Journal of Computational Mathematics, Vol.16, No.1, 1998, 27–39.

AN ACCURATE NUMERICAL SOLUTION OF A TWO DIMENSIONAL HEAT TRANSFER PROBLEM WITH A PARABOLIC BOUNDARY LAYER*

C. Clavero
(*Dpto. de Matematica Aplicada, Universidad de Zaragoza, Zaragoza, Spain*)
J.J.H. Miller
(*Mathematics Department, Trinity College, Dublin 2, Ireland*)
E. O'Riordan
(*School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland*)
G.I. Shishkin
(*Institute of Mathematics and Mechanics, Ekaterinburg, Russia*)

Abstract

A singularly perturbed linear convection-diffusion problem for heat transfer in two dimensions with a parabolic boundary layer is solved numerically. The numerical method consists of a special piecewise uniform mesh condensing in a neighbourhood of the parabolic layer and a standard finite difference operator satisfying a discrete maximum principle. The numerical computations demonstrate numerically that the method is ε-uniform in the sense that the rate of convergence and error constant of the method are independent of the singular perturbation parameter ε. This means that no matter how small the singular perturbation parameter ε is, the numerical method produces solutions with guaranteed accuracy depending solely on the number of mesh points used.

Key words: Linear convection-diffusion, parabolic layer, piecewise uniform mesh, finite difference.

1. Introduction

Singularly perturbed differential equations are characterised by the presence of a small parameter ε multiplying the highest order derivatives. Such problems arise in many areas of applied mathematics. The solutions of singularly perturbed differential equations typically have steep gradients, in thin regions of the domain, whose magnitude depends inversely on some positive power of ε. Such regions are called either interior or boundary layers, depending on whether their location is the interior or the boundary of the domain. The location and width of these layers depend on the local asymptotic nature of the solution of the differential equation. Layers described by

* Received April 15, 1996.
1) This research has been supported by the CICYT project num. AMB.94-0396 and by the contract with ENRESA num. 70.2.1.15.01

an ordinary, parabolic or elliptic differential equation are called respectively regular, parabolic or elliptic layers. Numerical methods for which the error bounds are independent of the singular perturbation parameter ε are called *ε-uniform* methods. In most previous work ε-uniform methods have been constructed and tested numerically only for singular perturbation problems with regular layers. In this paper, numerical results are presented for a singularly perturbed linear convection-diffusion problem with a parabolic layer. These results confirm numerically that numerical methods composed of a standard finite difference operator satisfying a maximum principle on a special piecewise-uniform mesh are ε-uniform. In fact it has been established theoretically in [9] that such numerical methods are ε-uniform for a wide class of singularly perturbed problems, including the problem considered here. Special piecewise-uniform meshes were first introduced and analyzed by Shishkin in [8]. The first computations using such methods were presented in [4].

2. Statement of the problem

Letting θ denote the temperature, $\vec{u} = (u_1, u_2)$ the velocity field of the fluid and $\varepsilon = \dfrac{1}{Pe}$ (where Pe is the Peclet number) the coefficient of diffusion, the transfer of heat in a two-dimensional region Ω is described by the following linear convection-diffusion equation (in dimensionless form)

$$\nabla \cdot (-\varepsilon \nabla \theta + \vec{u}\theta) = f \quad \text{in } \Omega \tag{2.1a}$$

where it is assumed that Ω is a bounded domain with Lipshitz continuous boundary Γ. Let Γ_D and Γ_N respectively denote the parts of Γ where Dirichlet and Neumann boundary conditions are specified, where $\Gamma = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Let $\vec{n}$ denote the outward unit normal on Γ. The inflow and outflow boundaries Γ_i and Γ_o are defined respectively by

$$\Gamma_o = \{(x,y) \in \Gamma : (\vec{u} \cdot \vec{n})(x,y) > 0\}, \qquad \Gamma_i = \{(x,y) \in \Gamma : (\vec{u} \cdot \vec{n})(x,y) < 0\}$$

It is assumed that $\Gamma_D \supset \Gamma_i$, that the diffusion coefficient ε is positive and that $\nabla \cdot \vec{u} = 0$. The latter condition means that the velocity of the fluid $\vec{u}$ corresponds to an incompressible flow. When $\varepsilon \ll 1$, the differential equation (2.1a) is singularly perturbed and the flow is said to be convection dominated.

The solution of this linear problem with general boundary conditions can be decomposed into the sum of a smooth and a singular part for each kind of singularity. In this paper we choose boundary conditions so that there is just one kind of singularity, namely a parabolic boundary layer. The purpose of this paper is to obtain ε-uniformly accurate solutions for this special problem. Because of the linearity, it is clear that the same numerical method will solve any general linear problem having this type of singularity.

The following boundary conditions are taken on Γ:

$$\theta = g \quad \text{on } \Gamma_D \tag{2.1b}$$

$$\frac{\partial \theta}{\partial n} = 0 \quad \text{on } \Gamma_N \tag{2.1c}$$

The streamlines of the reduced equation corresponding to (2.1a) are in the direction $\vec{u}$. The boundary is said to be characteristic at a point $(x, y) \in \Gamma$ if the tangent to Γ at that point is in the direction $\vec{u}$. Equivalently, Γ is characteristic at each point $(x, y) \in \Gamma$ such that

$$(\vec{u} \cdot \vec{n})(x, y) = 0$$

In what follows $\Omega = (-1, 1) \times (0, 1)$ and $\vec{u}, f$ are taken to be[10]

$$\vec{u}(x, y) = (2y(1 - x^2), -2x(1 - y^2))^T \tag{2.1d}$$

$$f(x, y) \equiv 0 \tag{2.1e}$$

(See Fig. 2.1). Consequently, $\Gamma_i = \{(x, 0) : -1 < x < 0\}$ and $\Gamma_o = \{(x, 0) : 0 < x < 1\}$. The Neumann part of the boundary is taken to be

$$\Gamma_N = \Gamma_o \tag{2.1f}$$

The Dirichlet boundary conditions on Γ_D are defined by

$$g(x, y) = \begin{cases} 1 - y, & \text{on } \Gamma_e \\ 0, & \text{on } \Gamma_D \setminus \Gamma_e q \end{cases} \tag{2.1g}$$

Fig. 2.1 Geometry specification and streamlines for the test problem (2.1, a–g)

where $\Gamma_e = \{(1, y) : 0 < y < 1\}$. This choice of boundary conditions ensures that they are compatible and that the only boundary layer in the solution is a parabolic layer in a neighbourhood of the edge Γ_e.

In [3] Hutton considered a similar problem with the Dirichlet boundary conditions

$$g(x, y) = \begin{cases} 1 + \tanh(20x + 10), & \text{on } \Gamma_i \\ 0, & \text{on } \Gamma_D \setminus \Gamma_i \end{cases}$$

With this choice the solution has no boundary layers. This problem demonstrates the effects of cross-stream diffusion when the streamlines are not parallel to the co-ordinate axes. A further study of the problem was made by Scotney in [5] with the Dirichlet boundary conditions

$$g(x, y) = \begin{cases} 1 + \tanh(20x + 10), & \text{on } \Gamma_i \\ 100, & \text{on } \Gamma_e \\ 0, & \text{on } \Gamma_D \setminus (\Gamma_i \cup \Gamma_e) \end{cases}$$

The solution of this problem exhibits a parabolic boundary layer in a neighbourhood of the edge Γ_e, but it also has an additional discontinuity arising from an incompatability of the boundary conditions at the corner point (1,1). Our choice of Dirichlet boundary conditions (2.1g) is more suitable for the investigation of the effect of a parabolic boundary layer, because it is not complicated by any further effect due to an incompatability in the boundary conditions, as is the case with Scotney's choice. Numerical experiments using higher-order finite difference schemes on problems similar to those considered by Hutton and Scotney were reported by Smith and Hutton in [10] and by Thompson and Wilkes in [11].

Both uniform and piecewise-uniform meshes $\Omega_{N,N} \equiv \{(x_i, y_j)\colon\ i = 0, 1, \cdots N;\ j = 0, 1, \cdots N\}$ are used to discretize the domain Ω. Mesh functions defined on the mesh $\Omega_{N,N}$ are denoted by θ^N, and various mesh parameters by

$$
\begin{aligned}
h_i &\equiv x_i - x_{i-1}, \quad \tilde{h}_{i+1} \equiv (h_{i+1} + h_i)/2, \\
k_j &\equiv y_j - y_{j-1}, \quad \tilde{k}_{j+1} \equiv (k_{j+1} + k_j)/2.
\end{aligned} \tag{2.2}
$$

We write $\theta_{i,j} = \theta^N(x_i, y_j)$. On $\Omega_{N,N}$ the following upwind finite difference operator is defined

$$
-\varepsilon(\delta_x^2 + \delta_y^2)\theta^N + \tilde{D}_x(u_1\theta^N) + \tilde{D}_y(u_2\theta^N) = 0, \quad \text{for } i = 1, \cdots, N-1; j = 1, \cdots, N-1; \tag{2.3a}
$$

where $\tilde{D}_x$ is the first order upwind finite difference operator

$$
\tilde{D}_x(u_1\theta^N) \equiv \frac{u_{1;i,j} - |u_{1;i,j}|}{2} D_x^+\theta_{i,j} + u_{1;i,j} + |u_{1;i,j}| 2 D_x^-\theta_{i,j} \tag{2.3b}
$$

$$
D_x^-\theta_{i,j} \equiv \frac{\theta_{i,j} - \theta_{i-1,j}}{h_i} \quad \text{and } D_x^+\theta_{i,j} \equiv \frac{\theta_{i+1,j} - \theta_{i,j}}{h_{i+1}} \tag{2.3c}
$$

The operator $\tilde{D}_y$ is defined analogously. The standard second order centered finite difference is

$$
\delta_x^2\theta_{i,j} \equiv \frac{D_x^+\theta_{i,j} - D_x^-\theta_{i,j}}{\tilde{h}_{i+1}} \tag{2.3d}
$$

and δ_y^2 is defined analogously. The boundary conditions are discretized as follows.

$$
\frac{\theta_{i,1} - \theta_{i,0}}{k_1} = 0 \quad \text{for } i = \frac{N}{2} + 1, \cdots, N-1 \tag{2.3e}
$$

$$
\theta_{i,0} = 0 \quad \text{for } i = 0, \cdots, \frac{N}{2} \tag{2.3f}
$$

$$
\theta_{i,N} = 0 \quad \text{for } i = 0, \cdots, N-1 \tag{2.3g}
$$

$$
\theta_{0,j} = 0 \quad \text{for } j = 1, \cdots, N-1 \tag{2.3h}
$$

$$
\theta_{N,j} = 1 - y_j \quad \text{for } j = 0, \cdots, N \tag{2.3i}
$$

Since the matrix associated with the numerical method (2.3 a-i) is an M-matrix, the upwind finite difference operator in (2.3a) satisfies a maximum principle and the finite difference method (2.3a-i) is stable. In this paper an iterative method is used to solve the discretized equations. The Relaxed Incomplete LU-Factorisation method[1] and the preconditioned conjugate gradient squared method are used, where the convergence criteria on the residuals is taken to be $\|r_k\|_\infty \leq 10^{-6}$.

3. Numerical results on a uniform mesh

In this section, the problem (2.1a-g) is solved using a numerical method composed of the upwind difference operator (2.3) on a sequence of uniform meshes $\Omega_{N,N}$, with $N = 8, 16, 32, 64, 128, 256, 512$. Since the exact solution of this problem is not known, the pointwise errors $|\theta(x_i, y_j) - \theta^N(x_i, y_j)|$ are approximated for successive values of ε on the five coarser meshes by $e_\varepsilon^N(i,j) = |\theta^N(x_i, y_j) - \theta^{512}(x_i, y_j)|$, where the superscript indicates the number of mesh elements used in the x-direction. For each ε the maximum nodal error is approximated by

$$E_{\varepsilon,N} = \max_{i,j} e_\varepsilon^N(i,j) \tag{3.1}$$

and for each N, the ε-uniform maximum nodal error is defined by

$$E_N = \max_{\varepsilon} E_{\varepsilon,N} \tag{3.2}$$

Computed values of $E_{\varepsilon,N}$ and E_N for problem (2.1a-g) are given in Table 3.1 for various values of ε and N.

It is seen from the table that, for all values of $\varepsilon \le 2^{-14}$, the error grows monotonically as the mesh is refined. This shows that for small ε an increase of the computational effort does not yield greater accuracy; on the contrary it increases the pointwise error. This behaviour is completely opposite to what is expected from a satisfactory numerical method. Also, note that the error E_N does not decrease monotonicaly as the mesh is refined. These results indicate numerically that this numerical method is not ε-uniform. It is important to note that the accuracy of the approximations to the error decreases as we move to the right in the table, because the exact solution has been replaced by θ^{512}, nevertheless we believe that the qualitative behaviour of the numerical method is correctly represented by this table of results (see also Remark at the end of §5).

In Fig. 3.1 a contour plot of the numerical solution for the problem with $\varepsilon = 2^{-10}$ is given in a neighbourhood of the edge of the domain where the parabolic boundary layer occurs. The solution itself on the whole domain is shown in Fig. 3.2. Notice that the solution is almost zero everywhere except in a neighbourhood of the parabolic boundary layer. The 0.1 contour intersects the x-axis close to the point $x = 0.94$.

Fig. 3.1 Contours of the numerical solution of problem (2.1, a–g) with $\varepsilon = 2^{-10}$ near the edge of the domain where the parabolic layer occurs using the upwind finite difference operator (2.3) and the uniform mesh $\Omega_{32\times 32}$

Fig. 3.2 Numerical solution of problem (2.1, a–g) with $\varepsilon = 2^{-10}$ using the upwind finite difference operator (2.3) and the uniform mesh $\Omega_{32\times 32}$

Table 3.1 Maximum Nodal Errors $E_{\varepsilon,N}$ and E_N using a uniform mesh

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$
2^{0}	4.961E−2	2.904E−2	1.484E−2	7.021E−3	3.079E−3
2^{-2}	3.318E−2	2.781E−2	1.561E−2	7.479E−3	3.176E−3
2^{-4}	6.459E−2	3.288E−2	1.793E−2	8.751E−3	3.852E−3
2^{-6}	5.454E−2	6.486E−2	3.126E−2	1.611E−2	7.215E−3
2^{-8}	1.631E−2	5.283E−2	6.068E−2	2.770E−2	1.323E−2
2^{-10}	4.122E−3	1.565E−2	5.117E−2	5.523E−2	2.291E−2
2^{-12}	1.036E−3	3.986E−2	1.545E−2	4.877E−2	4.695E−2
2^{-14}	2.594E−4	1.001E−3	3.931E−3	1.532E−2	4.391E−2
2^{-16}	6.486E−5	2.506E−4	9.873E−4	3.906E−3	1.505E−2
2^{-18}	1.666E−5	6.669E−5	2.471E−4	9.811E−4	3.894E−3
2^{-20}	4.169E−6	1.676E−5	6.673E−5	2.664E−4	9.782E−4
2^{-22}	4.172E−6	4.191E−6	1.680E−5	6.723E−5	2.658E−4
2^{-24}	2.992E−7	4.191E−6	1.680E−5	1.682E−5	6.720E−5
2^{-26}	2.944E−7	4.191E−6	1.383E−6	1.682E−5	6.721E−5
2^{-28}	2.944E−7	4.191E−6	1.383E−6	1.682E−5	5.672E−6
2^{-30}	2.733E−7	4.192E−6	1.383E−6	1.682E−5	5.674E−6
2^{-32}	2.733E−7	4.191E−6	1.383E−6	1.682E−5	5.672E−6
E_N	0.0646	0.0649	0.0607	0.0552	0.0470

4. Condensing Piecewise-Uniform Meshes

Condensing piecewise-uniform meshes $\Omega^*_{N,N}$ are constructed as follows. A piecewise uniform mesh is taken in the x-direction and a uniform mesh is taken in the y-direction. The mesh $\Omega^*_{N,N}$ is defined to be the tensor product of these two one-dimensional meshes. The condensing piecewise-uniform mesh is defined on the interval $[-1,1]$ by subdividing it into the three intervals $[-1,0],[0,1-\sigma_x]$, $[1-\sigma_x,1]$ where the transition point

$$\sigma_x \equiv \min\left\{\sqrt{\varepsilon}\ln N, \frac{1}{2}\right\} \tag{4.1}$$

On each subinterval a uniform mesh is constructed using

$$\frac{N}{2} \quad \text{mesh points in } [-1,0],$$

$$\text{and} \quad \frac{N}{4} \quad \text{mesh points in both } [0,1-\sigma_x], \quad \text{and } [1-\sigma_x,1].$$

The condensing piecewise uniform mesh is then defined by

$$\Omega^*_{N,N} = \{(x_i,y_j) : y_j = j/N, 0 \le i,j \le N\} \tag{4.2a}$$

where

$$x_i = \begin{cases} -1+\dfrac{2i}{N}, & \text{for } 0 \le i \le N/2 \\ \dfrac{4(1-\sigma_x)(i-N/2)}{N}, & \text{for } N/2 < i \le 3N/4 \\ 1-\sigma_x+\dfrac{4\sigma_x(i-3N/4)}{N}, & \text{for } 3N/4 < i \le N \end{cases} \tag{4.2b}$$

Note that $\Omega^*_{N,N}$ is a uniform mesh on the domain Ω when ε or N are sufficiently large.

The main theoretical result for numerical methods consisting of standard finite difference operators satisfying a maximum principle and these condensing piecewise uniform meshes is contained in the following theorem. It states that these methods are ε-uniform in the sense that the error bound is independent of the singular perturbation parameter ε.

In order to state a theoretical result, it is necessary to modify the definition of the problem so that in a neighbourhood of the corner $(1,1)$ the velocity field is

$$\vec{u} = (2a_1(1-x^2), -2a_2(1-y^2))$$

where a_1, a_2 are positive constants. Then for this slightly modified problem, the following theorem can be proved.

Theorem 1. *Let θ be the solution of the problem* (2.1a-g) *and let θ^N be the numerical approximation of θ computed using the upwind finite difference operator* (2.3) *on the condensing piecewise uniform mesh* (4.2). *Then the following pointwise error estimate holds*

$$\max_{i,j} |\theta(x_i,y_j) - \theta^N(x_i,y_j)| \leq C(N^{-1}\ln N)^{1/7} \tag{4.3}$$

where C is a constant independent of ε and N.

Proof. This may be proved using the analytical techniques described in Shishkin [6,7].

It is important to note that we are using the maximum norm in the entire domain to measure the pointwise error. Therefore, when this is small the error at each point of the domain is small and since approximately $\frac{1}{4}$ of the points are inside the boundary layer, the results we obtain are accurate not only outside boundary layers but comparably so within the boundary layers. This is in marked contrast to previous work on problems of this type, where qualitative rather than quantitative measures of the accuracy were employed.

5. Numerical results on condensing piecewise uniform meshes

The problem (2.1a-g) is solved using a numerical method composed of the upwind difference operator (2.3) on the sequence of condensing piecewise uniform meshes $\Omega^*_{N,N}$ with $N = 8, 16, 32, 64, 128, 256, 512$ The errors $|\theta^N(x_i,y_j) - \theta(x_i,y_j)|$ are approximated, for successive values of ε, on the five coarser meshes, by $e^N_\varepsilon(i,j) = |\theta^N(x_i,y_j) - \theta^{512}_I(x_i,y_j)|$, where the superscript indicates the number of mesh elements used and the subscript denotes bilinear interpolation. For each ε and N the maximum nodal error is approximated by

$$E^*_{\varepsilon,N} = \max_{i,j} e^N_\varepsilon(i,j) \tag{5.1}$$

and for each N the ε-uniform maximum nodal error is defined by

$$E^*_N = \max_{\varepsilon} E^*_{\varepsilon,N} \tag{5.2}$$

Computed values of $E^*_{\varepsilon,N}$ and E^*_N for problem (2.1 a-g) are given in Table 5.1 for various values of ε and N. Note that the superscript $*$ denotes the errors encountered using the condensing piecewise uniform meshes.

Table 5.1. Maximum Nodal Errors $E^*_{\varepsilon,N}$ and E^*_N using a piecewise uniform mesh

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$
2^{0}	4.961E−2	2.904E−2	1.484E−2	7.021E−3	3.079E−3
2^{-2}	3.318E−2	2.781E−2	1.561E−2	7.479E−3	3.176E−3
2^{-4}	6.459E−2	3.288E−2	1.793E−2	8.751E−3	3.852E−3
2^{-6}	6.599E−2	4.660E−2	2.842E−2	1.611E−2	7.215E−3
2^{-8}	6.318E−2	4.482E−2	2.680E−2	1.537E−2	7.483E−3
2^{-10}	6.172E−2	4.425E−2	2.663E−2	1.532E−2	7.470E−3
2^{-12}	6.085E−2	4.435E−2	2.654E−2	1.530E−2	7.462E−3
2^{-14}	6.039E−2	4.427E−2	2.650E−2	1.529E−2	7.457E−3
2^{-16}	6.015E−2	4.422E−2	2.647E−2	1.528E−2	7.455E−3
2^{-18}	6.003E−2	4.420E−2	2.646E−2	1.528E−2	7.454E−3
2^{-20}	5.997E−2	4.419E−2	2.646E−2	1.527E−2	7.453E−3
2^{-22}	5.990E−2	4.414E−2	2.641E−2	1.523E−2	7.415E−3
2^{-24}	5.986E−2	4.412E−2	2.639E−2	1.521E−2	7.415E−3
2^{-26}	5.985E−2	4.410E−2	2.638E−2	1.521E−2	7.415E−3
2^{-28}	5.984E−2	4.410E−2	2.638E−2	1.520E−2	7.382E−3
2^{-30}	5.984E−2	4.410E−2	2.638E−2	1.521E−2	7.382E−3
2^{-32}	5.984E−2	4.410E−2	2.638E−2	1.521E−2	7.382E−3
E_N	0.0660	0.0466	0.0284	0.0161	0.0075

The numerical behaviour indicated by Table 5.1 is quite different qualitatively from that in Table 3.1. Note that for each fixed value of ε, the errors decrease monotonically as the mesh is refined. Furthermore, the errors E^*_N also decrease monotonically for increasing N and at a much faster rate than in Table 3.1. Indeed, for $N = 128$ the result is an order of magnitude better. Hence, increasing the computational effort yields greater accuracy, which is the intuitively correct behaviour for a numerical method to be considered satisfactory.

Note also that the ε-uniform rate of convergence can be estimated using the double-mesh principle (see ,e.g., Hegarty et al.[2]). A numerical method for solving (2.1) is said to have an ε-uniform rate of convergence of order p on the sequence of meshes $\{\Omega_{N,N}\}_1^\infty$ if $\exists N_0$, independent of ε, such that for all $N \geq N_0$

$$\sup_{0<\varepsilon\leq 1} \|\theta - \theta^N\|_{\Omega_{N,N}} \leq CN^{-p}$$

($\|w\|_{\Omega_{N,N}} \equiv \max_{(x_i,y_j)\in\Omega_{N,N}} |w((x_i,y_j))|$) where θ is the solution of (2.1), θ^N is the numerical approximation to θ, C and $p > 0$ are independent of ε and N.

These numerical rates of convergence are in agreement with the ε-uniform rate given in 4.3.

Comparing the contours in Figs 3.1 and 5.1 it can be seen that the solution on the uniform mesh $\Omega_{32\times32}$ is significantly more diffuse than that on the special mesh

$\Omega^*_{32\times32}$. This numerical diffusion is seen much more clearly in the four graphs in Fig. 5.3 where in each case the dashed line in each graph is a small section of the line joining the values of the numerical solution at two points of a uniform mesh, while the continuous line is the piecewise linear interpolant of the numerical solution at five points of a special piecewise uniform mesh. Assuming that the contours in Fig. 5.4 are good approximations to the true contours, it follows that the contours in Fig. 5.1 are better approximations than those in Fig. 3.1.

Fig. 5.1 Controus of the numerical solution of problem (2.1, a–g) with $\varepsilon = 2^{-10}$ near the edge fo the domain where the parabolic layer occurs using the upwind finite difference operator (2.3) and the special piecewise uniform mesh $\Omega^*_{32\times32}$.

Fig. 5.2 Numerical solution of problem (2.1, a–g) with $\varepsilon = 2^{-10}$ using the upwind finite difference operator (2.3) and the special piecewise uniform mesh $\Omega^*_{32\times32}$

Fig. 5.3 Comparison of the numerical solution of problem (2.1, a–g) with $\varepsilon = 2^{-10}$ near the edge of the domain where the parabolic layer occurs using the uniform mesh $\Omega_{32\times32}$ (indicated by the dashes) and the special piecewise uniform mesh $\Omega^*_{32\times32}$ (indicated by the continuous line) on the interval $0.95 \le x \le 1$ with (a) $y = 0.0$ (b) $y = 0.25$ (c) $y = 0.5$ (d) $y = 0.75$.

Fig. 5.4 Contours of the numerical solution of problem (2.1, a–g) with $\varepsilon = 2^{-10}$ near the edge of the domain where the parabolic layer occurs using the upwind finite difference operator (2.3) and the special piecewise uniform mesh $\Omega^*_{256\times256}$.

Table 5.2. Numerical rates of convergence for a piecewise uniform mesh

ε	N= 8	N= 16	N= 32	N=64
2^0	0.772	0.969	1.080	1.189
2^{-2}	0.255	0.833	1.062	1.235
2^{-4}	0.974	0.875	1.035	1.184
2^{-6}	0.502	0.713	0.819	1.159
2^{-8}	0.495	0.742	0.802	1.039
2^{-10}	0.471	0.741	0.797	1.037
2^{-12}	0.456	0.741	0.795	1.036
2^{-14}	0.448	0.740	0.793	1.036
2^{-16}	0.444	0.740	0.793	1.036
2^{-18}	0.442	0.740	0.793	1.036
2^{-20}	0.440	0.740	0.792	1.036
2^{-22}	0.440	0.741	0.793	1.039
2^{-24}	0.440	0.741	0.794	1.037
2^{-26}	0.440	0.741	0.794	1.037
2^{-28}	0.440	0.741	0.794	1.044
2^{-30}	0.440	0.741	0.795	1.042
2^{-32}	0.440	0.741	0.795	1.042
Min:	0.255	0.713	0.792	1.036

Remark. In the following table, the maximum nodal errors on a uniform mesh are given where the exact solution is approximated by the numerical solution θ_I^{512} computed on the condensing piecewise uniform mesh $\Omega^*_{512,512}$.

Table 5.3. Maximum Nodal Errors E_N on a Uniform Mesh

ε	N= 8	N= 16	N= 32	N= 64	N= 128
E_N	0.0646	0.0649	0.0614	0.0598	0.0589

6. Conclusions

It has been shown by numerical computation that a numerical method consisting of an upwind finite difference operator on a uniform mesh gives inaccurate solutions to a singularly perturbed linear convection-diffusion problem of heat transfer in two dimensions when a parabolic boundary layer is present. Further numerical computations were presented which confirm the known theoretical result that ε-uniform methods can be constructed for this problem using an upwind finite difference operator on a special piecewise uniform mesh.

References

[1] G. Guiding, On The Solution of Non-Symmetric Linear Systems by The Preconditioned Conjugate Gradient Square Method, INCA preprint No. 7, Dublin, 1987.

[2] A.F. Hegarty, J.J.H. Miller, E. O'Riordan, G.I. Shishkin, Special meshes for finite difference approximations to an advection-diffusion equation with parabolic layers, *J. Comp. Phys.*, **117** (1995), 47–54.

[3] A.G. Hutton, The Numerical Representation of Convection, IAHR Working Group Meeting, 1981.

[4] J.J.H. Miller, E. Mullarkey, E. O'Riordan, G. Shishkin, A simple recipe for uniformly convergent finite difference schemes for singularly perturbed problems, *C. R. Acad. Sci. Paris*, t.312, Serie I, (1991), 643–648.

[5] B. Scotney, Numerical experiments and error analysis for Petrov-Galerkin methods, Numer.Anal. Report, 11/82, University of Reading, 1982.

[6] G.I. Shishkin, Approximation of solutions of singularly perturbed boundary value problems with a corner layer, *U.S.S.R. Comput. Maths. Math. Phys.,* **27** : 5 (1987), 54–63.

[7] G.I. Shishkin, Numerical solution of elliptic equations with a small parameter at the highest order derivatives, *Numerical Methods in the Mechanics of Continuous Media, Novosibirsk,* **10** : 4 (1979), 107–124. (in Russian)

[8] G.I. Shishkin, Difference scheme for singularly perturbed parabolic equation with discontinuous boundary condition, *U.S.S.R. Comput. Maths. Math. Phys.,* **28** : 11 (1988), 1679–1622. (in Russian)

[9] G.I. Shishkin, Discrete approximation of singularly perturbed elliptic and parabolic equations, Russian Academy of Sciences, Ural Section, Ekaterinburg, 1992 (in Russian).

[10] R.M. Smith, A.G. Hutton, The numerical treatment of advection: A performance comparison of current methods, *Numercial Heat Transfer,* **5** (1982), 439–461.

[11] C.P. Thompson, N.S. Wilkes, Experiments With Higher-Order Finite Difference Formulae, AERE-R10493, U.K. Atomic Energy Authority, Harwell, 1982.