Journal of Computational Mathematics, Vol.16, No.2, 1998, 129–140.

A FAMILY OF DIFFERENCE SCHEMES WITH FOUR NEAR-CONSERVED QUANTITIES FOR THE KdV EQUATION*1)

Zhen Han
(*Department of Computer Science and Technology, Northern Jiaotong University, Beijing 100044, China*)

Long-jun Shen
(*Institute of Applied Physics and Computational Mathematics, Beijing 100088, China*)

Abstract

We construct and analyze a family of semi-discretized difference schemes with two parameters for the Korteweg-de Vries (KdV) equation. The scheme possesses the first four near-conserved quantities for periodic boundary conditions. The existence and the convergence of its global solution in Sobolev space $\mathbf{L}_\infty(0,T;\mathbf{H}^3)$ are proved and the scheme is also stable about initial values. Furthermore, the scheme conserves exactly the first two conserved quantities in the special case.

Key words: Convergence, difference scheme, KdV equation, conserved quantity

1. Introduction

In this paper, we are concerned with the semi-discretized difference methods which are capable of approximating to the KdV equation to a considerable extent. Consider the periodic initial-boundary problem:

$$u_t + uu_x + u_{xxx} = 0, \qquad -\infty < x < +\infty, \quad t > 0 \tag{1.1}$$

$$u(x+1,t) = u(x,t), \qquad -\infty < x < +\infty, \quad t > 0 \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad -\infty < x < +\infty \tag{1.3}$$

where $u_0(x)$ is a given 1-periodic function and belongs to $\mathbf{H}^3$. Let J be a positive integer, put the spatial mesh length $h = 1/J$. Discrete periodic function $V_h = \{V_j | j = 0, \pm 1, \pm 2, \cdots\}$ takes the values on the net points $x_j = jh$. Denote Δ_o, Δ_+ and Δ_-, respectively, of the centered, the forward and the backward difference quotient operators, i.e.,

$$\Delta_o V_j = \frac{V_{j+1} - V_{j-1}}{2h}, \quad \Delta_+ V_j = \frac{V_{j+1} - V_j}{h}, \quad \Delta_- V_j = \frac{V_j - V_{j-1}}{h} \tag{1.4}$$

and E is a mean operator as follows

$$EV_j = \frac{1}{2}(V_{j+1} + V_{j-1}). \tag{1.5}$$

* Received May 17, 1994.
1) The research was supported by the National Science Fundation of China.

As similar as [13], for real $1 \le p \le \infty$, denote by $\mathbf{W}_p = \mathbf{W}_p(0,1)$ the usual discretized real Sobolev spaces on (0, 1) and by $\|\cdot\|_p$ the associated norm:

$$\|V_h\|_p = \Big(\sum_{j=1}^{J} |V_j|^p h\Big)^{1/p}, \quad 1 \le p \le \infty. \tag{1.6}$$

For integer $s \ge 0$, let $\mathbf{H}^s = \mathbf{W}_2^s$ and the inner product on $\mathbf{W}_2(0,1)$ is denoted by $(\cdot,\cdot)$.

The KdV equation (1.1) can be shown to have an infinite hierarchy of conservation quantities and the first four of them can be written as follows[7]:

$$F_0(u) = \int_0^1 3u\mathrm{d}x \tag{1.7a}$$

$$F_1(u) = \int_0^1 \frac{1}{2}u^2\mathrm{d}x \tag{1.7b}$$

$$F_2(u) = \int_0^1 \Big(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\Big)\mathrm{d}x \tag{1.7c}$$

$$F_3(u) = \int_0^1 \Big(\frac{5}{70}u^4 - \frac{5}{6}uu_x^2 + \frac{1}{2}u_{xx}^2\Big)\mathrm{d}x. \tag{1.7d}$$

Unfortunately, it is difficult for discretizations of (1.1) to preserve more than two exact conserved quantities. Although the numerical studies of the KdV equation have been largely developed since Zabusky and Kruskal used the second order accuracy Leap-Frog scheme to solve this evolution equation, there were seldom works discussing about multiple conservation laws of difference approximation of (1.1) or estimates of difference solutions under norm $\|\cdot\|_\infty$ and their higher order difference quotients. Recently, the various computational instabilities occurring in difference approximating to the KdV equation were observed by several scholars[1,10,11]. One was conscious that high order discrete conserved quantities are very significant for restraining numerical instabilities.

In authors' previous papers [3], [4] and [5], several semi-discrete difference schemes were studied. They have three or four near-conserved quantities. Recently, we presented a method in [9] to construct schemes with multiple near-conserved quantities. The method draws construction of infinite conservation laws in continuous situation, which can be refered in Lax [7], and utilizes discretizations of the gradients of the invariant functionals. By the way, we obtained a family of schemes with a real parameter β:

$$\begin{aligned} V_{jt} &+ \frac{1}{2}\Delta_o V_j^2 + \Delta_o\Delta_+\Delta_- V_j + \frac{1-\beta}{12}h^2\Delta_o V_j\Delta_+\Delta_- V_j \\ &+ \frac{\beta}{6}h^2 V_j\Delta_o\Delta_+\Delta_- V_j + \frac{\beta-2}{36}h^4\Delta_+\Delta_- V_j\Delta_o\Delta_+\Delta_- V_j = 0 \end{aligned} \tag{1.8}$$

for J odd, the condition required by the inverse operation of difference operators. The schemes presented in [3] and [5] are the special cases of (1.8) for $\beta = 1$ and $\beta = 0$ respectively.

In this paper, we prove that the scheme (1.8) possesses the first four near-conserved quantities for J any positive integer. We gain the estimates of the difference solution and its difference quotients up to order 3 using the theory of discrete functional analysis due to Zhou[13] and the technique of coupled priori estimating[2]. Thanks to these estimates, the convergence and the stability of the scheme (1.8) are proved.

In addition, we present a new scheme with two parameters based on (1.8)

$$\begin{aligned} &V_{jt}+\frac{1}{2}\Delta_o V_j^2+\Delta_o\Delta_+\Delta_- V_j+\frac{1-\beta}{12}h^2\Delta_o V_j\Delta_+\Delta_- V_j \\ &+\frac{\beta}{6}h^2 V_j\Delta_o\Delta_+\Delta_- V_j-\frac{\alpha}{6}h^4\Delta_o(|\Delta_+\Delta_- V_j|^2)=0. \end{aligned} \tag{1.9}$$

Because that the term $h^4\Delta_o(|\Delta_+\Delta_- V_j|^2)$ can be dominated by the forth conserved quantity, (1.9) also has the first four near-conserved quantities. In particular, (1.9) keeps the first two exact conserved quantities for $\alpha=\beta=\frac{3}{5}$. A numerical example is given, which shows that the conservation properties of the approximation solution agrees with our analysis.

2. Main Results

We set

$$Q_j=\frac{1-\beta}{12}h^2\Delta_o V_j\Delta_+\Delta_- V_j+\frac{\beta}{6}h^2 V_j\Delta_o\Delta_+\Delta_- V_j+\frac{\beta-2}{36}h^4\Delta_+\Delta_- V_j\Delta_o\Delta_+\Delta_- V_j \tag{2.1}$$

then the scheme (1.8) can be rewritten as a form of

$$V_{jt}+\frac{1}{2}\Delta_o V_j^2+\Delta_o\Delta_+\Delta_- V_j+Q_j=0. \tag{2.2}$$

We introduce the discrete periodic boundary condition

$$V_{j+J}(t)=V_j(t),\quad \forall j,\quad t>0 \tag{2.3}$$

and the initial value

$$V_j(0)=u_0(x_j),\quad j=0,\pm1,\cdots. \tag{2.4}$$

(2.2) is also a five-point scheme and has an equivalent form:

$$\begin{aligned} &V_{jt}+\Delta_o\Delta_+\Delta_- V_j+\Big[\frac{25-5\beta}{36}(V_{j+1}+V_{j-1})-\frac{7+\beta}{18}V_j\Big]\Delta_o V_j \\ &+\Big[\frac{\beta-2}{36}(V_{j+1}+V_{j-1})+\frac{1+\beta}{9}V_j\Big]\Delta_o(V_{j+1}+V_{j-1})=0. \end{aligned} \tag{2.5}$$

The scheme (2.2) approximates to the KdV equation (1.1) to a considerable extent. We obtain the following results about the priori estimates, the near-conservation, the convergence and the stability.

Theorem 1. *Suppose $u_0(x)\in\mathbf{H}^2$. For any given $T>0$, if h is small enough, the scheme (2.2) has the first four near-conserved quantities*

$$F_0^h(V_h(t))=\sum_{j=1}^{J}3V_j(t)h \tag{2.6a}$$

$$F_1^h(V_h(t)) = \frac{1}{2}(V_h(t) \quad V_h(t)) \tag{2.6b}$$

$$F_2^h(V_h(t)) = \frac{1}{6}(V_h^2,\ V_h) - \frac{1}{2}(\Delta_+ V_h,\ \Delta_+ V_h) \tag{2.6c}$$

$$F_3^h(V_h(t)) = \frac{5}{72}(V_h^2, \quad V_h^2) - \frac{5}{12}(V_h,\ (\Delta_+ V_h)^2 + (\Delta_- V_h)^2) + \frac{1}{2}(\Delta_+\Delta_- V_h,\ \Delta_+\Delta_- V_h) \tag{2.6d}$$

which satisfy the restrictions for any $t \in [0, T]$*:*

$$\frac{d}{dt}F_0^h(V_h(t)) = 0 \quad \textit{and} \quad \left|\frac{d}{dt}F_i^h(V_h(t))\right| \le C_i h^2, \quad i = 1, 2, 3 \tag{2.7}$$

where and below C_i $(i = 1, 2, \cdots)$ *are constants independent of* h *and* T*.*

Theorem 2. *Suppose* $u_0(x) \in \mathbf{H}^3$*. For any given* $T > 0$*, if* h *is small enough, the solution* $V_h(t)$ *of the difference method* (2.2)–(2.4) *satisfies the priori estimates:*

$$\max_{0\le t\le T} \|V_h(t)\|_{H^2} \le C_5 \tag{2.8}$$

$$\max_{0\le t\le T} \|V_h(t)\|_{H^3} \le \tilde{C}_1 \tag{2.9}$$

$$\max_{0\le t\le T} \|V_{ht}(t)\|_2 \le \tilde{C}_2 \tag{2.10}$$

where and below $\tilde{C}_i$ $(i = 1, 2, \cdots)$ *are constants independent of* h*.*

The proof of Theorem 1 and Theorem 2 will be given in the section 4. Having the priori estimations in Theorem 2, basing on the framework of Zhou in [13], we know that the global solution of the difference scheme (2.2)–(2.4) exists in Sobolev space $\mathbf{L}_\infty(0, T; \mathbf{H}^3)$ and get following convergence theorem:

Theorem 3. *Suppose* $u_0(x) \in \mathbf{H}^3$*. For any* $T > 0$*, the difference solution* $V_h(t)$ *of scheme* (2.2)–(2.4) *converges to the differential solution* $u(x, t)$ *of* (1.1)–(1.3) *in* $\mathbf{L}_\infty(0, T; \mathbf{H}^3)$ *as* $h \to 0$*.*

Having the bounded estimations in Theorem 2, we also get the stability of the scheme (2.2) and state it in theorem 4. Its proof is similar to that of Theorem 3 in Ref. [3].

Theorem 4. *Under the conditions of Theorem* 2*, the scheme* (2.2) *is stable about initial values in the sense of*

$$\|V_h(t) - \tilde{V}_h(t)\|_{H^2} \le \tilde{C}_3 \exp(\tilde{C}_4 t)\|u_0 - \tilde{u}_0\|_{H^2}, \quad \forall t \in [0, T] \tag{2.11}$$

where $\tilde{V}_h$ *is the solution of scheme* (2.2)–(2.4) *with another initial* $\tilde{u}_0(x) \in \mathbf{H}^3$*.*

3. Two-parameter Schemes

In the scheme (1.8), the last term $h^4\Delta_+\Delta_- V_j \Delta_o \Delta_+\Delta_- V_j$ is not by no means controled. We improve the scheme (1.8) through introducing another parameter α and write it as

$$V_{jt} + \frac{1}{2}\Delta_o V_j^2 + \Delta_o\Delta_+\Delta_- V_j + \frac{1-\beta}{12}h^2\Delta_o V_j \Delta_+\Delta_- V_j$$

$$+\frac{\beta}{6}h^2V_j\Delta_o\Delta_+\Delta_-V_j-\frac{\alpha}{3}h^4\Delta_+\Delta_-V_j\Delta_o\Delta_+\Delta_-V_j=0. \tag{1.8'}$$

It is easy to see that

$$\begin{aligned}h^4|(\Delta_+\Delta_-V_h\Delta_o\Delta_+\Delta_-V_h,\Delta_+\Delta_-\Delta_+\Delta_-V_h)|=&\frac{h^4}{2}|(1,(\Delta_+\Delta_+\Delta_-V_h)^3)|\\ \le 4h\|\Delta_+\Delta_-V_h\|_3^3\le 4h^{\frac{1}{2}}\|\Delta_+\Delta_-V_h\|_2^3.\end{aligned} \tag{3.1}$$

Thanks to (3.1), although it is quite rough, the proving process in section 4 is still tenable for the scheme (1.8′) and the corresponding results to Theorem 1-4 can be derived for smaller h, but

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\,t}F_3^h(V_h(t))\Big|\le C_3'h^{\frac{1}{2}},t\in[0,T] \tag{3.2}$$

where $V_h(t)$ is a solution of (1.8′) and C_3' is a constant independent of h and T.

Now, we replace $\Delta_+\Delta_-V_j\Delta_o\Delta_+\Delta_-V_j$ by $\frac{1}{2}\Delta_o(\Delta_+\Delta_-V_j)^2$ in the scheme (1.8′) and obtain a new scheme (1.9). Because that

$$h^4|(\Delta_o(\Delta_+\Delta_-V_h)^2,\Delta_+\Delta_-\Delta_+\Delta_-V_h)|\le 8h^{\frac{1}{2}}\|\Delta_+\Delta_-V_h\|_2^3, \tag{3.3}$$

the scheme (1.9) has the properties as same as (1.8′).

It is clear that there holds $\frac{\mathrm{d}}{\mathrm{d}t}F_0^h(V_h(t))=0$ for any solution $V_h(t)$ of equation (1.9).

Multiplying (1.9) by V_j and summing them up for j from 1 to J and considering the periodic boundary condition (2.3), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_1^h(V_h(t))=&-\frac{1}{2}(\Delta_oV_h^2,V_h)-\frac{1-\beta}{12}h^2(V_h,\Delta_oV_h\Delta_+\Delta_-V_h)\\ &-\frac{\beta}{6}h^2(V_h^2,\Delta_o\Delta_+\Delta_-V_h+\frac{\alpha}{6}h^4(\Delta_o(\Delta_+\Delta_-V_h)^2,V_h)\\ =&\frac{5\beta-3}{12}h^2(V_h,\Delta_oV_h\Delta_+\Delta_-V_h)+\frac{\beta-\alpha}{6}h^4(\Delta_oV_h,(\Delta_+\Delta_-V_h)^2).\end{aligned}$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}t}F_1^h(V_h(t))=0$, for $\beta=\alpha=\frac{3}{5}$. In this case, (1.9) can be rewritten as follows

$$V_{jt}+V_j\Delta_oV_j+\Big[1+\frac{h^2}{6}(V_{j+2}-2V_{j+1}+V_j-2V_{j-1}+V_{j-2})\Big]\Delta_o\Delta_+\Delta_-V_j=0. \tag{3.4}$$

The scheme (3.4) possesses the first two exact conserved quantities and the succeeding two near-conserved ones. A numerical example using scheme (3.4) to compute an initial monochromatic wave is given in section 5. Its momentum and energy (the first two invariants) are preserved indeed for ever and a better recurrence of the initial state is obtained.

4. The Proof of Theorem 1 and Theorem 2

To prove the theorems in section 2, several lemmas used in paper [3] are required again. For convenience, we list them here.

Lemma 1[14]**.** *Let V_h is a discrete function. For any constants p, q, r and integers k, n which satisfy $1 \le q,\ r \le \infty;\ 0 \le k < n,\ -\Big(n-k-\dfrac{1}{r}\Big) \le \dfrac{1}{p} \le 1$, there exists a constant K such that the following interpolation formula holds*

$$\|\Delta_+^k V_h\|_p \le K(\|V_h\|_q^{1-\alpha}\|\Delta_+^n V_h\|_r^{\alpha} + \|V_h\|_q) \tag{4.1}$$

where the constant α is fixed by

$$\frac{1}{p} - k = \frac{1-\alpha}{q} + \alpha\Big(\frac{1}{r} - n\Big).$$

Lemma 2[8]**.** *For any discrete periodic function V_h, there holds*

$$\|V_h\|_2^2 \le \frac{1}{4}\|\Delta_+ V_h\|_2^2 + \Big(\sum_{j=1}^{J} V_j h\Big)^2, \quad Jh = 1. \tag{4.2}$$

Lemma 3[5]**.** *Suppose $z(t)$ is a non-negative function on $[0,T]$ and satisfies the inequality:*

$$z(t) \le D_0 + D_1 \int_0^t |z(s)|^{8/3} ds, \quad \forall t \in [0,T] \tag{4.3}$$

where D_0, $D_1 > 0$. Then, if D_1 is small enough that

$$\frac{5}{3} D_0^{5/3} D_1 T \le \frac{1}{4}, \tag{4.4}$$

(4.3) implies the estimate

$$z(t) \le 2D_0, \quad \forall t \in [0,T]. \tag{4.5}$$

Lemma 4[5]**.** *Set A_h and B_h are any discrete functions.There are relationships:*

$$\begin{aligned}\Delta_+\Delta_- A_j^2 =& \Delta_+(A_j\Delta_- A_j) + \Delta_-(A_j\Delta_+ A_j)\\ =& 2A_j\Delta_+\Delta_- A_j + (\Delta_+ A_j)^2 + (\Delta_- A_j)^2,\end{aligned} \tag{4.6}$$

$$\Delta_+ A_j\Delta_+ B_j + \Delta_- A_j\Delta_- B_j = 2\Delta_o A_j\Delta_o B_j + \frac{1}{2}h^2\Delta_+\Delta_- A_j\Delta_+\Delta_- B_j, \tag{4.7}$$

$$\Delta_+ A_j\Delta_+ B_j - \Delta_- A_j\Delta_- B_j = h\{\Delta_o A_j\Delta_+\Delta_- B_j + \Delta_o B_j\Delta_+\Delta_- A_j\}, \tag{4.8}$$

$$\begin{aligned}&\Delta_+\Delta_+ A_j\Delta_+ B_j + \Delta_-\Delta_- A_j\Delta_- B_j\\ &= 2\Delta_+\Delta_- A_j\Delta_o B_j + h^2[\Delta_o\Delta_+\Delta_- A_j\Delta_+\Delta_- B_j + \Delta_o B_j\Delta_+\Delta_-\Delta_+\Delta_- A_j].\end{aligned} \tag{4.9}$$

Lemma 5[5]**.** *Set A_h and B_h are periodic discrete functions. The inner product satisfies the formulas:*

$$(A_h, \Delta_o B_h) = -(\Delta_o A_h, B_h), (A_h, \Delta_+ B_h) = -(\Delta_- A_h, B_h) \tag{4.10}$$

$$(A_h, B_h\Delta_o B_h) = -\frac{1}{2}(\Delta_o A_h, B_h^2) + \frac{1}{4}h^2(\Delta_+ A_h, (\Delta_+ B_h)^2) \tag{4.11}$$

$$(A_h, \Delta_+\Delta_- B_h \Delta_o B_h) = -\frac{1}{2}(\Delta_+ A_h, (\Delta_+ B_h)^2) \tag{4.12}$$

$$(A_h B_h, \Delta_o \Delta_+ \Delta_- B_h) = (\Delta_o A_h, \Delta_+ B_h \Delta_- B_h) + \frac{1}{2}(\Delta_+ E A_h, (\Delta_+ B_h)^2) + \frac{1}{2}(B_h, \Delta_o\Delta_+ A_h \Delta_+ B_h + \Delta_o \Delta_- A_h \Delta_- B_h) \tag{4.13}$$

and

$$((\Delta_+ A_h)^2 + (\Delta_- A_h)^2, \Delta_o\Delta_+\Delta_- A_h) = \frac{4}{5}(\Delta_+\Delta_- A_h^2, \Delta_o\Delta_+\Delta_- A_h) + \frac{1}{5}h^2(\Delta_o A_h \Delta_+\Delta_- A_h, \Delta_+\Delta_-\Delta_+\Delta_- A_h) - \frac{2}{15}h^4(\Delta_+\Delta_- A_h \Delta_o \Delta_+\Delta_- A_h, \Delta_+\Delta_-\Delta_+\Delta_- A_h) \tag{4.14}$$

Lemma 6[4]**.** *For any a, $b \geq 0$, $0 \leq \theta \leq 1$ and $\tau \geq 1$, the following inequalities are valid*

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b, \quad a^\tau + b^\tau \leq (a+b)^\tau.$$

Set

$$g_j = \Delta_+\Delta_-\Delta_+\Delta_- V_j + \frac{5}{6}\Delta_+\Delta_- V_j^2 - \frac{5}{12}[(\Delta_+ V_j)^2 + (\Delta_- V_j)^2] + \frac{5}{18}V_j^3. \tag{4.15}$$

Then

$$(V_{ht}, g_h) = \frac{\mathrm{d}}{\mathrm{d}t} F_3^h(V_h(t)). \tag{4.16}$$

Multiplying (2.2) by g_j and summing them up for j from 1 to J, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} F_3^h(V_h(t)) = -\frac{1}{2}(\Delta_o V_h^2, g_h) - (\Delta_o\Delta_+\Delta_- V_h, g_h) - (Q_h, g_h). \tag{4.17}$$

For convenience, we emit the foot-symbol $_h$ of discrete functions below. Thanks to the formula (4.14) in Lemma 5 and the formula below:

$$(\Delta_o V \Delta_+\Delta_- V, \Delta_+\Delta_-\Delta_+\Delta_- V) = (2V\Delta_o\Delta_+\Delta_- V + \frac{1}{3}h^2 \Delta_+\Delta_- V \Delta_o\Delta_+\Delta_- V, \Delta_+\Delta_-\Delta_+\Delta_- V) \tag{4.18}$$

we get

$$(\Delta_o\Delta_+\Delta_- V, g) = \frac{1}{2}(\Delta_+\Delta_- V^2, \Delta_o\Delta_+\Delta_- V) + \frac{5}{18}(V^3, \Delta_o\Delta_+\Delta_- V) - (Q, \Delta_+\Delta_-\Delta_+\Delta_- V) \tag{4.19}$$

Substituting (4.19) into (4.17), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} F_3^h(V(t)) = -\frac{5}{6}(Q, \Delta_+\Delta_- V^2) + \frac{5}{12}(Q, (\Delta_+ V)^2 + (\Delta_- V)^2) - \frac{5}{18}(Q, V^3) + \frac{5}{24}(\Delta_o V^2, (\Delta_+ V)^2 + (\Delta_- V)^2) - \frac{5}{36}(\Delta_o V^2, V^3) - \frac{5}{18}(V^3, \Delta_o\Delta_+\Delta_- V). \tag{4.20}$$

Applying the formulas in Lemma 3 and Lemma 4 again, there are

$$(V^3, \Delta_o\Delta_+\Delta_- V) = \frac{3}{4}(\Delta_o V^2, (\Delta_+ V)^2 + (\Delta_- V)^2) - \frac{1}{2}h^2(\Delta_o V^2, (\Delta_+\Delta_- V)^2) - \frac{1}{2}h^2(V\Delta_o V, (\Delta_+\Delta_- V)^2) \tag{4.21}$$

and

$$(\Delta_o V^2, V^3) = \frac{3}{10}h^2((\Delta_+ V)^2, \Delta_+ V^3) - \frac{1}{5}h^2((\Delta_+ V)^3 + (\Delta_- V)^3, V^2), \tag{4.22}$$

therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}F_3^h(V(t)) = I_1 + I_2 \tag{4.23}$$

with

$$\begin{aligned}
I_1 =& \frac{5}{36}h^2(\Delta_o V^2 + V\Delta_o V, (\Delta_+\Delta_- V)^2) - \Big(Q, \frac{5}{6}\Delta_+\Delta_- V^2 - \frac{5}{12}[(\Delta_+ V)^2 + (\Delta_- V)^2]\Big) \\
=& \frac{5}{36}\Big(1 + \frac{\beta}{2}\Big)h^2(\Delta_o V^2, (\Delta_+\Delta_- V)^2) + \frac{5\beta}{9}h^2(V\Delta_o V, (\Delta_+\Delta_- V)^2\Big) \\
&+ \frac{5(1+\beta)}{54}h^4(V\Delta_o\Delta_+\Delta_- V, (\Delta_+\Delta_- V)^2) + \frac{5\beta}{24}h^4(V\Delta_o\Delta_+\Delta_- V, \Delta_+\Delta_-\Delta_+\Delta_- V) \\
&- \frac{5(\beta-2)}{432}h^4(\Delta_+\Delta_- V\Delta_o\Delta_+\Delta_- V, (\Delta_+ V)^2 + (\Delta_- V)^2)
\end{aligned}$$

and

$$\begin{aligned}
I_2 =& -\frac{1}{24}h^2((\Delta_+ V)^2, \Delta_+ V^3) + \frac{1}{36}h^2((\Delta_+ V)^3 + (\Delta_- V)^3, V^2) - \frac{5}{18}(Q, V^3) \\
=& -\frac{13+5\beta}{432}h^2((\Delta_+ V)^2, \Delta_+ V^3) + \frac{1}{36}h^2((\Delta_+ V)^3 + (\Delta_- V)^3, V^2) \\
&+ \frac{5(\beta-2)}{1296}h^4(\Delta_+ V^3, \Delta_+\Delta_+ V\Delta_+\Delta_- V) - \frac{5\beta}{54}h^2((\Delta_+ V)^2, \Delta_+(EVEV^2)).
\end{aligned}$$

It is easy to see that I_1 and I_2 above can be estimated as follows:

$$\begin{aligned}
|I_1| &\le K_4 h^2\|V\|_\infty\|\Delta_+ V\|_\infty\|\Delta_+\Delta_- V\|_2^2 \\
|I_2| &\le K_5 h^2\|V\|_{2q}^2\|\Delta_+ V\|_{3p}^3, \quad \Big(\frac{1}{p} + \frac{1}{q} = 1\Big)
\end{aligned}$$

where, the absolute constants K_4 and K_5 are not large, for instance, $K_4 = \dfrac{25}{54}$ and $K_5 = \dfrac{1}{8}$ if $\beta = 0$. Furthermore, from the interpolation formula (4.1), we have

$$\begin{aligned}
\|V\|_\infty\|\Delta_+ V\|_\infty &\le 2K^2(\|V\|_4^{8/7}\|\Delta_+\Delta_- V\|_2^{6/7} + \|V\|_4^2) \\
\|V\|_{2q}^2\|\Delta_+ V\|_{3p}^3 &\le 16K^5(\|V\|_4^{22/7}\|\Delta_+\Delta_- V\|_2^{13/7} + \|V\|_4^5)
\end{aligned}$$

Thus, (4.23) results in the inequality:

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}F_3^h(V(t))\Big| \le K_6 h^2\{\|V\|_4^{22/7}\|\Delta_+\Delta_- V\|_2^{13/7} + \|V\|_4^5\}$$

$$+ K_7h^2\|\Delta_+\Delta_-V\|_2^2\{\|V\|_4^{8/7}\|\Delta_+\Delta_-V\|_2^{6/7} + \|V\|_4^2\}. \tag{4.24}$$

In the other hand, according to the interpolation formula again, there holds

$$\begin{aligned}\frac{5}{6}|(V,(\Delta_+V)^2 + (\Delta_-V)^2)| \leq &\frac{5}{6}\|V\|_4\|\Delta_+V\|_{8/3}^2\\ \leq &\frac{2}{72}\|V\|_4^4 + \frac{1}{4}\|\Delta_+\Delta_-V\|_2^2 + K_8\|V\|_2^{8/3} + K_9\|V\|_2^{14/3}.\end{aligned} \tag{4.25}$$

Making an integral of (4.23) for t from 0 to t, considering (4.25), and we get

$$\begin{aligned}\|\Delta_+\Delta_-V(t)\|_2^2 + \frac{1}{6}\|V(t)\|_4^4 \leq\ & C_0 + K_{10}\|V(t)\|_2^{8/3} + K_{11}\|V(t)\|_2^{14/3}\\ &+ 4K_6h^2\int_0^t\{\|V(s)\|_4^{22/7}\|\Delta_+\Delta_-V(s)\|_2^{13/7} + \|V(s)\|_4^5\}\mathrm{d}s\\ &+ 4K_7h^2\int_0^t\|\Delta_+\Delta_-V(s)\|_2^2\{\|V(s)\|_4^{8/7}\|\Delta_+\Delta_-V(s)\|_2^{6/7} + \|V(s)\|_4^2\}\mathrm{d}s\end{aligned} \tag{4.26}$$

where

$$C_0 = 2\|\Delta_+\Delta_-u_0\|_2^2 - \frac{5}{3}(u_0,(\Delta_+u_0)^2 + (\Delta_-u_0)^2) + \frac{5}{18}\|u_0\|_4^4.$$

Now, multiplying (2.2) by V_j and summing them up for j from 1 to J, we have

$$(V,V_t) + \frac{1}{6}h^2(V,\Delta_oV\Delta_+\Delta_-V) + (Q,V) = 0 \tag{4.27}$$

Using the formulas (4.11), (4.12) and (4.13), we get the estimate

$$\Big|\frac{1}{6}h^2(V,\Delta_oV\Delta_+\Delta_-V) + (Q,V)| \leq K_{12}h^2\|\Delta_+V(t)\|_3^3.$$

Hence, there is

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|_2^2\Big| \leq 2K_{12}h^2\|\Delta_+V(t)\|_3^3 \tag{4.28}$$

where the constant K_{12} is also very small and it is $\frac{17}{72}$ for $\beta = 0$.

From (4.28), we get

$$\|V(t)\|_2^2 \leq \|u_0\|_2^2 + 2K_{12}h^2\int_0^t\|\Delta_+V(s)\|_3^3\mathrm{d}s. \tag{4.29}$$

Using Lemma 1 and Lemma 6, we obtain the following inequalities from (4.29)

$$\|V(t)\|_2^{8/3} \leq \sqrt[3]{2}\|u_0\|_2^{4/3} + K_{13}h^{8/3}t^{1/3}\int_0^t\{\|V(s)\|_4^{40/21}\|\Delta_+\Delta_-V(s)\|_2^{44/21} + \|V(s)\|_4^4\}\mathrm{d}s \tag{4.30}$$

and

$$\|V(t)\|_2^{14/3} \leq 2\sqrt[3]{2}\|u_0\|_2^{7/3} + K_{14}h^{14/3}t^{4/3}\int_0^t\{\|V(s)\|_4^{10/3}\|\Delta_+\Delta_-V(s)\|_2^{11/3} + \|V(s)\|_4^7\}\mathrm{d}s \tag{4.31}$$

Substituting (4.30), (4.31) into (4.26) and standing by Lemma 6 again, we obtain the estimate for any $t \le T$:

$$\|\Delta_+\Delta_- V(t)\|_2^2 + \frac{1}{6}\|V(t)\|_4^4 \le C_9 + K_{15}Th^2[1 + (Th^2)^{1/3} + (Th^2)^{4/3}] + K_{16}h^2[1 + (Th^2)^{1/3} + (Th^2)^{4/3}] \int_0^t \{\|\Delta_+\Delta_- V(s)\|_2^2 + \frac{1}{6}\|V(s)\|_4^4\}^{8/3} \mathrm{d}s. \tag{4.32}$$

According to Lemma 2, for any $T > 0$, if h is small enough there is an estimate

$$\max_{0\le t\le T} \left\{\|\Delta_+\Delta_- V(t)\|_2^2 + \frac{1}{6}\|V(t)\|_4^4\right\} \le C_{10}, \tag{4.33}$$

therefore,

$$\max_{0\le t\le T}\{\|V(t)\|_\infty + \|\Delta_+ V(t)\|_\infty + \|\Delta_+\Delta_- V(t)\|_2\} \le C_{11}. \tag{4.34}$$

So, we gain the forth near-conserved quantity (2.6d) with (2.7) from (4.24) and (4.34). The first conserved quantity (2.6a) of scheme (2.2) is proved immediately because $\sum_{j=1}^{J} V_{jt} = 0$ and the second (2.6c) is obtained with (2.7) from (4.28) and (4.34). To derive the third one (2.6c), we make the inner product of $\Delta_+\Delta_- V_h + \frac{1}{2}V_h^2$ and equation (2.2). There is

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{6}(V^2, V) - \frac{1}{2}(\Delta_+ V, \Delta_+ V)\Big\}\Big| = \Big|(Q, \Delta_+\Delta_- V + \frac{1}{2}V^2)\Big|$$
$$\le K_{17}h^2[\|\Delta_+ V\|_\infty \|\Delta_+\Delta_- V\|_2^2 + \|V\|_\infty \|\Delta_+ V\|_2 \|\Delta_+\Delta_- V\|_2] \le C_{12}h^2$$

i.e., (2.6c) also satisfies (2.7). Theorem 1 is proved.

The estimate (2.8) in Theorem 2 is obtained directly from (4.33). To prove (2.9), (2.10), we set $V_j' = V_{jt}$ and make the derivation of (2.2) with respect to t, then get

$$V_{jt}' + \Delta_o(V_j' V_j) + \Delta_o\Delta_+\Delta_- V_j' + Q_{jt} = 0. \tag{4.35}$$

Equation (4.35) is linear with respect to V_h', therefore the follows estimate is valid because of (4.34):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V_t(t)\|_2^2 \le C_{13}\|V_t(t)\|_2^2$$

or, by Gronwall's inequality, $\max_{0\le t\le T} \|V_t(t)\|_2^2 \le \|V_t(0)\|_2^2 e^{C_{13}T} \equiv \tilde{C}_7$.

Finally, from the difference equation (2.2) and above estimate, we have

$$\max_{0\le t\le T} \|\Delta_o\Delta_+\Delta_- V(t)\|_2 \le \tilde{C}_8.$$

The proof of Theorem 2 is completed.

5. Numerical Result

We apply the scheme (3.4) to solve the periodic initial-boundary problem of the KdV equation:

$$\begin{cases} u_t + uu_x + \varepsilon u_{xxx} = 0 \\ u(x,t) = u(x+2,t) \\ u(x,0) = 2C_p\cos(\pi x) \end{cases}$$

where ε and C_p are constants. In the computation, we take $\varepsilon = 0.484 \times 10^{-3}$ and $C_p = 0.1$. Such a solution has the properties as same as the problem studied by Zabusky and Kruskal in [12]. And it has about four solitons and its recurrence time is not much large[15], therefore a quiet lager space mesh may be enough for accuracy and the total integration time is short.

For the time discretization procedure, we use implicit Runge-Kutta (IRK) method of Gauss-Legendre type which conserves numerically the first two invariants (see Ref. [16]). We write (3.4) as follows

$$\frac{\mathrm{d}V_h}{\mathrm{d}t} = -H_h(V_h(t)) \tag{5.1}$$

where $H_j(V_h) = V_j\Delta_o V_j + \Big[1+\dfrac{h^2}{6}(V_{j+2}-2V_{j+1}+V_j-2V_{j-1}+V_{j-2})\Big]\Delta_o\Delta_+\Delta_- V_j$. The IRK scheme is:

$$\tilde{V}_h(t+\Delta t) = V_h(t) - \frac{\Delta t}{2}H_h(\tilde{V}_h) \tag{5.2}$$

$$V_h(t+\Delta t) = V_h(t) - \Delta t H_h(\tilde{V}_h) \tag{5.3}$$

where Δt is the time step.

The nonlinear system (5.2) is solved by a simple fixed-point-type iteration of the form

$$\tilde{V}_h^{(k+1)} = V_h(t) - \frac{\Delta t}{2}H_h(\tilde{V}_h^{(k)}) \tag{5.4}$$

with an iteration initial $\tilde{V}_h^{(0)} = 1.5V_h(t) - 0.5V_h(t-\Delta t)$.

We take $h = 2.0\times 10^{-2}$ and $\Delta t = 5.0\times 10^{-3}/\pi$, in this case, the nonlinear iteration is convergent and the number of itaerations is not larger than 3 (the iteration accuracy criterion $\le 10^{-10}$). We made a numerical time integration up to the recurrence time $T_r = 67.4/\pi$ (the definition of T_r was referred in [12] and [15]).

The computation results show that the first several discretized conserved quantities are in good agreement with the theoretical analysis, $|F_0^h| < 10^{-15}$, $|F_1^h - 0.02| < 10^{-15}$, $|F_2^h + 0.00009451| < 3.0\times 10^{-5}$, $|F_3^h - 0.00008379| < 2.0\times 10^{-6}$.

Fig. 1. The temporal development of the wave form Fig. 2. Time evolution of the first Fourier cofficient

Figure 1 gives the numerical solutions, curve A for the initial, curve B for $t = 15.0/\pi$, and curve C for the recurrence time T_r. Figure 2 gives the time evolution of the first one Fourier coefficient $|a_1(t)|$, with $a_1(t) = \sum_{j=1}^{J} V_j(t) \exp(-\mathrm{i}\pi x_j)h$ and $a_1(0) = 0.2$. The amount $|a_1(t)|$ depicts the degree of the recurrence of the initial wave state. Here, $|a_1(T_r)| = 0.19877$ and a better recurrence is obtained.

References

[1] A. Aoyagi, K. Abe, Parametric excitation of computational modes inherent to leap-frog scheme applied to the KdV equation, *J. Comp. Phys.,* 83(1989), 447–462.

[2] Z. Han, Economical difference methods and parallel algorithms for parabolic partial differential equations (systems), *Ph.D. Thesis, IAPCM,* Beijing, 1992. (in chinese)

[3] Z. Han, L.J. Shen, A better difference scheme with four near-conserved quantities for the KdV equation, *J. Comp. Math.,* 12(1994), 224–234.

[4] Z. Han, L.J. Shen, and H.Y. Fu, Uniform Convergence of A Difference Solution for the KdV Equation, IAPCM Report 1992.

[5] Z. Han, L.J. Shen, H.Y. Fu, A Difference Scheme With Four Near-Conserved Quantities for the KdV Equation, 1992 Annual Report of Laboratory of Computational Physics, Beijing, China, 393-413.

[6] P.Y. Kuo, J.M. Sanz-Serna, Convergence of methods for the numerical solution of the KdV equation, *IMA J. Numer. Anal.,* 1(1981), 215–221.

[7] P.D. Lax, Almost periodic solutions of the KdV equation, *SIAM Review,* 18(1976), 351–375.

[8] T. Ortega, J.M. Sanz-Serna, Nonlinear stability and convergence of finite difference methods for the "Good" boussinesq equation, *Numer. Math.,* 58(1990), 215–229.

[9] L.J. Shen, Z. Han, Constructing of Difference Schemes with Multiple Near- Conserved Quantities for the KdV Equation, 1992 Annual Report of Laboratory of Computational Physics, Beijing, China, 91-104.

[10] D.M. Sloan, On modulational instabilities in discretizations of the KdV equation, *J. Comp. Phys.,* 79(1988), 167–183.

[11] A. Stuart, Nonlinear instability in dissipative finite difference schemes, *SIAM Review,* 31(1989), 191–220.

[12] N.J. Zabusky, M.D. Kruskal, Interaction of "Solitons" in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Letters,* 15(1965), 240–243.

[13] Y.L. Zhou, Applications of Discrete Functional Analysis to the Finite Difference Method, International Academic Publisher, Beijing, China, 1990.

[14] Y.L. Zhou, On the general interpolation formulas for discrete functional spaces (I), *J. Comp. Math.,* 11(1993), 188–192.

[15] K. Abe, N. Satofuka, Recurrence of initial state of nonlinear Ion waves, *Phys. Fluids,* 24(1981), 1045–1048.

[16] D.G. Cacuci, O.A. Karakashian, Benchmarking the propagator method for nonlinear systems: A burgers-korteweg-de vries equation, *J. Comp. Phys.,* 89(1990), 63–79.