Journal of Computational Mathematics, Vol.16, No.2, 1998, 97–106.

# VARIATIONS ON A THEME BY EULER*[1)]

Kang Feng    Dao-liu Wang
(*State Key Laboratory of Scientific and Engineering Computing, ICMSEC, Chinese Academy of Sciences, Beijing 100080, China*)

**Abstract**

The oldest and simplest difference scheme is the explicit Euler method. Usually, it is not symplectic for general Hamiltonian systems. It is interesting to ask: Under what conditions of Hamiltonians, the explicit Euler method becomes symplectic? In this paper, we give the class of Hamiltonians for which systems the explicit Euler method is symplectic. In fact, in these cases, the explicit Euler method is really the phase flow of the systems, therefore symplectic. Most of important Hamiltonian systems can be decomposed as the summation of these simple systems. Then composition of the Euler method acting on these systems yields a symplectic method, also explicit. These systems are called symplectically separable. Classical separable Hamiltonian systems are symplectically separable. Especially, we prove that any polynomial Hamiltonian is symplectically separable.

*Key words:* Hamiltonian systems, symplectic difference schemes, explicit Euler method, nilpotent, symplectically separable

## 1. Introduction

A Hamiltonian system of differential equations on $\mathbf{R}^{2n}$ is given by

$$\dot{p} = -H_q(p,q), \quad \dot{q} = H_p(p,q), \tag{1}$$

where $p = (p_1, \cdots, p_n)$, $q = (q_1, \cdots, q_n) \in \mathbf{R}^n$ are the generalized coordinates and momenta respectively and $H(p,q)$ is the energy of the system. The system (1) can be rewritten as the compact form

$$\dot{z} = JH_z(z), \quad J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \tag{2}$$

where $z = (z_1, \cdots, z_n, z_{n+1}, \cdots, z_{2n}) = (p,q) \in \mathbf{R}^{2n}$, $H(z) = H(p,q)$. The phase flow, denoted by $e_H^t$, of the Hamiltonian system is symplectic, i.e., it preserves the differential 2-form on $\mathbf{R}^{2n}$ ${e_H^t}^*\omega = \omega$, where $\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$, or $(e_H^t)_z^T(z)J(e_H^t)_z(z) = J$, $\forall z \in \mathbf{R}^{2n}$.

For the Hamiltonian system (2), a single step numerical method can be characterized by a map $g_H^\tau$, $\tau$ is the time step size, $z^{n+1} = g_H^\tau z^n$, or $\hat{z} = g_H^\tau z$. If $g_H^\tau$ is symplectic, i.e., $(g_H^\tau)_z^T(z)J(g_H^\tau)_z(z) = J$, $\forall z \in \mathbf{R}^{2n}$, then, the method $g_H^\tau$ is called symplectic.

* Received December 15, 1995.
[1)] This work was supported by National Natural Science Foundation of China and State Commission of Science and Technology.

The oldest and simplest difference scheme for Hamiltonian system (2) is the explicit Euler method

$$\hat{z} = E_H^\tau z := z + \tau J H_z(z), \quad E_H^\tau = 1 + \tau J H_z. \tag{3}$$

Usually, it is not symplectic for general Hamiltonian systems. But, it is symplectic for a kind of specific Hamiltonian systems, i.e., systems with nilpotent of degree 2 (see Section 2). In fact, it is the exact phase flow for these systems, therefore is symplectic. Many important Hamiltonian systems can be decomposed as the summation of Hamiltonian systems with nilpotent of degree 2, which are called symplectically separable. Then explicit symplectic schemes can be derived by composition of explicit Euler methods acting on these systems (exact phase flows). This kind of Hamiltonians is not too rare but can cover most important cases. Usual Hamiltonian systems in classical mechanics are symplectically separable. Especially, classical separable Hamiltonian systems are symplectically separable. At last, we proved that any polynomial Hamiltonian is symplectically separable.

This paper is devoted to the construction of explicit symplectic algorithms for symplectically separable Hamiltonian systems. In section 2, we give the definition of symplectically separable Hamiltonian systems. We list possible symplectically separable Hamiltonian systems in section 3 and give the construction of explicit symplectic algorithms for these systems in section 4. In section 5 we prove that all polynomials in $\mathbf{R}^{2n}$ are symplectically separable, therefore there exist explicit symplectic algorithms for these systems in principle. More detail materials about Hamiltonian systems, symplectic geometry and symplectic algorithms can be referred to [1–23].

## 2. Systems with Nilpotent of Degree 2

**Definition 1.** A Hamiltonian $H$ is nilpotent of degree 2 if $H$ satisfies

$$JH_{zz}(z)JH_z(z) = 0, \quad \forall z \in \mathbf{R}^{2n}. \tag{4}$$

Evidently, $H(p,q) = \phi(p)$ or $H(p,q) = \psi(q)$, which presents inertial flow and standing flow, are nilpotent of degree 2 since for $H(p,q) = \phi(p)$,

$$H_{zz}(z)JH_z(z) = \begin{bmatrix} \phi_{pp} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \phi_p \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_{pp} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \phi_p \end{bmatrix} = 0$$

and for $H(p,q) = \psi(q)$,

$$H_{zz}(z)JH_z(z) = \begin{bmatrix} 0 & 0 \\ 0 & \psi_{qq} \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \psi_q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \psi_{qq} \end{bmatrix} \begin{bmatrix} -\psi_q \\ 0 \end{bmatrix} = 0.$$

**Theorem 1.** *If $H$ is nilpotent of degree 2 then the explicit Euler method $E_H^\tau$ is the exact phase flow of the Hamiltonian, therefore symplectic.*

*Proof.* Let $z = z(0)$. From the condition (4) it follows that

$$\ddot{z}(t) = \frac{d}{dt} JH_z(z(t)) = (JH_z(z(t)))_z \dot{z}(t) = JH_{zz}(z(t))JH_z(z(t)) = 0.$$

Therefore,

$$\dot{z}(t) = \dot{z}(0) = JH_z(z(0)).$$

Hence

$$z(t) = z(0) + tJH_z(z(0)) = z + tJH_z(z) = E_H^t(z).$$

It is just the explicit Euler method $E_H^t$. This shows that for such a system, explicit Euler method $E_H^\tau$ is the exact phase flow, therefore symplectic.

**Theorem 2.** *Let $\phi : \mathbf{R}^n \to \mathbf{R}$ be a scalar function on $n$ variables $u$, $\phi(u) = \phi(u_1, \cdots, u_n)$. Let $C_{n\times 2n} = (A, B)$ be a linear transformation from $\mathbf{R}^{2n}$ to $\mathbf{R}^n$. Then the Hamiltonian $H(z) = \phi(Cz)$ satisfies*

$$JH_{zz}(z)JH_z(z) = 0, \quad \forall \phi, z \tag{5}$$

*if and only if*

$$CJC^T = 0. \tag{6}$$

*Proof.* Since

$$JH_{zz}(z)JH_z(z) = JC^T\phi_{uu}(Cz)CJC^T\phi_u(Cz), \tag{7}$$

the sufficient condition is trivial.

We now prove the necessity. If

$$JH_{zz}(z)JH_z(z) = 0, \quad \forall \phi, z$$

then from (7) it follows that

$$JC^T\phi_{uu}(Cz)CJC^T\phi_u(Cz) = 0, \quad \forall \phi, z.$$

Especially, take $\phi(u) = \frac{1}{2}u^Tu$. Then

$$JC^TCJC^TCz = 0, \quad \forall z.$$

i.e.,

$$JC^TCJC^TC = 0.$$

Left multiplying $C$ and right multiplying $JC^T$ this equation, we get

$$(CJC^T)^3 = 0.$$

The anti-symmetry of $CJC^T$ implies $CJC^T = 0$.

**Lemma 1.** *Let $C = (A, B)$. Then $CJC^T = 0$ if and only if $AB^T = BA^T$.*

**Theorem 3.** *For any Hamiltonian*

$$H(z) = H(p, q) = \phi(Cz) = \phi(Ap + Bq), \quad AB^T = BA^T,$$

*where $\phi(u)$ is any $n$ variable function, the explicit Euler method*

$$\hat{z} = E_H^\tau z = E_\phi^\tau z = z + \tau JH_z(z) = z + \tau JC^T\phi_u(Cz)$$

*is the exact phase flow, i.e.,*

$$e_\phi^\tau = E_\phi^\tau = 1 + \tau JH_z = 1 + \tau JC^T\phi_u \circ C.$$

*Hence, $E_\phi^\tau$ is symplectic.*

## 3. Symplectically Separable Hamiltonian Systems

**Definition 2.** Hamiltonian $H(z)$ is symplectically separable if

$$H(z) = \sum_{i=1}^{m} H_i(z), \quad H_i(z) = \phi_i(C_i z) = \phi(A_i p + B_i q), \tag{8}$$

where $\phi_i$ are the functions on $n$ variables and $C_i = (A_i, B_i)$ with the condition $AB^T = BA^T$, $i = 1, \dots, m$.

**Proposition 1.** *Linear combination of symplectic separable Hamiltonians is symplectically separable.*

For symplectically separable Hamiltonian (8), the explicit composition scheme

$$g_H^\tau = E_m^\tau \circ E_{m-1}^\tau \circ \dots \circ E_2^\tau \circ E_1^\tau := E_{H_m}^\tau \circ E_{H_{m-1}}^\tau \circ \dots \circ E_{H_2}^\tau \circ E_{H_1}^\tau \tag{9}$$

is symplectic and of order 1. As a matter of fact

$$\begin{aligned} E_{H_2}^\tau \circ E_{H_1}^\tau &= (1 + \tau J H_{2,z}) \circ (1 + \tau J H_{1,z}) = 1 + \tau J H_{2,z} + \tau J H_{1,z} + O(\tau^2) \\ &= 1 + \tau J (H_{2,z} + H_{1,z}) + O(\tau^2) \\ &\dots \end{aligned}$$

$$\begin{aligned} g_H^\tau = E_m^\tau \circ E_{m-1}^\tau \circ \dots \circ E_2^\tau \circ E_1^\tau &= (1 + \tau J H_{m,z}) \circ \Big(1 + \tau J \sum_{i=1}^{m-1} H_{i,z} + O(\tau^2)\Big) \\ &= 1 + \tau J \sum_{i=1}^{m} H_{i,z} + O(\tau^2) = 1 + \tau J H_z + O(\tau^2). \end{aligned}$$

The symplecticity of $g_H^\tau$ follows from the fact that symplectic maps on $\mathbf{R}^{2n}$ forms a group under composition.

Similarly,

$$\check{g}_H^\tau = E_1^\tau \circ E_2^\tau \circ \dots \circ E_{m-1}^\tau \circ E_m^\tau$$

is symplectic and of order 1. Composite schemes of arbitrary ordering

$$E_{i_m}^\tau \circ E_{i_{m-1}}^\tau \circ \dots \circ E_{i_2}^\tau \circ E_{i_1}^\tau$$

and

$$E_{i_1}^\tau \circ E_{i_2}^\tau \circ \dots \circ E_{i_{m-1}}^\tau \circ E_{i_m}^\tau,$$

where $i_1$, $i_2$, $\cdots$, $i_m$ is a permutation of $1, 2, \dots, m$, are also symplectic and of order 1.

## 4. Construction of Explicit Symplectic Methods with High Order

Let $g^t$ be a one-parameter family of diffeomorphisms, $g^0 = 1$. Define its reversion as

$$\check{g}^t = (g^{-t})^{-1}.$$

$g^t$ is revertible (or self-adjoint, or symmetric) if $g^t = \check{g}^t$, i.e., $g^t \circ g^{-t} =$ identity. Evidently, the phase flow of Hamiltonian systems is revertible.

The step transition operator $g^\tau$ of a difference scheme is said to be of order $r$ if it is an $r$-th approximation to the phase flow $e^t$, i.e.,

$$g^\tau = e^\tau + O(\tau^{r+1}).$$

If $g^\tau$ is revertible, then $g^\tau$ is of even order of accuracy.

Let $g^\tau$ be revertible and of order $2r$, then its revertible composition

$$g^{\alpha\tau} \circ g^{\beta\tau} \circ g^{\alpha\tau}$$

is of order $2(r+1)$ if

$$2\alpha + \beta = 1, \quad \beta^{2r+1} + 2\alpha^{2r+1} = 0, \tag{10}$$

i.e.,

$$\alpha = \frac{1}{2 - 2^{1/(2r+1)}} > 0, \quad \beta = 1 - 2\alpha < 0.$$

The proof can refer to Yoshida[23] and Qin/Zhu[19]. Here we give a more direct proof.

Since $g^\tau$ is of order $2r$, then it has the expansion

$$g^\tau = e^\tau + O(\tau^{2r+1}) = e^\tau + \tau^{2r+1} g_{2r+1} + O(\tau^{2r+2}).$$

Therefore,

$$\begin{aligned} g^{\alpha\tau} \circ g^{\beta\tau} =& (e^{\alpha\tau} + \alpha^{2r+1}\tau^{2r+1} g_{2r+1} + O(\tau^{2r+2})) \circ (e^{\beta\tau} + \beta^{2r+1}\tau^{2r+1} g_{2r+1} + O(\tau^{2r+2})) \\ =& e^{\alpha\tau} \circ (e^{\beta\tau} + \beta^{2r+1}\tau^{2r+1} g_{2r+1} + O(\tau^{2r+2})) \\ &+ \alpha^{2r+1}\tau^{2r+1} g_{2r+1} \circ (e^{\beta\tau} + \beta^{2r+1}\tau^{2r+1} g_{2r+1} + O(\tau^{2r+2})) + O(\tau^{2r+2}) \\ =& e^{\alpha\tau} e^{\beta\tau} + \beta^{2r+1}\tau^{2r+1} g_{2r+1} + \alpha^{2r+1}\tau^{2r+1} g_{2r+1} + O(\tau^{2r+2}) \\ =& e^{(\alpha+\beta)\tau} + (\beta^{2r+1} + \alpha^{2r+1})\tau^{2r+1} g_{2r+1} + O(\tau^{2r+2}) \end{aligned}$$

Similarly,

$$g^{\alpha\tau} \circ g^{\beta\tau} \circ g^{\alpha\tau} = e^{(2\alpha+\beta)\tau} + (\beta^{2r+1} + 2\alpha^{2r+1})\tau^{2r+1} g_{2r+1} + O(\tau^{2r+2}).$$

Therefore, $g^{\alpha\tau} \circ g^{\beta\tau} \circ g^{\alpha\tau}$ is of order $2r+1$ if and only if

$$2\alpha + \beta = 1, \quad \beta^{2r+1} + 2\alpha^{2r+1} = 0.$$

The revertibility of $g^{\alpha\tau} \circ g^{\beta\tau} \circ g^{\alpha\tau}$ implies that it is of order $2(r+1)$.

Since the phase flow of a Hamiltonian system is revertible, the Euler method $E_H^\tau$ for Hamiltonians with nilpotent of degree 2 is also revertible. Hence,

$$\check{g}_H^\tau = E_1^\tau \circ E_2^\tau \circ \ldots \circ E_{m-1}^\tau \circ E_m^\tau$$

is the reversion of

$$g_H^\tau = E_m^\tau \circ E_{m-1}^\tau \circ \ldots \circ E_2^\tau \circ E_1^\tau.$$

The composition

$$g_2^\tau = \check{g}_H^{\tau/2} \circ g_H^{\tau/2} = E_{H_1}^{\tau/2} \circ E_{H_2}^{\tau/2} \circ \ldots \circ E_{H_{m-1}}^{\tau/2} \circ E_{H_m}^{\tau} \circ E_{H_{m-1}}^{\tau/2} \circ \ldots \circ E_{H_2}^{\tau/2} \circ E_{H_1}^{\tau/2}$$

is revertible and of order 2. The composition

$$g_4^\tau = g_2^{\alpha\tau} \circ g_2^{\beta\tau} \circ g_2^{\alpha\tau}$$

gives a revertible explicit symplectic scheme of order 4 when

$$2\alpha + \beta = 1, \quad 2\alpha^3 + \beta^3 = 0,$$

i.e.,

$$\alpha = \frac{1}{2 - 2^{1/3}} > 0, \qquad \beta = 1 - 2\alpha < 0, \tag{11}$$

which was derived by Qin/Wang/Zhang[18] and Yoshida[23] etc. by different ways. Similarly, we can get high order symplectic schemes by this procedure.

**Example 1.** Since $\phi(p)$ and $\psi(q)$ are nilpotent of degree 2, the classical separable Hamiltonian $H(p,q) = \phi(p) + \psi(q)$ is also symplectically separable. The composition of $E_\phi^\tau$ and $E_\psi^\tau$ gives explicit symplectic schemes of order 1

$$\begin{aligned} E_\psi^\tau \circ E_\phi^\tau : &\quad \hat{p} = p - \tau\psi_q(\hat{q}), \quad \hat{q} = q + \tau\phi_p(p), \\ E_\phi^\tau \circ E^\tau\psi : &\quad \hat{p} = p - \tau\psi_q(q), \quad \hat{q} = q + \tau\phi_p(\hat{p}). \end{aligned}$$

The revertible schemes

$$\begin{aligned} &g^\tau := (E_\psi^{\tau/2} \circ E_\phi^{\tau/2})^\vee \circ (E_\psi^{\tau/2} \circ E_\phi^{\tau/2}) = E_\phi^{\tau/2} \circ E_\psi^\tau \circ E_\phi^{\tau/2} : \\ &q^1 = q + \frac{\tau}{2}\phi_p(p), \quad \hat{p} = p - \tau\psi_q(q_1), \quad \hat{q} = q^1 + \frac{\tau}{2}\phi_p(\hat{p}). \end{aligned}$$

and

$$\begin{aligned} &f^\tau := (E_\phi^{\tau/2} \circ E_\psi^{\tau/2})^\vee \circ (E_\phi^{\tau/2} \circ E_\psi^{\tau/2}) = E_\psi^{\tau/2} \circ E_\phi^\tau \circ E_\psi^{\tau/2} : \\ &p^1 = p - \frac{\tau}{2}\psi_q(q), \quad \hat{q} = q + \tau\phi_p(p_1), \quad \hat{p} = p^1 - \frac{\tau}{2}\psi_q(\hat{q}). \end{aligned}$$

are symplectic and of order 2. The revertible composition

$$g^{\alpha\tau} \circ g^{\beta\tau} \circ g^{\alpha\tau} \quad f^{\alpha\tau} \circ f^{\beta\tau} \circ f^{\alpha\tau}$$

give symplectic schemes of order 4 with the parameters (11), i.e. ([18], [23]),

$$\begin{aligned} p^1 &= p - c_1\tau\psi_q(q), & q^1 &= q + d_1\tau\phi_p(p^1), \\ p^2 &= p^1 - c_2\tau\psi_q(q^1), & q^2 &= q^1 + d_2\tau\phi_p(p^2), \\ p^3 &= p^2 - c_3\tau\psi_q(q^2), & q^3 &= q^2 + d_3\tau\phi_p(p^3), \\ \hat{p} &= p^3 - c_4\tau\psi_q(q^3), & \hat{q} &= q^3 + d_4\tau\phi_p(\hat{p}), \end{aligned}$$

with the parameters $\alpha = (2 - 2^{1/3})^{-1}$, $\beta = 1 - 2\alpha$ and either

$$c_1 = 0, \quad c_2 = c_4 = \alpha, \quad c_3 = \beta, \quad d_1 = d_4 = \alpha/2, \quad d_2 = d_3 = (\alpha + \beta)/2,$$

or

$$c_1 = c_4 = \alpha/2, \quad c_2 = c_3 = (\alpha + \beta)/2, \quad d_1 = d_3 = \alpha, \quad d_2 = \beta, \quad d_4 = 0.$$

**Example 2.** The Hamiltonian

$$H_k(p,q)=\sum_{i=0}^{k-1}\cos\Big(p\cos\frac{2\pi i}{k}+q\sin\frac{2\pi i}{k}\Big)$$

with $k$-fold rotational symmetry in phase plane[2,4] are not separable in the conventional sense for $k\neq 1,2,4$, but symplectically separable, since every term

$$\cos\Big(p\cos\frac{2\pi i}{k}+q\sin\frac{2\pi i}{k}\Big)$$

is nilpotent of degree 2 according to Theorem 2. Such as, for $k=3$,

$$\begin{aligned}H_3(p,q)&=\cos p+\cos\Big(p\cos\frac{2\pi}{3}+q\sin\frac{2\pi}{3}\Big)+\cos\Big(p\cos\frac{4\pi}{3}+q\sin\frac{4\pi}{3}\Big)\\&=\cos p+\cos\Big(\frac{1}{2}p-\frac{\sqrt{3}}{2}q\Big)+\cos\Big(-\frac{1}{2}p-\frac{\sqrt{3}}{2}q\Big)\end{aligned}$$

The explicit symplectic scheme of order 1 is

$$\begin{aligned}&q^1=q-\frac{1}{2}\tau\sin\Big(\frac{1}{2}p+\frac{\sqrt{3}}{2}q\Big), && p^1=p+\frac{\sqrt{3}}{2}\tau\sin\Big(\frac{1}{2}p+\frac{\sqrt{3}}{2}q\Big),\\&q^2=q^1-\frac{1}{2}\tau\sin\Big(\frac{1}{2}p^1-\frac{\sqrt{3}}{2}q^1\Big), && \hat{p}=p^1-\frac{\sqrt{3}}{2}\tau\sin\Big(\frac{1}{2}p-\frac{\sqrt{3}}{2}q\Big),\\&\hat{q}=q^2-\tau\sin\hat{p}.\end{aligned}$$

The explicit revertible symplectic scheme of order 2 is

$$\begin{aligned}&q^1=q-\frac{1}{4}\tau\sin\Big(\frac{1}{2}p+\frac{\sqrt{3}}{2}q\Big), && p^1=p+\frac{\sqrt{3}}{4}\tau\sin\Big(\frac{1}{2}p+\frac{\sqrt{3}}{2}q\Big),\\&q^2=q^1-\frac{1}{4}\tau\sin\Big(\frac{1}{2}p^1-\frac{\sqrt{3}}{2}q^1\Big), && p^2=p^1-\frac{\sqrt{3}}{4}\tau\sin\Big(\frac{1}{2}p^1-\frac{\sqrt{3}}{2}q^1\Big),\\&q^3=q^2-\tau\sin p^2, && p^3=p^2-\frac{\sqrt{3}}{4}\tau\sin\Big(\frac{1}{2}p^2-\frac{\sqrt{3}}{2}q^2\Big),\\&q^4=q^3-\frac{1}{4}\tau\sin\Big(\frac{1}{2}p^3-\frac{\sqrt{3}}{2}q^3\Big), && \hat{p}=p^3+\frac{\sqrt{3}}{4}\tau\sin\Big(\frac{1}{2}p^3+\frac{\sqrt{3}}{2}q^4\Big),\\&\hat{q}=q^4-\frac{1}{4}\tau\sin\Big(\frac{1}{2}p^3+\frac{\sqrt{3}}{2}q^4\Big)\end{aligned}$$

Similarly, we can get explicit symplectic schemes of order 4.

## 5. Separability of All Polynomials in $\mathbf{R}^{2n}$

**Theorem 4.** *Every monomial $x^{n-k}y^k$ of degree $n$ in 2 variables $x$ and $y$, $n\geq 2$, $0\leq k\leq n$ can be expanded as a linear combination of $n+1$ terms*

$$\{(x+y)^n,\ (x+2y)^n,\ \ldots,\ (x+2^{n-2}y)^n,\ x^n,\ y^n\}$$

*Proof.* Using binomial expansion

$$(x+y)^n=x^n+C_n^1x^{n-1}y^1+C_n^2x^{n-2}y^2+\ldots+C_n^2x^2y^{n-2}+C_n^1x^1y^{n-1}+y^n,$$

define

$$\begin{aligned} P_1(x,y) &= (x+y)^n - x^n - y^n \\ &= C_n^1 x^{n-1}y^1 + C_n^2 x^{n-2}y^2 + \ldots + C_n^2 x^2 y^{n-2} + C_n^1 x^1 y^{n-1}, \end{aligned}$$

which is separable, and the right hand side consists of "mixed terms". $P_1$ is a linear combination of 3 terms $(x+y)^n$, $x^n$ and $y^n$. Then

$$\begin{aligned} P_1(x,2y) =& 2C_n^1 x^{n-1}y^1 + 2^2 C_n^2 x^{n-2}y^2 + \ldots + 2^{n-2}C_n^2 x^2 y^{n-2} + 2^{n-1}C_n^1 x^1 y^{n-1}, \\ 2P_1(x,2y) =& 2C_n^1 x^{n-1}y^1 + 2C_n^2 x^{n-2}y^2 + \ldots + 2C_n^2 x^2 y^{n-2} + 2C_n^1 x^1 y^{n-1}. \end{aligned}$$

Define

$$\begin{aligned} P_2(x,y) =& P_1(x,2y) - 2P_1(x,y) \\ =& (2^2-2)C_n^2 x^{n-2}y^2 + \ldots + (2^{n-2}-2)C_n^2 x^2 y^{n-2} + (2^{n-1}-2)C_n^1 x^1 y^{n-1}. \end{aligned}$$

It is separable in 4 terms $(x+y)^n$, $(x+2y)^n$, $x^n$ and $y^n$. Define

$$\begin{aligned} P_3(x,y) =& P_2(x,2y) - 2^2 P_2(x,y) = (2^3-2^2)(2^3-2)C_n^3 x^{n-3}y^3 + \ldots \\ &+ (2^{n-2}-2^2)(2^{n-2}-2)C_n^2 x^2 y^{n-2} + (2^{n-1}-2^2)(2^{n-1}-2)C_n^1 x^1 y^{n-1}. \end{aligned}$$

It is separable in 5 terms $(x+y)^n$, $(x+2y)^n$, $(x+2^2y)^n$, $x^n$ and $y^n$.

Define

$$\begin{aligned} P_{n-2}(x,y) =& P_{n-3}(x,2y) - 2^{n-3}P_{n-3}(x,y) \\ =& (2^{n-2}-2^{n-3})\cdots(2^{n-2}-2)C_n^2 x^2 y^{n-2} \\ &+ (2^{n-1}-2^{n-3})\cdots(2^{n-1}-2)C_n^1 x^1 y^{n-1}, \end{aligned}$$

separable in $n$ terms $(x+y)^n$, $(x+2y)^n$, $\cdots$, $(x+2^{n-3}y)^n$, $x^n$ and $y^n$. Finally, we get

$$\begin{aligned} P_{n-1}(x,y) &= P_{n-2}(x,2y) - 2^{n-2}P_{n-2}(x,y) \\ &= (2^{n-1}-2^{n-2})(2^{n-1}-2^{n-3})\cdots(2^{n-1}-2)C_n^1 x^1 y^{n-1} \\ &= \gamma_{n-1}x^1 y^{n-1}, \quad \gamma \neq 0 \end{aligned}$$

separable in $n+1$ terms $(x+y)^n$, $(x+2y)^n$, $\ldots$, $(x+2^{n-2}y)^n$, $x^n$ and $y^n$.

Hence, the mixed term $xy^{n-1}$ is separable in $n+1$ terms. Then from the separability of $P_{n-2}(x,y)$ and $xy^{n-1}$ we know that $x^2y^{n-2}$ is separable in $n+1$ terms. Similarly, $x^3y^{n-3}$, $x^4y^{n-4}$, $\cdots$, $x^{n-2}y^2$ and $x^{n-1}y$ are separable in $n+1$ terms.

**Remark.** We can also work with formula

$$\begin{aligned} &\frac{1}{2}(x+y)^{2m+1} + \frac{1}{2}(x-y)^{2m+1} - x^{2m+1} \\ &\qquad = C_{2m+1}^2 x^{2m-1}y^2 + C_{2m+1}^4 x^{2m-3}y^4 + \cdots + C_{2m+1}^{2m} xy^{2m} \\ &\frac{1}{2}(x+y)^{2m+1} - \frac{1}{2}(x-y)^{2m+1} - y^{2m+1} \\ &\qquad = C_{2m+1}^1 x^{2m}y + C_{2m+1}^3 x^{2m-2}y^3 + \cdots + C_{2m+1}^{2m-1} x^2 y^{2m-1} \end{aligned}$$

$$\begin{aligned}&\frac{1}{2}(x+y)^{2m}+\frac{1}{2}(x-y)^{2m}-x^{2m}-y^{2m}\\&\qquad=C_{2m}^{2}x^{2m-2}y^{2}+C_{2m}^{4}x^{2m-4}y^{4}+\cdots+C_{2m}^{2m-2}x^{2}y^{2m-2}\\&\frac{1}{2}(x+y)^{2m}-\frac{1}{2}(x-y)^{2m}=C_{2m}^{1}x^{2m-1}y+C_{2m}^{3}x^{2m-3}y^{3}+\cdots+C_{2m}^{2m-1}xy^{2m-1}\end{aligned}$$

by means of elimination to get more symmetric and economic expansions, e.g.,

$$xy=\frac{1}{2}(x+y)^2-\frac{1}{2}x^2-\frac{1}{2}y^2=\frac{1}{4}(x+y)^2-\frac{1}{4}(x-y)^2.$$

**Theorem 5.** *Every polynomial $P(p,q)$ of degree $n$ in variables $p$ and $q$ can be expanded as $n+1$ terms*

$$P_1(x,y),P_2(x,y),\ldots,P_{n-1}(x,y),P_n(x),P_{n+1}(y),$$

*where each $P_i(u)$ is a polynomial of degree $n$ in one variable or more generally every polynomial $P(p,q)$ can be expanded as*

$$P(p,q)=\sum_{i=1}^{m}P_i(a_ip+b_iq),\quad m\le n+1,$$

*where $P_i(u)$ are polynomial of degree $n$ in one variable.*

**Theorem 6.** *Every monomial in $2n$ variables is of the form*

$$f(p,q)=(p_1^{m_1-k_1}q_1^{k_1})(p_2^{m_2-k_2}q_2^{k_2})\cdots(p_n^{m_n-k_n}q_n^{k_n})$$

*and can be expanded as a linear combination of the terms in the form*

$$\phi(Ap+Bq)=(a_1p_1+b_1q_1)^{m_1}(a_2p_2+b_2q_2)^{m_2}\ldots(a_np_n+b_nq_n)^{m_n},$$

*where $\phi(u)=\phi(u_1,\ldots,u_n)=u_1^{m_1}u_2^{m_2}\ldots u_n^{m_n}$ is the monomial in $n$ variables with total degree $m=\sum\limits_{i=1}^{m}m_i$ and with degree $m_i$ in variable $u_i$, $A$ and $B$ are diagonal matrices of order $n$*

$$A=\begin{pmatrix}a_1&0&\cdots&0\\0&a_2&\cdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\cdots&a_n\end{pmatrix},\qquad B=\begin{pmatrix}b_1&0&\cdots&0\\0&b_2&\cdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\cdots&b_n\end{pmatrix},$$

*which automatically satisfies $AB^T=BA^T$. The elements $a_i$, $b_i$ can be chosen integers.*

**Theorem 7.** *Every polynomial $P(p_1,q_1,\cdots,p_n,q_n)$ of degree $m$ in $2n$ variables can be expanded as*

$$P(p,q)=\sum_{i=1}^{m}P_i(A_ip+B_iq),$$

*each $P_i$ is a polynomial of degree $m$ in $n$ variables, $A_i$ and $B_i$ are diagonal matrices, (satisfying $A_iB_i^T=B_iA_i^T$). So for polynomial Hamiltonians, the symplectic explicit Euler composite schemes of order 1, 2 or 4 can be easily constructed.*

## References

[1] V.I. Arnold, Mathematical Methods of Classical Mechanics, New York: Springer. 1978.

[2] V.V. Beloshapkin, A.A. Chernikov, M. Natenzon, B.A. Petrovichev, R.Z. Sagdeev, G.M. Zaslavsky, Chaotic streamlines in pre-turbulent states, *Nature,* **337**:12 (1989), 133–137.

[3] P.J. Channell, J.C. Scovel, Symplectic Integration of Hamiltonian Systems, *Nonlinearity,* **3** (1990), 213.

[4] A.A. Chernikov, R.Z. Sagdeev, G.M. Zaslavsky, Stochastic Webs, *Physica D,* **33** (1988), 65–76.

[5] K. Feng, On difference schemes and symplectic geometry, In *Feng K, ed. Proc 1984 Beijing Symp Diff Geometry and Diff Equations. Beijing: Science Press,* (1985), 42–58.

[6] K. Feng, Difference schemes for Hamiltonian formalism and symplectic geometry, *J. Comp. Math.,* **4**:3: (1986), 279–289.

[7] K. Feng, The Hamiltonian way for computing Hamiltonian dynamics, In *Spigler R, ed. Applied and Industrial Mathematics. Netherlands: Kluwer,* (1991), 17–35.

[8] K. Feng, How to compute properly Newton's equation of motion, In Ying L A, Guo B Y, ed. Proc of 2nd Conf on Numerical Methods for Partial Differential Equations, Singapore: World Scientific, (1992), 15–22.

[9] K. Feng, Formal dynamical systems and numerical algorithms, In *Feng K, Shi Z C, ed. Proc Inter Conf on Computation of Differential Equations and Dynamical Systems,* Singapore: World Scientific, (1993), 1–10.

[10] K. Feng, The step transition operators for multi-step methods of ODE's: [Preprint]. Beijing: CAS. Computing Center, 1991.

[11] K. Feng, The calculus of generating functions and the formal energy for Hamiltonian algorithms: [Preprint]. Beijing: CAS. Computing Center, 1991.

[12] K. Feng, M.Z. Qin, The symplectic methods for computation of Hamiltonian equations. In Zhu Y L, Guo Ben-yu, ed. Proc Conf on Numerical Methods for PDE's. Berlin: Springer, 1987. 1–37. Lect Notes in Math 1297.

[13] K. Feng, M.Z. Qin, Hamiltonian algorithms and a comparative numerical study, *Comput Phys Comm.,* **65** (1991), 173–187.

[14] K. Feng, D.L. Wang, A note on conservation laws of symplectic difference schemes for Hamiltonian systems, *J. Comp. Math.,* **9**:3 (1991), 229–237.

[15] K. Feng, H.M. Wu, M.Z. Qin, D.L. Wang, Construction of canonical difference schemes for Hamiltonian formalism via generating functions, *J. Comp. Math.,* **7**:1, (1989), 71–96.

[16] Z. Ge, K. Feng, On the approximation of Hamiltonian systems, *J. Comp. Math.,* **6**:1 (1988), 88–97.

[17] Z. Ge, J. Marsden, Lie-Poisson Hamilton-Jacobi theory and Lie-Poisson integrators, *Phys Lett A.,* **133**:3 (1988), 137–139.

[18] M.Z. Qin, D.L. Wang, M.Q. Zhang, Explicit symplectic difference schemes for separable Hamiltonian systems, *J. Comp. Math.,* **9**:3 (1991), 211–221.

[19] M.Z. Qin, W.J. Zhu, Construction of higher order symplectic schemes by composition, *Computing,* **47** (1992), 309–321.

[20] D.L. Wang, Some aspects of Hamiltonian systems and symplectic algorithms, *Physica D,* **73** (1994), 1–16.

[21] Y.H. Wu, The generating function of the solution of ODE and its discrete methods, *Computer Math. Applic.,* **15**:12 (1988), 1041–1050.

[22] Y.H. Wu, Symplectic transformations and symplectic difference schemes, *Chinese J. Numer. Math. Applic.,* **12**:1 (1990), 23–31.

[23] H. Yoshida, Construction of higher order symplectic integrators, *Phys. Letters A,* **150** (1990), 262–268.