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A PARALLEL ITERATIVE DOMAIN DECOMPOSITION ALGORITHM FOR ELLIPTIC PROBLEMS

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Abstract

An iterative nonoverlapping domain decomposition procedure is proposed and analyzed for linear elliptic problems. At the interface of two subdomains, one subdomain problem requires that Dirichlet data be passed to it from the previous iteration level, while the other subdomain problem requires that Neumann data be passed to it. This procedure is suitable for parallel processing. A convergence analysis is established. Standard and mixed finite element methods are employed to give discrete versions of this domain decomposition algorithm. Numerical experiments are conducted to show the effectiveness of the method.

Key words: Domain decomposition methods, finite element methods, parallel computing

1. Introduction

Nonoverlapping domain decomposition methods have received a lot of attention during the past few years, since they have advantages of dealing with transmission problems and allow efficient parallelism. For a recent development of these methods, we refer to the papers by Funaro, Quarteroni and Zanolli[7], Marini and Quarteroni[11,12], Lions[10], Després[4], Douglas, Paes Leme, Roberts and Wang[5], and the author[13,14].

In this paper, we propose an iterative nonoverlapping domain decomposition procedure for second order partial differential equations. At the interface of two subdomains, one subdomain problem requires that Dirichlet data be passed to it from the previous iteration level, while the other subdomain problem requires that Neumann data be passed to it. Thus, this procedure can be efficiently implemented on computers with parallel architecture, as an improvement of the method in [7], [11], [12]. Both the method and the convergence analysis in this paper are closely related to and based on the techniques given in [7], [11], [12]. However, we will introduce a Galerkin approximation with Lagrange multipliers and a hybridized mixed finite element method, which were not dealt with in [7], [11], [12]. We will also prove that the error reduction factors per iteration in Galerkin approximations and hybridized mixed finite element approximations are independent of the grid size.

In §2, the domain decomposition method is described for general elliptic problems. In §3, a convergence analysis is carried out for general linear elliptic problems in multidimensions. In §4, a finite element approximation is employed. In §5, a finite element approximation with Lagrange multipliers is considered. Then, in §6, a hybridized mixed

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finite element method is applied. Finally in §7, numerical experiments are provided to check the correctness of the theory.

2. Domain Decomposition Method

Let Ω be a smooth bounded domain or a convex polygon in $\mathcal{R}^2$ with boundary $\partial\Omega$. Consider the following boundary value problem: find $u \in H^1(\Omega)$ such that

$$Lu = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega, \tag{1}$$

where $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$ are given, and the operator L is defined by

$$Lu = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + a_0(x)u. \tag{2}$$

The coefficients $\{a_{ij}\}$ are assumed to be symmetric, uniformly positive definite, bounded, and piecewise smooth in Ω, and $a_0 \geq 0$.

For simplicity, we partition the domain Ω into two nonoverlapping subdomains Ω_1 and Ω_2 such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, $\Omega_1 \cap \Omega_2 = 0$ We denote the interface by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. The following argument makes it possible to include more than two computational subdomains: If either Ω_1 or Ω_2 is not a connected set, then decompose them into connected components:

$$\Omega_1 = \bigcup_{j=1}^{N_1} \Omega_{1j}, \quad \Omega_2 = \bigcup_{j=1}^{N_2} \Omega_{2j}, \tag{3}$$

where N_1 and N_2 are some positive integers. However, we must confine ourselves to the case in which no interior vertices are allowed. That is, only strip-type domain decompositions are considered here. Domain decompositions with cross points will be treated later [6].

We now define the following domain decomposition method: Choose $u_k^0 \in H^1(\Omega_k)$ with $u_k^0|_{\partial\Omega_k \cap \partial\Omega} = g$, $k = 1, 2$. For $n = 0, 1, 2, \cdots$, we construct the sequence $u_k^{n+1} \in H^1(\Omega_k)$ with $u_k^{n+1}|_{\partial\Omega \cap \partial\Omega_k} = g$ satisfying

$$Lu_1^{n+1} = f \text{ in } \Omega_1, \qquad \frac{\partial u_1^{n+1}}{\partial \nu_A^1} = \theta\frac{\partial u_1^n}{\partial \nu_A^1} + (1-\theta)\frac{\partial u_2^n}{\partial \nu_A^1} \text{ on } \Gamma, \tag{4}$$

$$Lu_2^{n+1} = f \text{ in } \Omega_2, \qquad u_2^{n+1} = \delta u_1^n + (1-\delta)u_2^n \text{ on } \Gamma, \tag{5}$$

where $\frac{\partial u_k^n}{\partial \nu_A^k} = \sum_{i,j=1}^{2} a_{ij}\frac{\partial u_k^n}{\partial x_j}\nu_i^k$, $\nu^k = \{\nu_1^k, \nu_2^k\}$ is the outward unit normal vector to $\partial\Omega_k$, and $\theta, \delta \in (0, 1)$ are relaxation parameters that will be determined to ensure and to accelerate the convergence of the iterative procedure.

The differences between this method and the one in [7], [11], [12] lie in that the former method gives parallelizable subdomain problems at each iteration level, while the latter method leads to sequential subdomain problems, and that the former utilizes the Neumann boundary values on an interface from two neighboring subdomains, while

the latter just uses the Neumann boundary value from its neighboring subdomain. In other words, our method introduces an underrelaxation mechanism in the iterative process.

3. Convergence Analysis

In this section, we give a convergence analysis of the iterative procedure (4)–(5) for general self-adjoint linear elliptic problems. Keep in mind that we only treat strip-type domain decompositions in this work. We first introduce some notation. Define the Hilbert space by interpolation: $H_{00}^{\frac{1}{2}}(\Gamma) = [H_0^1(\Gamma), L^2(\Gamma)]_{1/2}$. The space $H_{00}^{\frac{1}{2}}(\Gamma)$ is strictly contained in $H_0^{\frac{1}{2}}(\Gamma)$; see Lions and Magenes [9, page 66], or Bjorstad and Widlund[2]. We introduce the Hilbert spaces: $\mathcal{V}_k = \{v \in H^1(\Omega_k)\colon\ v|_{\partial\Omega\cap\partial\Omega_k} = 0\}$, $k = 1, 2$. Let γ_0 denote the trace operator from $H^1(\Omega_k)$ onto $H^{\frac{1}{2}}(\Gamma)$. For any $\phi \in H_{00}^{\frac{1}{2}}(\Gamma)$, we denote by $R_1\phi$ and $R_2\phi$ the following extensions to Ω_1 and Ω_2, respectively, such that

$$R_k\phi \in \mathcal{V}_k: \quad L(R_k\phi) = 0 \text{ in } \Omega_k, \quad \gamma_0(R_k\phi) = \phi \text{ on } \Gamma, \quad k = 1, 2. \tag{6}$$

Introduce the bilinear form

$$a_k(u, w) = \sum_{i,j=1}^{2} \int_{\Omega_k} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial w}{\partial x_i} dx + \int_{\Omega_k} a_0 uw \, dx, \qquad k = 1, 2. \tag{7}$$

We define an operator $T_{21} \in \mathcal{L}(H_{00}^{\frac{1}{2}}(\Gamma),\ H_{00}^{\frac{1}{2}}(\Gamma))$ as follows: $T_{21} : \phi \to T_{21}\phi = \gamma_0 w$, where $w \in \mathcal{V}_1$ satisfies

$$Lw = 0 \text{ in } \Omega_1, \qquad \frac{\partial w}{\partial \nu_A^1} = -\frac{\partial (R_1\phi)}{\partial \nu_A^2} \text{ on } \Gamma. \tag{8}$$

The variational form of (8) can be written as:

$$a_1(R_1 T_{21}\phi, w) = -a_2(R_2\phi, R_2\gamma_0 w), \qquad \forall w \in \mathcal{V}_1. \tag{9}$$

It is readily seen that if $w \in H^1(\Omega_k)$ and $Lw = 0$ in Ω_k, then $\frac{\partial w}{\partial \nu_A^k} \in H^{-1/2}(\partial\Omega_k)$. This ensures the existence of the function $w \in H^1(\Omega_1)$ in the definition of the operator T_{21}. This actually requires a nontangential approach along Γ to $\partial\Omega$; i.e., no cusps are allowed where Γ intersects $\partial\Omega$. For convenience we will use the following norms in $\mathcal{V}_1$ and $\mathcal{V}_2$:

$$\|w\|_k^2 = a_k(w, w), \qquad \forall w \in \mathcal{V}_k, \quad k = 1, 2. \tag{10}$$

We now begin the convergence analysis for the scheme (4)–(5) by noting that the error $e_k^n \equiv u_k - u_k^n \in \mathcal{V}_k$ satisfies the following equations:

$$Le_1^{n+1} = 0 \text{ in } \Omega_1, \qquad \frac{\partial e_1^{n+1}}{\partial \nu_A^1} = \theta \frac{\partial e_1^n}{\partial \nu_A^1} + (1-\theta)\frac{\partial e_2^n}{\partial \nu_A^1} \text{ on } \Gamma, \tag{11}$$

$$Le_2^{n+1} = 0 \text{ in } \Omega_2, \qquad e_2^{n+1} = \delta e_1^n + (1-\delta) e_2^n \text{ on } \Gamma \tag{12}$$

Combining (8) and (11) we have

$$e_1^{n+1} = R_1\gamma_0 e_1^{n+1} = \theta e_1^n + (1-\theta)T_{21}\gamma_0 e_2^n. \tag{13}$$

Combining (6) and (12) we have

$$e_2^{n+1} = R_2\gamma_0 e_2^{n+1} = \delta R_2\gamma_0 e_1^n + (1-\delta)e_2^n. \tag{14}$$

Theorem 4.1. *Define the positive number* $\tau = \sup_{\phi\in H_{00}^{1/2}(\Gamma)}\{\|R_2\phi\|_2^2/\|R_1\phi\|_1^2\}$. *Then, the following error reduction estimate holds:*

$$\begin{aligned}\|e_1^{n+1}\|_1^2 + \|e_2^{n+1}\|_2^2 \le{}& [\theta^2 + \tau\delta^2 + \tau|\delta(1-\delta) - \theta(1-\theta)|]\|e_1^n\|_1^2 \\ &+ [\tau(1-\theta)^2 + (1-\delta)^2 + |\delta(1-\delta) - \theta(1-\theta)|]\|e_2^n\|_2^2.\end{aligned} \tag{15}$$

Proof. From (13) we see that

$$\begin{aligned}\|e_1^{n+1}\|_1^2 &= a_1(e_1^{n+1}, e_1^{n+1}) \\ &= \theta^2\|e_1^n\|_1^2 + 2\theta(1-\theta)a_1(e_1^n, R_1T_{21}\gamma_0 e_2^n) + (1-\theta)^2\|R_1T_{21}\gamma_0 e_2^n\|_1^2.\end{aligned} \tag{16}$$

We now estimate (16) term-by-term. Letting $w = R_1T_{21}\phi$ in (9) we have

$$a_1(R_1T_{21}\phi, R_1T_{21}\phi) = -a_2(R_2\phi, R_2T_{21}\phi) \le \|R_2\phi\|_2\|R_2T_{21}\phi\|_2,$$

which, together with the definition of τ, yields

$$\|R_1T_{21}\phi\|_1 \le \sqrt{\tau}\|R_2\phi\|_2, \qquad \forall\phi \in H_{00}^{\frac{1}{2}}(\Gamma). \tag{17}$$

From (9) we have

$$a_1(R_1\phi, R_1T_{21}\psi) = -a_2(R_2\phi, R_2\psi), \qquad \forall\phi, \quad \psi \in H_{00}^{\frac{1}{2}}(\Gamma). \tag{18}$$

Substituting (17) and (18) into (16) we have

$$\begin{aligned}\|e_1^{n+1}\|_1^2 &\le \theta^2\|e_1^n\|_1^2 + \tau(1-\theta)^2\|e_2^n\|_2^2 - 2\theta(1-\theta)a_2(R_2\gamma_0 e_1^n, e_2^n) \\ &= \theta^2\|e_1^n\|_1^2 + \tau(1-\theta)^2\|e_2^n\|_2^2 - 2\theta(1-\theta)a_2(R_2\gamma_0 e_1^n, e_2^n).\end{aligned} \tag{19}$$

Next, from (14) we have

$$\begin{aligned}\|e_2^{n+1}\|_2^2 = a_2(e_2^{n+1}, e_2^{n+1}) &= \delta^2\|R_2\gamma_0 e_1^n\|_2^2 + (1-\delta)^2\|e_2^n\|_2^2 + 2\delta(1-\delta)a_2(R_2\gamma_0 e_1^n, e_2^n) \\ &\le \tau\delta^2\|e_1^n\|_1^2 + (1-\delta)^2\|e_2^n\|_2^2 + 2\delta(1-\delta)a_2(R_2\gamma_0 e_1^n, e_2^n).\end{aligned} \tag{20}$$

Combining (19) and (20) and applying Schwarz inequality gives (15). This completes the proof.

Corollary 4.2. *The domain decomposition method* (4)–(5) *is convergent in the energy norm if* $\theta = 1-\delta$ *and* $\delta < \min\{1, \frac{2}{1+\tau}\}$. *The optimal parameters are* $\delta = \frac{1}{1+\tau}$ *and* $\theta = \frac{\tau}{1+\tau}$. *In particular, if* $\theta = \delta = \frac{1}{2}$, *when* $\tau = 1$, *which is true when* $\Omega_1 = \Omega_2$ *and the coefficients* a_{ij}, a_0 *are constants, then*

$$\|e_1^{n+1}\|_1^2 + \|e_2^{n+1}\|_2^2 \le \frac{1}{2}(\|e_1^n\|_1^2 + \|e_2^n\|_2^2). \tag{21}$$

4. Finite Element Approximation

Suppose that the domain Ω is decomposed into nonoverlapping subdomains Ω_1 and Ω_2 with interface Γ. For an easy presentation we assume that $g = 0$ in (1). Let $T_h = \{T\}$ be a regular triangulation of Ω with no elements crossing the interface Γ, where h is the grid size. Define a conforming Lagrange finite element space[12]

$$W^h = \{w \in C(\bar{\Omega}) : \quad w|_T \in P_r(T) \quad \forall T \in T_h, w|_{\partial\Omega} = 0\}, \tag{22}$$

where $P_r(T)$ denotes the space of polynomials of degree $\leq r$ on T, and define the finite element spaces over the subdomains:

$$W_k^h = \{w \in H^1(\Omega_k) : \quad w|_T \in P_r(T) \quad \forall T \in T_h,\ w|_{\partial\Omega\cap\partial\Omega_k} = 0\}, \tag{23}$$

$$M_k^h = \{w \in W_k^h : w|_{\partial\Omega_k} = 0\}. \tag{24}$$

Let Σ_h denote the subdivision of Γ induced by T_h. Set

$$\Phi^h = \{\phi \in C(\Gamma) :\ \phi|_I \in P_r(I) \text{ for } \forall I \in \Sigma_h,\ \phi|_{\partial\Gamma} = 0\}. \tag{25}$$

For $k = 1, 2$ we define the discrete extension operators $R_k^h : \Phi^h \to W_k^h$ by

$$R_k^h\phi \in W_k^h :\ a_k(R_k^h\phi, w) = 0, \quad \forall w \in M_k^h; \quad R_k^h\phi = \phi \ \text{ on } \ \Gamma. \tag{26}$$

Let U_k^n be the finite element approximation of u_k^n. We define the following discrete version of our domain decomposition method. Choose $U_k^0 \in W_k^h$, $k = 1, 2$. For $n = 0, 1, 2, \cdots$, find $U_k^{n+1} \in W_k^h$ such that

$$\begin{aligned} a_1(U_1^{n+1}, w) =& (1-\theta)(f, w)_{\Omega_1} + (1-\theta)(f, R_2^h\gamma_0 w)_{\Omega_2} \\ &+ \theta a_1(U_1^n, w) - (1-\theta)a_2(U_2^n, R_2^h\gamma_0 w), \quad \forall w \in W_1^h, \end{aligned} \tag{27}$$

$$a_2(U_2^{n+1}, w) = (f, w)_{\Omega_2}, \quad \forall w \in M_2^h; \quad U_2^{n+1} = \delta U_1^n + (1-\delta)U_2^n \text{ on } \Gamma, \tag{28}$$

where $(f, g)_{\Omega_k} = \displaystyle\int_{\Omega_k} fg\, dx$.

Note that, in obtaining (27), the conormal derivatives at the interface at the iteration level n were replaced by the bilinear forms defined over the subdomains. This replacement avoids requiring continuity of the conormal derivatives upon convergence. This is necessary in the finite element approximation, since when applying piecewise linear interpolation polynomials, for example, the continuity of conormal derivatives would result in a global linear polynomial in the elements adjacent to the interface.

Note that the approximation can also be applied to nonconforming finite elements. See [8] for details. The analysis follows readily from §3.

5. Finite Element Approximation with Lagrange Multipliers

Suppose that the domain Ω is decomposed into nonoverlapping subdomains Ω_1 and Ω_2 with interface Γ. Without loss of generality we assume that $g = 0$ in (1). Then, the variational formulation of the scheme (4)–(5) can be written as:

$$a_1(u_1^{n+1}, w) = \theta\left\langle \frac{\partial u_1^n}{\partial \nu_A^1}, w \right\rangle_\Gamma + (1-\theta)\left\langle \frac{\partial u_2^n}{\partial \nu_A^1}, w \right\rangle_\Gamma + (f, w)_{\Omega_1}, \quad \forall w \in \mathcal{V}_1, \tag{29}$$

$$a_2(u_2^{n+1},w)-\left\langle\frac{\partial u_2^{n+1}}{\partial\nu_A^2},w\right\rangle_\Gamma=(f,w)_{\Omega_2},\quad\forall w\in\mathcal{V}_2;\ u_2^{n+1}=\delta u_1^n+(1-\delta)u_2^n\text{ on }\Gamma, \tag{30}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product over the interface.

Replace the conormal derivatives $\dfrac{\partial u_k^n}{\partial\nu_A^k}$ by Lagrange multipliers λ_k^n; (29)–(30) now becomes:

$$a_1(u_1^{n+1},w)=\langle\theta\lambda_1^n-(1-\theta)\lambda_2^n,w\rangle_\Gamma+(f,w)_{\Omega_1},\quad\forall w\in\mathcal{V}_1, \tag{31}$$

$$a_2(u_2^{n+1},w)-\langle\lambda_2^{n+1},w\rangle_\Gamma=(f,w)_{\Omega_2},\quad\forall w\in\mathcal{V}_2;\quad u_2^{n+1}=\delta u_1^n+(1-\delta)u_2^n\text{ on }\Gamma, \tag{32}$$

$$\lambda_1^{n+1}=\theta\lambda_1^n-(1-\theta)\lambda_2^n. \tag{33}$$

The procedure (31)–(32) is the domain decomposition method with Lagrange multipliers at the continuous level, a variant of (4)–(5). The convergence of the scheme (31)–(32) can be easily established under the same hypothesis as assumed earlier and the proof is omitted here. We now formulate the finite element version of this procedure. Let $T_h=\{T\}$ be a regular triangulation of Ω with no elements crossing the interface Γ. Define the finite element spaces

$$W_k^h=\{w\in H^1(\Omega_k):\quad w|_T\in P_r(T)\quad\forall T\in T_h,\ w|_{\partial\Omega\cap\partial\Omega_k}=0\},\quad k=1,2, \tag{34}$$

where $P_r(T)$ denotes the space of polynomials of degree $\le r$ on T. Let Z^h be the space of the restrictions on the interface Γ of the functions in W_k^h. Note that there are two copies of such space assigned: one from Ω_1 and the other from Ω_2. We denote them by Z_1^h and Z_2^h, respectively. Let $\{U_k^n,\Lambda_k^n\}\in W_k^h\times Z_k^h$ denote the finite element approximation of $\{u_k^n,\lambda_k^n\}$. Then, the finite element domain decomposition method with Lagrange multipliers is constructed as follows:

$$a_1(U_1^{n+1},w)=\langle\theta\Lambda_1^n-(1-\theta)\Lambda_2^n,w\rangle_\Gamma+(f,w)_{\Omega_1},\quad\forall w\in W_1^h, \tag{35}$$

$$\begin{cases}a_2(U_2^{n+1},w)-\langle\Lambda_2^{n+1},w\rangle_\Gamma=(f,w)_{\Omega_2}, & \forall w\in W_2^h;\\ crU_2^{n+1}=\delta U_1^n+(1-\delta)U_2^n\text{ on }\Gamma,\end{cases} \tag{36}$$

$$\Lambda_1^{n+1}=\theta\Lambda_1^n-(1-\theta)\Lambda_2^n. \tag{37}$$

Note that the finite element approximation with Lagrange multipliers on the interface for problem (1)–(2) reads: find $\{u_k^h,\,\lambda_k^h\}\in W_k^h\times Z_k^h$ such that

$$a_k(u_k^h,w)-\langle\lambda_k^h,w\rangle_\Gamma=(f,w)_{\Omega_k}\qquad\forall w\in W_k^h,\quad k=1,2, \tag{38}$$

$$u_1^h=u_2^h\text{ on }\Gamma. \tag{39}$$

It follows easily that the function u^h, where $u^h|_{\Omega_k}=u_k^h$, solves the standard finite element problem without domain decomposition. Indeed, if u^h is the standard finite element solution, let $u_k^h=u^h|_{\Omega_k}$, $k=1,2$, and define $\lambda_k^h\in Z_k^h$ satisfying

$$\langle\lambda_1^h,w\rangle_\Gamma=(f,R_2^h\gamma_0w)_{\Omega_2}-a_2(u_2^h,R_2^h\gamma_0w),\quad\forall w\in W_1^h, \tag{40}$$

$$\langle\lambda_2^h,w\rangle_\Gamma=(f,R_1^h\gamma_0w)_{\Omega_1}-a_1(u_1^h,R_1^h\gamma_0w),\quad\forall w\in W_2^h, \tag{41}$$

where R_1^h and R_2^h are given by (26). Note that $u^h|_{\Omega_k}$ satisfies

$$a_1(u^h|_{\Omega_1}, w) + a_2(u^h|_{\Omega_2}, R_2^h\gamma_0 w) = (f, w)_{\Omega_1} + (f, R_2^h\gamma_0 w)_{\Omega_2}, \quad \forall w \in W_1^h, \tag{42}$$

$$a_1(u^h|_{\Omega_1}, R_1^h\gamma_0 w) + a_2(u^h|_{\Omega_2}, w) = (f, R_1^h\gamma_0 w)_{\Omega_1} + (f, w)_{\Omega_2}, \quad \forall w \in W_2^h. \tag{43}$$

Then, $\{u_k^h, \lambda_k^h\}$ is the solution of (38)–(39). Using this fact, we can easily make the convergence analysis for the scheme (35)–(37).

6. Mixed Finite Element Approximation

As seen in the last two sections, the continuity of the flux variable at the interface cannot be included explicitly in the finite element approximations. Similarly, the continuity of the scalar (pressure) variable cannot be imposed explicitly in a mixed finite element approximation. In this section, we will introduce Lagrange multipliers in place of the scalar variable at the interface.

Denote the matrix $(a_{ij})_{d\times d}$ by A and the flux $-A\nabla u$ by q. The Dirichlet problem (1) with homogeneous boundary condition has the weak form: seeking $\{q, u\} \in V \times W \equiv H(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

$$(A^{-1}q, v)_\Omega - (u, \mathrm{div}\ v)_\Omega = 0, \qquad \forall \quad v \in V, \tag{44}$$

$$(\mathrm{div}\ q, w)_\Omega + (a_0 u, w)_\Omega = (f, w)_\Omega, \quad \forall \quad w \in W. \tag{45}$$

Let us decompose the domain Ω into nonoverlapping subdomains Ω_1 and Ω_2 with interface Γ, and let $V_k = H(\mathrm{div}, \Omega_k)$, $W_k = L^2(\Omega_k)$. Assume that T_h is a subdivision of the domain Ω with no elements crossing the interface Γ, and $V^h \times W^h \subset V \times W$ be a mixed finite element space[3], which is defined through local spaces $V_k^h \times W_k^h$. Since functions in W^h are allowed to be discontinuous across the interface Γ, the pressure continuity condition cannot be imposed directly. Instead, we introduce Lagrange multipliers[1] at the interface Γ. Assume that, when $q_k = q^h|_{\Omega_k}$, $q^h \in V^h$, its normal component $q_k \cdot \nu_k$ on the interface Γ is a polynomial of some fixed degree τ, where for simplicity we assume τ independent of the edges of the elements. Let Λ^h be the space of all such polynomials. Note that again there are copies of such space assigned from Ω_1 and from Ω_2. We denote them by Λ_1^h and Λ_2^h, respectively. For notational convenience, we drop the superscript h on all of the approximate quantities. Then, the hybridized mixed finite element method is given by seeking $\{q_k, u_k, \lambda_k\} \in V_k \times W_k \times \Lambda_k$ such that

$$(A^{-1}q_k, v)_{\Omega_k} - (u_k, \mathrm{div}\ v)_{\Omega_k} + \langle \lambda_k, v \cdot \nu_k \rangle_\Gamma = 0, \quad \forall v \in V_k, \tag{46}$$

$$(\mathrm{div}\ q_k, w)_{\Omega_k} + (a_0 u_k, w)_{\Omega_k} = (f, w)_{\Omega_k}, \quad \forall w \in W_k, \tag{47}$$

$$q_1 \cdot \nu_1 + q_2 \cdot \nu_2 = 0 \quad \text{on} \quad \Gamma. \tag{48}$$

It follows easily that the pair $\{q, u\}$, where $q|_{\Omega_k} = q_k$ and $u|_{\Omega_k} = u_k$, solves the original mixed finite element problem.

We are now in a position to formulate the hybrid mixed finite element version of our iterative procedure (4)–(5). Choose an initial guess $\{q_k^0, u_k^0, \lambda_k^0\} \in V_k \times W_k \times \Lambda_k$ arbitrarily and compute $\{q_k^{n+1}, u_k^{n+1}, \lambda_k^{n+1}\} \in V_k \times W_k \times \Lambda_k$ such that

$$(A^{-1}q_1^{n+1}, v)_{\Omega_1} - (u_1^{n+1}, \text{div } v)_{\Omega_1} + \langle \lambda_1^{n+1}, v \cdot \nu_1 \rangle_\Gamma = 0, \quad \forall\, v \in V_1, \tag{49}$$

$$(\text{div } q_1^{n+1}, w)_{\Omega_1} + (a_0 u_1^{n+1}, w)_{\Omega_1} = (f, w)_{\Omega_1}, \quad \forall\, w \in W_1, \tag{50}$$

$$q_1^{n+1} \cdot \nu_1 = \theta q_1^n \cdot \nu_1 + (1-\theta) q_2^n \cdot \nu_1 \quad \text{on} \quad \Gamma, \tag{51}$$

and

$$(A^{-1}q_2^{n+1}, v)_{\Omega_2} - (u_2^{n+1}, \text{div } v)_{\Omega_2} = -\langle \delta\lambda_1^n + (1-\delta)\lambda_2^n, v \cdot \nu_2 \rangle_\Gamma, \quad \forall\, v \in V_2, \tag{52}$$

$$(\text{div } q_2^{n+1}, w)_{\Omega_2} + (a_0 u_2^{n+1}, w)_{\Omega_2} = (f, w)_{\Omega_2}, \quad \forall w \in W_2, \tag{53}$$

$$\lambda_2^{n+1} = \delta\lambda_1^n + (1-\delta)\lambda_2^n. \tag{54}$$

We now briefly discuss the error analysis. Let $r_k^n = q_k - q_k^n$, $e_k^n = u_k - u_k^n$, $\mu_k^n = \lambda|_k - \lambda_k^n$, $k = 1, 2$. Then, by combining (46)–(48) and (49)–(54), we have the following error equations:

$$(A^{-1}r_1^{n+1}, v)_{\Omega_1} - (e_1^{n+1}, \text{div } v)_{\Omega_1} + \langle \mu_1^{n+1}, v \cdot \nu_1 \rangle_\Gamma = 0, \quad \forall v \in V_1, \tag{55}$$

$$(\text{div } r_1^{n+1}, w)_{\Omega_1} + (a_0 e_1^{n+1}, w)_{\Omega_1} = 0, \qquad \forall w \in W_1, \tag{56}$$

$$r_1^{n+1} \cdot \nu_1 = \theta r_1^n \cdot \nu_1 + (1-\theta) r_2^n \cdot \nu_1 \quad \text{on} \quad \Gamma, \tag{57}$$

and

$$(A^{-1}r_2^{n+1}, v)_{\Omega_2} - (e_2^{n+1}, \text{div} v)_{\Omega_2} = -\langle \delta\mu_1^n + (1-\delta)\mu_2^n, v \cdot \nu_2 \rangle_\Gamma, \quad \forall v \in V_2, \tag{58}$$

$$(\text{div } r_2^{n+1}, w)_{\Omega_2} + (a_0 e_2^{n+1}, w)_{\Omega_2} = 0, \qquad \forall w \in W_2, \tag{59}$$

$$\nu_2^{n+1} = \delta\nu_1^n + (1-\delta)\nu_2^n. \tag{60}$$

In order to make error analysis, we now introduce some notation. Let $\Phi = \{\psi \cdot \nu_k : \psi \in V_k|_\Gamma,\ k = 1, 2\}$. Define the extension operators: $R_k : \phi \in \Phi \to \{R_k^1\phi, R_k^2\phi, R_k^3\phi\} \in V_k \times W_k \times \Lambda_k$ by

$$(A^{-1}R_k^1\phi, v)_{\Omega_k} - (R_k^2\phi, \text{div } v)_{\Omega_k} + \langle R_k^3\phi, v \cdot \nu_k \rangle_\Gamma = 0, \quad \forall v \in V_k, \tag{61}$$

$$(\text{div } R_k^1\phi, w)_{\Omega_k} + (a_0 R_k^2\phi, w)_{\Omega_k} = 0, \quad \forall w \in W_k, \tag{62}$$

$$R_k^1\phi \cdot \nu_k = \phi \quad \text{on} \quad \Gamma. \tag{63}$$

We introduce the following operator from Φ into itself: T_{12}: $\phi \in \Phi \to T_{12}\phi = \psi_1 \cdot \nu_2 \in \Phi$, where $\{\psi_1, \psi_2\} \in V_2 \times W_2$ satisfies:

$$(A^{-1}\psi_1, v)_{\Omega_2} - (\psi_2, \text{div } v)_{\Omega_2} = -\langle R_1^3\phi, v \cdot \nu_2 \rangle_\Gamma, \quad \forall v \in V_2, \tag{64}$$

$$(\text{div } \psi_1, w)_{\Omega_2} + (a_0\psi_2, w)_{\Omega_2} = 0, \quad \forall w \in W_2. \tag{65}$$

Again, let γ_0 denote the restriction operator on Γ. From (55)–(57) we have

$$\gamma_0 r_1^{n+1} \cdot \nu_1 = \theta\gamma_0 r_1^n \cdot \nu_1 - (1-\theta)\gamma_0 r_2^n \cdot \nu_2. \tag{66}$$

By the superposition principle and from (58), we have

$$\gamma_0 r_2^{n+1} \cdot \nu_2 = \delta T_{12}(\gamma_0 r_1^n \cdot \nu_1) + (1-\delta)\gamma_0 r_2^n \cdot \nu_2. \tag{67}$$

For convenience we define the following norm on the interface Γ :

$$|\phi|_k^2 = (A^{-1}R_k^1\phi, R_k^1\phi)_{\Omega_k} + (a_0 R_k^2\phi, R_k^2\phi)_{\Omega_k}, \quad \text{for} \quad \phi \in \Phi, \quad k = 1, 2, \tag{68}$$

and introduce the least finite and positive quantity σ that does not depend on the grid size, such that: $\sup_{\phi\in\Phi}\{|\phi|_1^2/|\phi|_2^2\} \le \sigma$. We can easily obtain the following error estimate.

Theorem 2. *For the hybrid mixed domain decomposition method* (49)–(54),

$$\begin{aligned}(A^{-1}r_1^{n+1},r_1^{n+1})_{\Omega_1} + (a_0 e_1^{n+1}, e_1^{n+1})_{\Omega_1} + (A^{-1}e_2^{n+1}, e_2^{n+1})_{\Omega_2} + (a_0 e_2^{n+1}, e_2^{n+1})_{\Omega_2} \\ \le [\theta^2 + \delta^2\sigma + |\delta(1-\delta) - \theta(1-\theta)|][(A^{-1}r_1^n, r_1^n)_{\Omega_1} + (a_0 e_1^n, e_1^n)_{\Omega_1}] \\ + [(1-\theta)^2\sigma + (1-\delta)^2 + \sigma|\delta(1-\delta) - \theta(1-\theta)|] \\ \cdot [(A^{-1}e_2^n, e_2^n)_{\Omega_2} + (a_0 e_2^n, e_2^n)_{\Omega_2}].\end{aligned} \tag{69}$$

7. Numerical Experiments

In this section, we conduct some numerical experiments for our domain decomposition method (4)–(5) for the example

$$\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) + \frac{\partial^2 u}{\partial y^2} = f, \qquad (x,y) \in \Omega, \tag{70}$$

$$u = g, \qquad (x,y) \in \partial\Omega, \tag{71}$$

where $\Omega = (0,1)\times(0,1)$. The functions f and g are chosen such that the exact solution is: $u(x,y) = y(1-y)\sin\left(\frac{\pi}{2}x\right)$.

First, we take $a(x) = 1$ in (70) and check the correctness of the estimate (21). We apply the finite element scheme without Lagrange multipliers in each subdomain. The interpolation polynomials consist of piecewise linears in x tensored with piecewise linears in y. The initial guesses are always taken to be zero. Let the domain Ω be divided into two equal subdomains $\Omega_1 = (0, 0.5)\times(0,1)$, and $\Omega_2 = (0.5,1)\times(0,1)$. The interface is $\Gamma = \{0.5\}\times(0,1)$. The optimal relaxation parameters in this case are $\theta = \delta = \frac{1}{2}$. We choose the grid sizes to be $\frac{1}{20}\times\frac{1}{20}$, $\frac{1}{50}\times\frac{1}{50}$, and $\frac{1}{80}\times\frac{1}{80}$, respectively. The errors between the exact solution of the differential problem and the domain decomposition solution are evaluated in the L^∞-norm over the two subdomains at each iteration and are shown in Table 1. Now we decompose the domain into four subdomains: $\Omega_1 = (0,1)\times(0,0.25)$, $\Omega_2 = (0,1)\times(0.25,0.5)$, $\Omega_3 = (0,1)\times(0.5,0.75)$, and $\Omega_4 = (0,1)\times(0.75,1)$. By coloring Ω_1 and Ω_3 red, and Ω_2 and Ω_4 black, we still can view this as a two-subdomain case, which is convergent by our theory. The results are given in Table 2.

Table 1 Numerical results with interface at $x = 0.5$, and grid size $\frac{1}{20} \times \frac{1}{20}$, $\frac{1}{50} \times \frac{1}{50}$, $\frac{1}{80} \times \frac{1}{80}$, respectively. The errors are shown in the L^∞-norm.

Iterations	Grid size $= \frac{1}{20} \times \frac{1}{20}$	Grid size $= \frac{1}{50} \times \frac{1}{50}$	Grid size $= \frac{1}{80} \times \frac{1}{80}$
1	1.2196E-1	1.2967E-1	1.2969E-1
2	5.2741E-2	5.3061E-2	5.3087E-2
5	7.7585E-3	7.6459E-3	7.6219E-3
8	8.1109E-4	1.0476E-3	9.4343E-4
11	3.4268E-4	1.0113E-4	9.1895E-5

Table 2 Numerical results with four subdomains. The interfaces are at $y = 0.25$, $y = 0.5$, $y = 0.75$. The grid size is chosen to be $\frac{1}{20} \times \frac{1}{20}$, $\frac{1}{50} \times \frac{1}{50}$, $\frac{1}{80} \times \frac{1}{80}$, respectively. The errors are shown in the L^∞-norm.

Iterations	Grid size $= \frac{1}{20} \times \frac{1}{20}$	Grid size $= \frac{1}{50} \times \frac{1}{50}$	Grid size $= \frac{1}{80} \times \frac{1}{80}$
1	1.7287E-1	1.7670E-1	1.7674E-1
2	6.5033E-2	6.5118E-2	9.5140E-2
5	9.7234E-3	8.7285E-3	8.7448E-3
8	1.2004E-3	1.1434E-3	1.1245E-3
9	7.6107E-4	7.1918E-4	7.8171E-4
13	1.1371E-4	9.3102E-5	5.3943E-5

Table 3 Numerical results with interface at $x = 0.3, 0.5, 0.7$ and grid size $\frac{1}{20} \times \frac{1}{20}$, $\frac{1}{70} \times \frac{1}{70}$, respectively. The errors are shown in the L^∞-norm.

	Grid size $\frac{1}{20} \times \frac{1}{20}$			Grid size $\frac{1}{70} \times \frac{1}{70}$		
Iteration	Γ at 0.3	Γ at 0.5	Γ at 0.7	Γ at 0.3	Γ at 0.5	Γ at 0.7
1	1.08E-1	1.47E-1	1.56E-1	1.08E-1	1.48E-1	1.56E-1
2	6.06E-2	6.12E-2	1.03E-1	6.10E-2	6.15E-2	1.03E-1
3	2.29E-2	4.81E-2	7.71E-2	2.32E-2	4.80E-2	7.72E-2
5	9.40E-3	9.50E-3	1.12E-2	9.14E-3	9.23E-3	1.15E-2
6	3.97E-3	4.26E-3	9.56E-3	3.67E-3	3.94E-3	9.86E-3
9	5.17E-4	2.12E-4	1.44E-3	4.31E-4	5.53E-4	1.25E-3
10	2.84E-4	4.19E-4	5.69E-4	2.71E-4	2.30E-4	3.55E-4
13	3.38E-4	3.86E-4	1.74E-4	1.97E-5	5.41E-5	4.73E-5

From Table 1 and Table 2 we see that the error reduction factor is approximately equal to $\frac{1}{2}$, and is independent of the grid size. Thus the estimates (21) and their discrete counterparts are correct.

Next, we take $a(x) = 2 + \sin(\frac{\pi}{2}x)$. Let $\Gamma = \{0.3\} \times (0,1)$, $\{0.5\} \times (0,1)$, and $\{0.7\} \times (0,1)$, respectively. This time the second-order finite difference method is implemented in each subdomain. The relaxation parameters are still chosen to be $\theta = \delta = \frac{1}{2}$. The grid sizes are $\frac{1}{20} \times \frac{1}{20}$ and $\frac{1}{70} \times \frac{1}{70}$, respectively. The errors are evaluated in the L^∞-norm over the two subdomains at each iteration and are shown in Table 3.

From Tables 1, 2, and 3 we see that the convergence of our method is pretty fast and independent of the grid size. Table 3 also tells us that different domain decompositions yield different rates of convergence and different accuracies. This may suggest that different domain decompositions at different times be necessary to deal with time-dependent problems with moving local phenomena; an optimal domain decomposition should be made at each time level according to the location of local phenomena. In [13], both theoretical analysis and numerical experiments showed that it is the case.

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