+\infty} (\|u(s)\|_3^2 + \|h_x(s)\|^2) ds\Big), \quad t > 0.\end{aligned}$$

Finally according to Theorem 3, we take $H = H_p^2(\Omega)$ and $I = [T_1, T_2] \subset R^+$, then we have

Theorem 4. *$d(\mathcal{A}_N, \mathcal{A}) \to 0$, as $N \to +\infty$.*

Next we discuss $d(\mathcal{A}_N^k, \mathcal{A}) \to 0$, as $N \to +\infty$, $k \to 0$.

For this, we need some priori estemates of u_t and u_{tt}. Taking the inner product of (1.1) with u_t, we obtain

Lemma 6. *If $f \in C^1$, ϕ, $g \in C^2$, $h \in C^0 \cap L^2(Q_\infty)$, $u_0 \in H_p^2(\Omega)$, then we have*

$$\sup_{0\le t<+\infty} \|u_{xx}(t)\| \le C_4^*, \quad \int_0^{+\infty} \|u_t(s)\|^2 ds \le C_5^*,$$

where C_4^*, C_5^* are constants independent of t.

Differentiating (1.1) with respect to t and taking the inner product of the result identity with u_t, we have

Lemma 7. *Under the conditions of Lemma* 6, *we suppose that* $f \in C^2$, $\phi \in C^3$, $h, h_t \in L^2(Q_\infty)$, *then we have*

$$t\|u_t(t)\|^2 + \gamma \int_0^t s\|u_{txx}(s)\|^2 ds \le \int_0^t \|u_t(s)\|^2 ds + C_6^* \int_0^t s(\|u_t(s)\|^2 + \|h_t\|^2) ds, \quad t > 0,$$

where the constant C_6^* *is independent of* t.

Differentiating (1.1) with respect to t and taking the inner product of result identity with u_{txx} and u_{tt} respectively, we have

Lemma 8. *Under the conditions of Lemma* 7, *we have*

$$\begin{aligned} t^2\|u_{xt}(t)\|^2 + \gamma \int_0^t s^2\|u_{xxxt}(s)\|^2 ds &\le 2\int_0^t s\|u_{xt}(s)\|^2 ds + 2\int_0^t s\|u_t(s)\|^2 ds \\ &+ C_7^* \int_0^t s^2[\|u_t(s)\|^2 + \|u_{xx}(s)\|^2\|u_t(s)\|^2 + \|h_t(s)\|^2] ds, \\ \gamma t^2\|u_{xxt}(t)\|^2 + \int_0^t s^2\|u_{tt}(s)\|^2 ds &\le \alpha t^2\|u_{xt}(t)\|^2 + 2\gamma \int_0^t s\|u_{xxt}(s)\|^2 ds \\ &+ C_8^* \int_0^t s^2(\|u_{xt}(s)\|^2 + \|u_{xx}(s)\|^2\|u_t(s)\|^2 + \|u_{xx}(s)\|^4\|u_t(s)\|^2 \\ &+ \|u_{xx}(s)\|^2\|u_{xt}(s)\|^2 + \|u_{xxxt}(s)\|^2 + \|h_t(s)\|^2) ds, \end{aligned}$$

where the constants C_7^*, C_8^* *are independent of* t.

These Lemmas can be proved as Lemma 5.

Thus, for any $T > 0$, $\displaystyle\int_0^T s^2\|u_{tt}(s)\|^2 ds \le E_T$.

Lemma 9. *Under the conditions of Lemma* 3, *we suppose that* $f \in C^2$, $\phi \in C^3$, $g \in C^1$, h_x, $h_{xt} \in L^2(Q_\infty)$, *then we have*

$$\|u_{Nxxx}^M\|^2 \le \frac{C}{Mk} k \sum_{n=0}^{M-1} \|u_{Nxxx}^n\|^2 + Ck \sum_{n=1}^{M-1} (\|u_{Nxx}^{n+1}\|^2 + \|u_{Nx}^{n+1}\|^2 + \|h_x^{n+1}\|^2), \ \forall M \ge 1,$$

where the constant C *is independent of* N *and* M.

Proof. Setting $\chi = (n-1)^2 k^2 u_{Nx^6}^n$ in (2.1), we obtain

$$\begin{aligned} \Big(\frac{1}{k}(u_N^n - u_N^{n-1}) + \alpha u_{Nxx}^n + \beta u_{Nxxx}^n + \gamma u_{Nxxxx}^n + f(u_N^n)_x \\ + \phi(u_N^n)_{xx} - g(u_N^n) - h(x, t_n), (n-1)^2 k^2 u_{Nx^6}^n\Big) = 0, \end{aligned} \tag{4.2}$$

Because

$$\begin{aligned} \Big(\frac{1}{k}(u_N^n - u_N^{n-1}), (n-1)^2 k^2 u_{Nx^6}^n\Big) &= -(n-1)^2 k (u_{Nxxx}^n - u_{Nxxx}^{n-1}, u_{Nxxx}^n) \\ &= -\frac{1}{2k}[\|nku_{Nxxx}^n\|^2 - \|(n-1)ku_{Nxxx}^{n-1}\|^2 - (2n-1)\|ku_{Nxxx}^n\|^2 \\ &+ \|(n-1)k(u_{Nxxx}^n - u_{Nxxx}^{n-1})\|^2] \end{aligned}$$

and using the interpolating inequality and Hölder inequality

$$\begin{aligned}|(\alpha u^n_{Nxx},(n-1)^2k^2u^n_{Nx^6})| \le&\alpha(n-1)^2k^2\|u^n_{Nxxx}\|\ \|u^n_{Nx^5}\|\\ \le& C(n-1)^2k^2\|u^n_{Nx^5}\|^{\frac{4}{3}}\|u^n_{Nxx}\|^{\frac{2}{3}}\\ \le&\ \eta\|(n-1)ku^n_{Nx^5}\|^2+C\|(n-1)ku^n_{Nxx}\|^2.\end{aligned}$$

By using the imbedding theorem $\|u^n_{Nx}\|_\infty\le C\|u^n_{Nxx}\|$ and Lemma 3

$$\begin{aligned}|(f(u^n_N)_x,(n-1)^2k^2u^n_{Nx^6})| =& |(f''(u^n_N)(u^n_{Nx})^2+f'(u^n_N)u^n_{Nxx},(n-1)^2k^2u^n_{Nx^5})|\\ \le&\ \eta\|(n-1)ku^n_{Nx^5}\|^2\\ &+C(\|u^n_{Nx}\|^2\|(n-1)ku^n_{Nxx}\|^2+\|(n-1)ku^n_{Nxx}\|^2)\\ \le&\ \eta\|(n-1)ku^n_{Nx^5}\|^2+C\|(n-1)ku^n_{Nxx}\|^2.\end{aligned}$$

Similarly

$$\begin{aligned}|(\phi(u^n_N)_{xx},(n-1)^2k^2u^n_{Nx^6})| =&|(\phi'''(u^n_N)(u^n_{Nx})^3+3\phi''(u^n_N)u^n_{Nx}u^n_{Nxx}\\ &+\phi'(u^n_N)u^n_{Nxxx},(n-1)^2k^2u^n_{Nx^5})|\\ \le&\eta\|(n-1)ku^n_{Nx^5}\|^2+C\sum_{j=1}^{3}\|(n-1)kD^j_xu^n_N\|^2,\\ |(g(u^n_N),(n-1)^2k^2u^n_{Nx^6})| =&\ |(g'(u^n_N)u^n_{Nx},(n-1)^2k^2u^n_{Nx^5})|\\ \le&\ \eta\|(n-1)ku^n_{Nx^5}\|^2+C\|(n-1)ku^n_{Nx}\|^2.\end{aligned}$$

Substituting above estimates into (2.1), we obtain

$$\begin{aligned}&\frac{1}{2k}(\|nku^n_{Nxxx}\|^2-\|(n-1)ku^{n-1}_{Nxxx}\|^2)\\ &\quad+\|(n-1)k(u^n_{Nxxx}-u^{n-1}_{Nxxx})\|^2+\frac{\gamma}{2}\|(n-1)ku^n_{Nx^5}\|^2\\ &\le\frac{2n-1}{2}k\|u^n_{Nxxx}\|^2+C(\|(n-1)ku^n_{Nxx}\|^2+\|(n-1)ku^n_{Nx}\|^2+\|(n-1)kh^n_x\|^2).\end{aligned}$$

Summing up for n from 1 to M

$$\begin{aligned}\frac{1}{2k}\|Mku^M_{Nxxx}\|^2 \le& k\sum_{n=1}^{M}\frac{2n-1}{2}\|u^n_{Nxxx}\|^2\\ &+C\sum_{n=1}^{M-1}(\|nku^{n+1}_{Nxx}\|^2+\|nku^{n+1}_{Nx}\|^2+\|nkh^{n+1}_x\|^2),\quad\forall M\ge1,\end{aligned}$$

i.e.

$$\|u^M_{Nxxx}\|^2\le\frac{C}{Mk}k\sum_{n=1}^{M-1}\|u^n_{Nxxx}\|^2+Ck\sum_{n=1}^{M-1}(\|u^{n+1}_{Nxx}\|^2+\|u^{n+1}_{Nx}\|^2+\|h^{n+1}_x\|^2),\quad\forall M\ge1.$$

In (2.13), the substitution of (2.12) for (2.11) and similar to (2.16) we have ($m=3$)

$$\|\zeta^n\|^2\le CkN^{-6}\sum_{p=1}^{n}(\|u^p\|_1^2+\|u^p_N\|_1^2)\|u^p_{xxx}\|^2+C\sum_{p=1}^{n}\frac{1}{p^2}\int_{t_{p-1}}^{t_p}s^2\|u_{tt}(s)\|^2ds$$

$$\leq CN^{-6}\int_0^T \|u_{xxx}(s)\|^2 ds + C\sum_{p=1}^{n}\frac{1}{p^2}\int_{t_{p-1}}^{t_p} s^2\|u_{tt}(s)\|^2 ds.$$

Since

$$\sum_{p=1}^{\infty}\frac{1}{p^2} < +\infty, \quad \int_{t_{p-1}}^{t_p} s^2\|u_{tt}(s)\|^2 ds \to 0,$$

as $k \to 0$, we can obtain $\|\zeta^n\| \to 0$, as $N \to \infty$, $k \to 0$, $T_1 \leq nk \leq T_2$. If setting $\chi = \zeta^n_{x^4}$ on (2.10), similarly to the estimate of $\|\zeta^n\|$ we have $\|\zeta^n_{xx}\| \to 0$, as $N \to \infty$, $k \to 0$, $T_1 \leq nk \leq T_2$. Finally, by using the triangle inequality

$$\begin{aligned}\|S(nk)u_0 - S_N^k(n)u_0\|_2^2 = \|u(nk) - u_N^n\|_2^2 &\leq 2[\|u(nk) - F_N u(nk)\|_2^2 + \|F_N u(nk) - u_N^n\|_2^2] \\ &\leq C(N^{-2}\|u^n_{xxx}\|^2 + \|\zeta^n\|^2) \to 0\end{aligned}$$

uniformly on $[T_1, T_2]$ as $N \to \infty$, $k \to 0$ and $u_0 \in S_N$, $\|u_0\|_2 \leq R$, i.e.

$$\delta_N(I^*) = \sup_{u_0\in S_N, \|u_0\|_2\leq R}\ \sup_{nk\in[T_1,T_2]} d(S_N^k(n)u_0,\ S(nk)u_0) \to 0, \quad N \to \infty,\ k \to 0.$$

According to Theorem 3, we immediately obtain

Theorem 5. *Under the conditions of Lemma* 3*, suppose that* $f \in C^2$*,* $\phi \in C^3$*,* $g, h \in C^2$*,* $h \in C^0 \cap L^2(Q_\infty)$*,* $h_t \in L^2(Q_\infty)$*,* $u_0 \in H_p^2(\Omega)$ *then we have*

$$d(\mathcal{A}_N^k, \mathcal{A}) \to 0, \quad N \to \infty,\ k \to 0.$$

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