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CONTACT ALGORITHMS FOR CONTACT DYNAMICAL SYSTEMS*

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Abstract

In this paper, we develop a general way to construct contact algorithms for contact dynamical systems. Such an algorithm requires the corresponding step-transition map preserve the contact structure of the underlying contact phase space. The constructions are based on the correspondence between the contact geometry of $\mathbf{R}^{2n+1}$ and the conic symplectic one of $\mathbf{R}^{2n+2}$ and therefore, the algorithms are derived naturally from the symplectic algorithms of Hamiltonian systems.

Key words: Contact algorithms, contact systems, conic symplectic geometry, generating functions.

1. Introduction

Contact structure is an analog of a symplectic one for odd-dimensional manifolds, it stems from manifolds of contact elements of configuration spaces in mechanics and, therefore, it is also of basic importance in physical and engineering sciences. We apply, in this paper, the ideas of preserving Lie group and Lie algebra structure of dynamical systems in constructing symplectic algorithms for Hamiltonian systems to the study of numerical algorithms for contact dynamical systems and present so-called contact algorithms, i.e., algorithms preserving contact structures, for solving numerically contact systems.

A contact structure on a manifold is defined as a nondegenerate field of tangent hyperplanes and, therefore, it is determined by a differential 1-form, uniquely up to an everywhere non-vanishing multiplier function, such that the zero set of the 1-form at a point on the manifold is the tangent hyperplane of the field at the point. So, contact structures occur only on manifolds of odd-dimensions. In this paper, we simply consider the Euclidean space $\mathbf{R}^{2n+1}$ of $2n+1$ dimensions as our basic manifold with the contact structure given by the normal form

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[1)] Professor K. Feng was the former director of Computing Center (now it is renamed ICMSEC) of Chinese Academy of Sciences. This paper is arranged by Zai-jiu Shang, based on the notes of Professor K. Feng.

$$\alpha = \sum_{i=1}^{n} x_i dy_i + dz =: xdy + dz = (0, x^T, 1) \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \tag{1.1}$$

here we have used 3-symbol notation to denote the coordinates and vectors on $\mathbf{R}^{2n+1}$

$$x = (x_1, \cdots, x_n)^T, \; y = (y_1, \cdots, y_n)^T, \; z = (z). \tag{1.2}$$

A contact dynamical system on $\mathbf{R}^{2n+1}$ is governed by a contact vector field $f = (a^T, b^T, c^T)$: $\mathbf{R}^{2n+1} \to \mathbf{R}^{2n+1}$ through equations

$$\dot{x} = a(x, y, z), \quad \dot{y} = b(x, y, z), \quad \dot{z} = c(x, y, z), \quad \dot{} =: \frac{d}{dt}, \tag{1.3}$$

where the contactivity condition of the vector field f is

$$L_f \alpha = \lambda_f \alpha \tag{1.4}$$

with some function $\lambda_f : \mathbf{R}^{2n+1} \to \mathbf{R}$, called the multiplier of f. In (1.4), $L_f \alpha$ denotes the Lie derivative of α with respect to f and is usually calculated by the formula[10]

$$L_f \alpha = i_f d\alpha + d i_f \alpha. \tag{1.5}$$

It is easy to show from (1.4) and (1.5) that to any contact vector field f on $\mathbf{R}^{2n+1}$, there corresponds a function $K(x, y, z)$, called contact Hamiltonian, such that

$$a = -K_y + K_z x, \quad b = K_x, \quad c = K - x^T K_x =: K_e. \tag{1.6}$$

In fact, (1.6) represents the general form of a contact vector field. Its multiplier, denoted as λ_K from now, is equal to K_z.

A contact transformation g is a diffeomorphism on $\mathbf{R}^{2n+1}$

$$g : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} \hat{x}(x, y, z) \\ \hat{y}(x, y, z) \\ \hat{z}(x, y, z) \end{pmatrix}$$

conformally preserving the contact structure, i.e., $g^* \alpha = \mu_g \alpha$, that means

$$\sum_{i=1}^{n} \hat{x}_i d\hat{y}_i + d\hat{z} = \mu_g \Big(\sum_{i=1}^{n} x_i dy_i + dz \Big) \tag{1.7}$$

for some everywhere non-vanishing function $\mu_g : \mathbf{R}^{2n+1} \to \mathbf{R}$, called the multiplier of g. The explicit expression of (1.7) is

$$(0, \hat{x}^T, 1) \begin{pmatrix} \hat{x}_x & \hat{x}_y & \hat{x}_z \\ \hat{y}_x & \hat{y}_y & \hat{y}_z \\ \hat{z}_x & \hat{z}_y & \hat{z}_z \end{pmatrix} = \mu_g (0, x^T, 1).$$

A fundamental fact is that the phase flow g_K^t of a contact dynamical system associated with contact Hamiltonian $K : \mathbf{R}^{2n+1} \to \mathbf{R}$ is a one parameter (local) group of contact transformations on $\mathbf{R}^{2n+1}$, i.e., g_K^t satisfies

$$g_K^0 = \text{identity map on } \mathbf{R}^{2n+1}; \tag{1.8}$$

$$g_K^{t+s} = g_K^t \circ g_K^s, \quad \forall t, s \in \mathbf{R}; \tag{1.9}$$

$$\left(g_K^t\right)^* \alpha = \mu_{g_K^t} \alpha \tag{1.10}$$

for some everywhere non-vanishing function $\mu_{g_K^t} : \mathbf{R}^{2n+1} \to \mathbf{R}$. Moreover, we have the following relation between $\mu_{g_K^t}$ and the Hamiltonian K:

$$\mu_{g_K^t} = \exp \int_0^t (K_z \circ g_K^s) ds. \tag{1.11}$$

For general contact systems, condition (1.9) is stringent for algorithmic approximations to phase flows because only the phase flows themselves satisfy it. We will construct algorithms for contact systems such that the corresponding algorithmic approximations to the phase flows satisfy the condition (1.10), of course probably with different, but everywhere non-vanishing, multipliers from $\mu_{g_K^t}$. We call such algorithms contact ones.

2. Contactization and Symplectization

There is a well known correspondence between the contact geometry on $\mathbf{R}^{2n+1}$ and the conic (or homogeneous) symplectic geometry on $\mathbf{R}^{2n+2}$. To establish this correspondence, we introduce two spaces $\mathbf{R}_+^{2n+2}$ and $\mathbf{R}_+ \times \mathbf{R}^{2n+1}$.

a. We use the 4-symbol notation for the coordinates on $\mathbf{R}^{2n+2}$

$$\begin{pmatrix} p_0 \\ p_1 \\ q_0 \\ q_1 \end{pmatrix}, \quad p_0 = (p_0), \quad q_0 = (q_0), \quad p_1 = \begin{pmatrix} p_{11} \\ \vdots \\ p_{1n} \end{pmatrix}, \quad q_1 = \begin{pmatrix} q_{11} \\ \vdots \\ q_{1n} \end{pmatrix}. \tag{2.1}$$

Consider

$$\mathbf{R}_+^{2n+2} = \{(p_0, p_1, q_0, q_1) \in \mathbf{R}^{2n+2} | p_0 > 0\}, \tag{2.2}$$

a conic symplectic space with the standard symplectic form

$$\omega = dp_0 \wedge dq_0 + dp_1 \wedge dq_1. \tag{2.3}$$

Definition 1. *Function $\phi : \mathbf{R}_+^{2n+2} \to \mathbf{R}$ is called a conic function if it satisfies*

$$\phi(p_0, p_1, q_0, q_1) = p_0 \phi\Big(1, \frac{p_1}{p_0}, q_0, q_1\Big), \quad \forall p_0 > 0. \tag{2.4}$$

So, a conic function on $\mathbf{R}^{2n+2}$ depends essentially only on $2n+1$ variables.

Definition 2. *A map* $F : \mathbf{R}_+^{2n+2} \to \mathbf{R}_+^{2n+2}$ *is called a conic map if*

$$F \circ T_\lambda = T_\lambda \circ F, \quad \forall \lambda > 0, \tag{2.5}$$

where T_λ *is the linear transformation on* $\mathbf{R}^{2n+2}$

$$T_\lambda \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda p \\ q \end{pmatrix}, \quad p = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}, \quad q = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}. \tag{2.6}$$

The conic condition (2.5) for the map $F : (p_0, p_1, q_0, q_1) \to (P_0, P_1, Q_0, Q_1)$ can be expressed as follows

$$\begin{aligned} P_0(p_0, p_1, q_0, q_1) &= p_0 P_0\Big(1, \frac{p_1}{p_0}, q_0, q_1\Big) > 0, \quad \forall p_0 > 0, \\ P_1(p_0, p_1, q_0, q_1) &= p_0 P_1\Big(1, \frac{p_1}{p_0}, q_0, q_1\Big), \\ Q_0(p_0, p_1, q_0, q_1) &= Q_0\Big(1, \frac{p_1}{p_0}, q_0, q_1\Big), \\ Q_1(p_0, p_1, q_0, q_1) &= Q_1(1, \frac{p_1}{p_0}, q_0, q_1\Big). \end{aligned} \tag{2.7}$$

So, a conic map is essentially depending only on $2n+2$ functions in $2n+1$ variables.

It should be noted that, in some cases, we also consider conic functions and conic maps defined on the whole Euclidean space. The following lemma gives a criterion of a conic symplectic map.

Lemma 1. $F : (p_0, p_1, q_0, q_1) \to (P_0, P_1, Q_0, Q_1)$ *is a conic symplectic map if and only if* $(0, 0, P_0^T, P_1^T)F_* - (0, 0, p_0^T, p_1^T) = 0$*, where* F_* *is the Jacobi matrix of* F *at the point* (p_0, p_1, q_0, q_1)*.*

Proof. For $F : (p_0, p_1, q_0, q_1) \to (P_0, P_1, Q_0, Q_1)$, the condition

$$(0, 0, P_0^T, P_1^T)F_* - (0, 0, p_0^T, p_1^T) = 0 \tag{2.8}$$

is equivalent to the condition

$$P_0\, dQ_0 + P_1\, dQ_1 = p_0\, dq_0 + p_1\, dq_1 \quad \text{or} \quad PdQ = pdq, \tag{2.9}$$

where $P = (P_0, P_1)$, $Q = (Q_0, Q_1)$, $p = (p_0, p_1)$, $q = (q_0, q_1)$. Hence in matrix form, it can be written as

$$Q_p^T P = 0, \qquad Q_q^T P = p. \tag{2.10}$$

Notice that a function $f(x_1, x_2, \cdots, x_n)$ is homogeneous of degree k, i.e., $f(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) = \lambda^k f(x_1, x_2, \cdots, x_n)$, if and only if $\sum_{i=1}^n x_i f_{x_i}(x_1, x_2, \cdots, x_n) = k f(x_1, x_2, \cdots, x_n)$. Therefore, the condition (2.7) is equivalent to

$$P_p(p, q)p = P(p, q), \quad Q_p(p, q)p = 0. \tag{2.11}$$

If F is conic symplectic then $Q_p^T P_p - P_p^T Q_p = 0$, $Q_q^T P_q - P_q^T Q_q = 0$, $Q_q^T P_p - P_q^T Q_p = I$. Combining with (2.11) we get $p = Q_q^T P_p p - P_q^T Q_p p = Q_q^T P$, $0 = Q_p^T P_p p - P_p^T Q_p p = Q_p^T P$. This proves the “only if” part.

Conversely, if F satisfies the condition (2.8), then it satisfies (2.9), which means that it is symplectic. We know that if a matrix is symplectic then its transpose is also symplectic. Therefore, $P_q P_p^T - P_p P_q^T = 0$, $Q_q Q_p^T - Q_p Q_q^T = 0$, $P_p Q_q^T - P_q Q_p^T = I$. Combining with (2.10), we get $P = P_p Q_q^T P - P_q Q_p^T P = P_p p$, $0 = Q_q Q_p^T P - Q_p Q_q^T P = Q_q p$. This means that F is conic. This finishes the proof.

b. Consider $\mathbf{R}_+ \times \mathbf{R}^{2n+1}$, the product of the positive real line $\mathbf{R}_+$ and the contact space $\mathbf{R}^{2n+1}$. We use (w, x, y, z) to denote the coordinates of $\mathbf{R}_+ \times \mathbf{R}^{2n+1}$ with $w > 0$ and with x, y, z as before.

Definition 3. *A map $G : \mathbf{R}_+ \times \mathbf{R}^{2n+1} \to \mathbf{R}_+ \times \mathbf{R}^{2n+1}$ is called a positive product map if it is composed by a map $g : \mathbf{R}^{2n+1} \to \mathbf{R}^{2n+1}$ and a positive function $\gamma : \mathbf{R}^{2n+1} \to \mathbf{R}_+$ in the form*

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix}, \quad W = w\gamma(x, y, z), \quad (X, Y, Z) = g(x, y, z). \tag{2.16}$$

We denote by $\gamma \otimes g$ the positive product map composed by map g and function γ.

c. Define map $S : \mathbf{R}_+ \times \mathbf{R}^{2n+1} \to \mathbf{R}_+^{2n+2}$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} p_0 = w \\ p_1 = wx \\ q_0 = z \\ q_1 = y \end{pmatrix}. \tag{2.17}$$

Then the inverse $S^{-1} : \mathbf{R}_+^{2n+2} \to \mathbf{R}_+ \times \mathbf{R}^{2n+1}$ is given by

$$\begin{pmatrix} p_0 \\ p_1 \\ q_0 \\ q_1 \end{pmatrix} \to \begin{pmatrix} w = p_0 \\ x = \dfrac{p_1}{p_0} \\ y = q_1 \\ z = q_0 \end{pmatrix}. \tag{2.18}$$

Lemma 2. *Given a transformation $F : (p_0, p_1, q_0, q_1) \to (P_0, P_1, Q_0, Q_1)$ on $\mathbf{R}_+^{2n+2}$ and let $G = S^{-1} \circ F \circ S$. Then we have*

(1) F is a conic map on $\mathbf{R}_+^{2n+2}$ if and only if G is a positive product map on $\mathbf{R}_+ \times \mathbf{R}^{2n+1}$; in this case, if we write $G = \gamma \otimes g$, then

$$\gamma(x, y, z) = P_0(1, x, z, y), \tag{2.19}$$

and $g : (x, y, z) \to (X, Y, Z)$ is given by

$$X = \frac{P_1(1, x, z, y)}{P_0(1, x, z, y)}, \quad Y = Q_1(1, x, z, y), \quad Z = Q_0(1, x, z, y). \tag{2.20}$$

(2) *F is a conic symplectic map if and only if G is a positive product map, say $\gamma \otimes g$, on $\mathbf{R}_+ \times \mathbf{R}^{2n+1}$ with g also a contact map on $\mathbf{R}^{2n+1}$. Moreover, in this case, the multiplier of the contact map g is just equal to $\gamma^{-1} = P_0^{-1}(1, x, z, y)$.*

Proof. The conclusion (1) is easily proved by some simple calculations. Below we devote to the proof of (2). Let F send (p_0, p_1, q_0, q_1) to (P_0, P_1, Q_0, Q_1) and G send (w, x, y, z) to (W, X, Y, Z). Then by using the conclusion (1), we have

$$P_0 \circ S = wP_0(1, x, z, y) = w\gamma, \quad P_1 \circ S = wP_1(1, x, z, y) = w\gamma X(x, y, z),$$

$$S_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & wI_n & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & I_n & 0 \end{pmatrix}, \quad G_* = \frac{\partial(W, X, Y, Z)}{\partial(w, x, y, z)} = \begin{pmatrix} \gamma & w\gamma_x & w\gamma_y & w\gamma_z \\ 0 & & & \\ 0 & & g_* & \\ & & & \end{pmatrix},$$

$$S_* \circ G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ X & WI_n & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & I_n & 0 \end{pmatrix}$$

and compute

$$\begin{aligned}
&([(0, 0, P_0^T, P_1^T)F_* - (0, 0, p_0^T, p_1^T)] \circ S)S_* = ((0, 0, P_0^T, P_1^T) \circ S)(F_* \circ S)S_* \\
&\qquad - ((0, 0, p_0^T, p_1^T) \circ S)S_* = (0, 0, w\gamma, w\gamma X^T)(F_* \circ S)S_* - (0, 0, w, wx^T)S_* \\
&\quad =(0, 0, w\gamma, w\gamma X^T)(S_* \circ G)G_* - (0, 0, w, wx^T)S_* \\
&\quad =w\gamma[0, (0, X^T, 1)g_*] - w\gamma[0, \gamma^{-1}(0, x^T, 1)].
\end{aligned}$$

Noting that S is a diffeomorphism, S_* is non-singular, $w > 0$, $\gamma > 0$, we obtain $(0, 0, P_0^T, P_1^T)F_* - (0, 0, p_0^T, p_1^T) \equiv 0 \Leftrightarrow (0, X^T, 1)g_* - \gamma^{-1}(0, x^T, 1) \equiv 0$, which proves the conclusion (2).

Lemma 2 establishes correspondences between the conic symplectic space and the contact space and between conic symplectic maps and contact maps. We call the transform from F to $G = S^{-1} \circ F \circ S = \gamma \otimes g$ contactization of conic symplectic maps, the transform from $G = \gamma \otimes g$ to $F = S \circ G \circ S^{-1}$ symplectization of contact maps and call the transform $S : \mathbf{R}_+ \times \mathbf{R}^{2n+1} \to \mathbf{R}_+^{2n+2}$ symplectization of the contact space, and the transform $C = S^{-1} : \mathbf{R}_+^{2n+2} \to \mathbf{R}_+ \times \mathbf{R}^{2n+1}$ contactization of the conic symplectic space.

3. Contact Generating Functions for Contact Maps

With the preliminaries of the last section, it is natural to derive contact generating function theory for contact maps from the well known symplectic analog.

The following two lemmas are easily proved.

Lemma 3. *Hamiltonian $\phi : \mathbf{R}^{2n+2} \to \mathbf{R}$ is a conic function if and only if the associated Hamiltonian vector field $a_\phi = J\nabla\phi$ is conic, i.e., $a(T_\lambda z) = T_\lambda a(z)$ for $\lambda \neq 0$ with $z \in \mathbf{R}^{2n+2}$, where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.*

Lemma 4. *Linear map* $\begin{pmatrix} p \\ q \end{pmatrix} \to C \begin{pmatrix} p \\ q \end{pmatrix}$ *is a conic transformation on* $\mathbf{R}^{2n+2}$, *i.e.,* $C \circ T_\lambda = T_\lambda \circ C$, *if and only if the matrix* C *has the diagonal form* $C = \begin{pmatrix} C_0 & 0 \\ 0 & C_1 \end{pmatrix}$ *with* $(n+1) \times (n+1)$ *matrices* C_0 *and* C_1.

Noting that the matrix in $gl(2n+2)$

$$C = \frac{1}{2}(I + JB), \quad B = B^T \in sm(2n+2)^{1)}, \tag{3.1}$$

establishes a 1-1 correspondence between near-zero Hamiltonian vector fields $z \to a(z) = J\nabla\phi(z)$ and near-identity symplectic maps $z \to g(z)$ via generating relation

$$g(z) - z = J\nabla\phi(Cg(z) + (I - C)z), \tag{3.2}$$

and combining Lemmas 3 and 4, we find that matrix

$$C = \begin{pmatrix} C_0 & 0 \\ 0 & I - C_0^T \end{pmatrix}, \quad C_0 \in gl(n+1) \tag{3.3}$$

establishes a 1-1 correspondence between near-zero conic Hamiltonian vector fields $z \to a(z) = J\nabla\phi(z)$ and near-identity conic symplectic maps $z \to g(z)$ via generating relation (3.2).

Write $C_0 = \begin{pmatrix} \alpha & \beta^T \\ \gamma & \delta \end{pmatrix}$ with $\alpha \in \mathbf{R}$, $\beta, \gamma \in \mathbf{R}^n$ and $\delta \in gl(n)$. Then the generating relation (3.2) with generating matrix C given by (3.3) can be expressed as

$$\begin{cases} \hat{p}_0 - p_0 = -\phi_{q_0}(\bar{p}, \bar{q}), \\ \hat{p}_1 - p_1 = -\phi_{q_1}(\bar{p}, \bar{q}), \\ \hat{q}_0 - q_0 = \phi_{p_0}(\bar{p}, \bar{q}), \\ \hat{q}_1 - q_1 = \phi_{p_1}(\bar{p}, \bar{q}), \end{cases} \tag{3.4}$$

where $\bar{p} = \begin{pmatrix} \bar{p}_0 \\ \bar{p}_1 \end{pmatrix}$ and $\bar{q} = \begin{pmatrix} \bar{q}_0 \\ \bar{q}_1 \end{pmatrix}$ are given by

$$\begin{cases} \bar{p}_0 = \alpha\hat{p}_0 + (1-\alpha)p_0 + \beta^T(\hat{p}_1 - p_1), \\ \bar{p}_1 = \delta\hat{p}_1 + (1-\delta)p_1 + \gamma(\hat{p}_0 - p_0), \\ \bar{q}_0 = (1-\alpha)\hat{q}_0 + \alpha q_0 - \gamma^T(\hat{q}_1 - q_1), \\ \bar{q}_1 = (1-\delta^T)\hat{q}_1 + \delta^T q_1 - \beta(\hat{q}_0 - q_0). \end{cases} \tag{3.5}$$

Every conic function ϕ can be contactized as an arbitrary function $\psi(x, y, z)$ as follows

$$\psi(x, y, z) = \phi(1, x, z, y). \tag{3.6}$$

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1) $sm(2n+2)$ denotes the set of all $(2n+2) \times (2n+2)$ real symmetric matrices.

i.e., $\phi(p_0, p_1, q_0, q_1) = p_0\phi(1, p_1/p_0, q_0, q_1) = p_0\psi(p_1/p_0, q_1, q_0)$ for $p_0 \neq 0$ and we have the partial derivative relation

$$\begin{cases} \phi_{q_0}(p_0, p_1, q_0, q_1) = p_0\psi_z(x, y, z), \\ \phi_{q_1}(p_0, p_1, q_0, q_1) = p_0\psi_y(x, y, z), \\ \phi_{p_0}(p_0, p_1, q_0, q_1) = \psi(x, y, z) - x^T\psi_x(x, y, z) = \psi_e(x, y, z), \\ \phi_{p_1}(p_0, p_1, q_0, q_1) = \psi_x(x, y, z) \end{cases} \tag{3.7}$$

with $x = \dfrac{p_1}{p_0}$, $y = q_1, z = q_0$ on the right hand sides. So, under contactizing transforms

$$S: \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} p_0 \\ p_1 \\ q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} w \\ wx \\ z \\ y \end{pmatrix}, \begin{pmatrix} \hat{w} \\ \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \to \begin{pmatrix} \hat{p}_0 \\ \hat{p}_1 \\ \hat{q}_0 \\ \hat{q}_1 \end{pmatrix} = \begin{pmatrix} \hat{w} \\ \hat{w}\hat{x} \\ \hat{z} \\ \hat{y} \end{pmatrix},$$
$$\begin{pmatrix} \bar{w} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \to \begin{pmatrix} \bar{p}_0 \\ \bar{p}_1 \\ \bar{q}_0 \\ \bar{q}_1 \end{pmatrix} = \begin{pmatrix} \bar{w} \\ \bar{w}\bar{x} \\ \bar{z} \\ \bar{y} \end{pmatrix}, \tag{3.8}$$

the generating relation (3.4) turns into

$$\begin{cases} \hat{w} - w = -\bar{w}\psi_z(\bar{x}, \bar{y}, \bar{z}), \\ \hat{w}\hat{x} - wx = -\bar{w}\psi_y(\bar{x}, \bar{y}, \bar{z}), \\ \hat{z} - z = \psi_e(\bar{x}, \bar{y}, \bar{z}), \\ \hat{y} - y = \psi_x(\bar{x}, \bar{y}, \bar{z}), \end{cases} \tag{3.9}$$

and Eq. (3.5) turns into

$$\begin{cases} \bar{w} = \alpha\hat{w} + (1-\alpha)w + \beta^T(\hat{w}\hat{x} - wx), \\ \bar{w}\bar{x} = \delta\hat{w}\hat{x} + (1-\delta)wx + \gamma(\hat{w} - w), \\ \bar{z} = (1-\alpha)\hat{z} + \alpha z - \gamma^T(\hat{y} - y), \\ \bar{y} = (1-\delta^T)\hat{y} + \delta^T y - \beta(\hat{z} - z). \end{cases} \tag{3.10}$$

Since the p_0-axis is distinguished for the contactization in which we should always take $p_0 \neq 0$, it is natural to require $\beta = 0$ in Eq. (3.5). Let $\hat{\mu} = w/\hat{w} = p_0/\hat{p}_0$ and $\bar{\mu} = w/\bar{w} = p_0/\bar{p}_0$, we obtain from Eq. (3.9) and (3.10)

$$\hat{\mu} = \frac{1 + \alpha\psi_z(\bar{x}, \bar{y}, \bar{z})}{1 - (1-\alpha)\psi_z(\bar{x}, \bar{y}, \bar{z})}, \quad \bar{\mu} = 1 + \alpha\psi_z(\bar{x}, \bar{y}, \bar{z}), \tag{3.11}$$

and the induced contact transformation on the contact (x, y, z) space $\mathbf{R}^{2n+1}$ is

$$\begin{cases} \hat{x} - x = -\psi_y(\bar{x}, \bar{y}, \bar{z}) + \psi_z(\bar{x}, \bar{y}, \bar{z})((1-\alpha)\hat{x} + \alpha x), \\ \hat{y} - y = \psi_x(\bar{x}, \bar{y}, \bar{z}), \\ \hat{z} - z = \psi_e(\bar{x}, \bar{y}, \bar{z}), \end{cases} \tag{3.12}$$

with the bar variables on the right hand sides given by

$$\begin{cases} \bar{x} = d_1\hat{x} + d_2 x + d_0, \\ \bar{y} = (1-\delta^T)\hat{y} + \delta^T y, \\ \bar{z} = (1-\alpha)\hat{z} + \alpha z - \gamma^T(\hat{y} - y), \end{cases} \tag{3.13}$$

where

$$d_1 = (1-(1-\alpha)\psi_z(\bar{x},\bar{y},\bar{z}))\delta, d_2 = (1+\alpha\psi_z(\bar{x},\bar{y},\bar{z}))(I-\delta), d_0 = -\psi_z(\bar{x},\bar{y},\bar{z})\gamma. \tag{3.14}$$

Summarizing the above discussions, we have

Theorem 1. *Relations* (3.12)-(3.14) *give a contact map* $(x,y,z) \to (\hat{x},\hat{y},\hat{z})$ *via contact generating function* $\psi(x,y,z)$ *under the type* $C_0 = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$.

However, the difficulty in the algorithmic implementation lies in the fact that, unlike $\bar{y}$ and $\bar{z}$, which are linear combinations of $\hat{y}$, y and $\hat{z}$, z with constant matrix coefficients, since $\bar{x} = d_1\hat{x} + d_2 x + d_0$ and d_1, d_2 are matrices with coefficients depending on $\bar{\psi}_z = \psi_z(\bar{x},\bar{y},\bar{z})$ which in turn depends on $\bar{x},\bar{y},\bar{z}$, the combination of $\bar{x}$ from $\hat{x}$ and x is not explicitly given, the entire equations for solving $\hat{x},\hat{y},\hat{z}$ in terms of x,y,z are highly implicit. The exceptional cases are the following.

(E1) $\alpha = 0$, $\delta = 0_n$, $\gamma = 0$.

$$\hat{\mu} = 1 - \psi_z(x,\hat{y},\hat{z}), \quad \bar{\mu} = 1, \tag{3.15$_1$}$$

$$\begin{cases} \hat{x} - x = -\psi_y(x,\hat{y},\hat{z}) + \hat{x}\psi_z(x,\hat{y},\hat{z}), \\ \hat{y} - y = \psi_x(x,\hat{y},\hat{z}), \\ \hat{z} - z = \psi_e(x,\hat{y},\hat{z}) = \psi(x,\hat{y},\hat{z}) - x^T\psi_x(x,\hat{y},\hat{z}). \end{cases} \tag{3.15$_2$}$$

(E2) $\alpha = 1$, $\delta = I_n$, $\gamma = 0$.

$$\hat{\mu} = \bar{\mu} = 1 + \psi_z(\hat{x},y,z), \tag{3.16$_1$}$$

$$\begin{cases} \hat{x} - x = -\psi_y(\hat{x},y,z) + x\psi_z(\hat{x},y,z), \\ \hat{y} - y = \psi_x(\hat{x},y,z), \\ \hat{z} - z = \psi_e(\hat{x},y,z) = \psi(\hat{x},y,z) - x^T\psi_x(\hat{x},y,z). \end{cases} \tag{3.16$_2$}$$

(E3) $\alpha = \frac{1}{2}$, $\delta = \frac{1}{2}I_n$, $\gamma = 0$.

$$\hat{\mu} = \frac{1+\frac{1}{2}\psi_z(\bar{x},\bar{y},\bar{z})}{1-\frac{1}{2}\psi_z(\bar{x},\bar{y},\bar{z})}, \quad \bar{\mu} = 1 + \frac{1}{2}\psi_z(\bar{x},\bar{y},\bar{z}), \tag{3.17$_1$}$$

$$\begin{cases} \hat{x} - x = -\psi_y(\bar{x},\bar{y},\bar{z}) + \psi_z(\bar{x},\bar{y},\bar{z}) \cdot \dfrac{\hat{x}+x}{2}, \\ \hat{y} - y = \psi_x(\bar{x},\bar{y},\bar{z}), \\ \hat{z} - z = \psi_e(\bar{x},\bar{y},\bar{z}) = \psi(\bar{x},\bar{y},\bar{z}) - \bar{x}^T\psi_x(\bar{x},\bar{y},\bar{z}), \end{cases} \tag{3.17$_2$}$$

with

$$\bar{x} = \frac{\hat{x}+x}{2} - \frac{1}{4}\psi_z(\bar{x},\bar{y},\bar{z})(\hat{x}-x), \quad \bar{y} = \frac{\hat{y}+y}{2}, \quad \bar{z} = \frac{\hat{z}+z}{2}. \tag{3.17$_3$}$$

For $\psi_z = \lambda =$ constant, the case (E3) reduces to

$$\hat{\mu} = \frac{1+\frac{1}{2}\lambda}{1-\frac{1}{2}\lambda}, \quad \bar{\mu} = 1 + \frac{1}{2}\lambda, \tag{3.18$_1$}$$

$$\begin{cases} \hat{x} - x = -\psi_y(\bar{x},\bar{y},\bar{z}) + \psi_z(\bar{x},\bar{y},\bar{z}) \cdot \dfrac{\hat{x}+x}{2}, \\ \hat{y} - y = \psi_x(\bar{x},\bar{y},\bar{z}), \\ \hat{z} - z = \psi_e(\bar{x},\bar{y},\bar{z}) = \psi(\bar{x},\bar{y},\bar{z}) - \bar{x}^T\psi_x(\bar{x},\bar{y},\bar{z}), \end{cases} \tag{3.18$_2$}$$

with

$$\bar{x} = \frac{\hat{x}+x}{2} - \frac{1}{4}\lambda(\hat{x}-x), \quad \bar{y} = \frac{\hat{y}+y}{2}, \quad \bar{z} = \frac{\hat{z}+z}{2}. \tag{3.18$_3$}$$

Note that the symplectic map induced by generating function ϕ from the relation (3.2) can be represented as the composition of the maps, non-symplectic generally, $z \to \bar{z}$ and $\bar{z} \to \hat{z}$

$$\bar{z} = z + CJ\nabla\phi(\bar{z}), \quad \hat{z} = \bar{z} + (I-C)J\nabla\phi(\bar{z}).$$

Theorem 2. *Contact map $(x,y,z) \to (\hat{x},\hat{y},\hat{z})$ induced by contact generating function ψ from the relations* (3.12)-(3.14) *can be represented as the composition of the maps $(x,y,z) \to (\bar{x},\bar{y},\bar{z})$ and $(\bar{x},\bar{y},\bar{z}) \to (\hat{x},\hat{y},\hat{z})$ which are not contact generally and given, respectively, as follows*

$$\begin{cases} \bar{x} - x = -\delta\psi_y(\bar{x},\bar{y},\bar{z}) + \alpha\psi_z(\bar{x},\bar{y},\bar{z})x - \gamma\psi_z(\bar{x},\bar{y},\bar{z}), \\ \bar{y} - y = (I-\delta^T)\psi_x(\bar{x},\bar{y},\bar{z}), \\ \bar{z} - z = (1-\alpha)\psi_e(\bar{x},\bar{y},\bar{z}) - \gamma^T\psi_x(\bar{x},\bar{y},\bar{z}), \end{cases} \tag{3.19}$$

and

$$\begin{cases} \hat{x} - \bar{x} = -(I-\delta)\psi_y(\bar{x},\bar{y},\bar{z}) + (1-\alpha)\psi_z(\bar{x},\bar{y},\bar{z})\hat{x} + \gamma\psi_z(\bar{x},\bar{y},\bar{z}), \\ \hat{y} - \bar{y} = \delta^T\psi_x(\bar{x},\bar{y},\bar{z}), \\ \hat{z} - \bar{z} = \alpha\psi_e(\bar{x},\bar{y},\bar{z}) + \gamma^T\psi_x(\bar{x},\bar{y},\bar{z}). \end{cases} \tag{3.20}$$

(3.19) and (3.20) are the 2-stage form of the generating relation (3.12) of the contact map induced by generating function ψ under the type $C_0 = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$. Corresponding to the exceptional cases (E1), (E2) and (E3), the above 2-stage representation has simpler forms, we no longer write them here.

4. Contact Algorithms for Contact Systems

Consider contact system (1.3) with the vector field a defined by contact Hamiltonian K according to Eq. (1.6). Take $\psi(x,y,z)=sK(x,y,z)$ in (3.12)-(3.14) as the generating function, we then obtain contact difference schemes with 1st order of accuracy of the contact system (1.3) associated with all possible types $C_0=\begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}$. The simplest and important cases are (write $\bar{K}_x=K_x(\bar{x},\bar{y},\bar{z})$, etc.) as follows.

$\widetilde{Q}$. Contact analog of symplectic method (p,Q)[1)] ($\alpha=0$, $\delta=0_n$, $\gamma=0$).

$$\begin{aligned}
\text{1-stage form:}\quad & \hat{x}=x+s(-K_y(x,\hat{y},\hat{z})+\hat{x}K_z(x,\hat{y},\hat{z}))\\
& \hat{y}=y+sK_x(x,\hat{y},\hat{z})\\
& \hat{z}=z+sK_e(x,\hat{y},\hat{z}),\\
\text{2-stage form:}\quad & \bar{x}=x,\quad \bar{y}=y+s\bar{K}_x,\quad \bar{z}=z+s\bar{K}_e,\\
& \hat{x}=\bar{x}+s(-\bar{K}_y+\hat{x}\bar{K}_z),\quad \hat{y}=\bar{y},\quad \hat{z}=\bar{z}.
\end{aligned}\tag{4.1}$$

$\widetilde{P}$. Contact analog of symplectic method (P,q) ($\alpha=1,\delta=I_n,\gamma=0$).

$$\begin{aligned}
\text{1-stage form:}\quad & \hat{x}=x+s(-K_y(\hat{x},y,z)+xK_z(\hat{x},y,z))\\
& \hat{y}=y+sK_x(\hat{x},y,z)\\
& \hat{z}=z+sK_e(\hat{x},y,z),\\
\text{2-stage form:}\quad & \bar{x}=x+s(-\bar{K}_y+x\bar{K}_z),\quad \bar{y}=y,\quad \bar{z}=z,\\
& \hat{x}=\bar{x},\quad \hat{y}=\bar{y}+s\bar{K}_x,\quad \hat{z}=\bar{z}+s\bar{K}_e.
\end{aligned}\tag{4.2}$$

$\widetilde{C}$. Contact version of centered Euler method $\Big(\alpha=\dfrac{1}{2},\delta=\dfrac{1}{2}I_n,\gamma=0\Big)$.
2-stage form:

$$\begin{aligned}
&\bar{x}=x+\frac{s}{2}(-\bar{K}_y+x\bar{K}_z),\quad \bar{y}=y+\frac{s}{2}\bar{K}_x,\quad \bar{z}=z+\frac{s}{2}\bar{K}_e,\\
&\hat{x}=\bar{x}+\frac{s}{2}(-\bar{K}_y+\hat{x}\bar{K}_x)=(\bar{x}-\frac{s}{2}\bar{K}_y)\Big(1-\frac{s}{2}\bar{K}_z\Big)^{-1},\\
&\hat{y}=\bar{y}+\frac{s}{2}\bar{K}_x=2\bar{y}-y,\quad \hat{z}=\bar{z}+\frac{s}{2}\bar{K}_e=2\bar{z}-z.
\end{aligned}\tag{4.3}$$

One might suggest, for example, the following scheme for (1.3): $\hat{x}=x+sa(\hat{x},y,z)$, $\hat{y}=y+sb(\hat{x},y,z)$, $\hat{z}=z+sc(\hat{x},y,z)$. It differs from (4.2) only in one term for $\hat{x}$, i.e., $\hat{x}K(\hat{x},y,z)$ instead of $xK(\hat{x},y,z)$. This minute but delicate difference makes (4.2) contact and other non-contact!

It should be noted that the $\widetilde{Q}$ and $\widetilde{P}$ methods are of order one of accuracy and the $\widetilde{C}$ method is of order two. The proof is similar to that for symplectic case. In principle, one can construct the contact difference schemes of arbitrarily high order of accuracy

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[1)] For Hamiltonian system $\dot{p}=-H_q(p,q)$, $\dot{q}=H_p(p,q)$, the difference scheme $\hat{p}=p-sH_q(p,\hat{q})$, $\hat{q}=q+sH_p(p,\hat{q})$ is symplectic and we call it (p,Q) method because the pair $(p,\hat{q})$, composed by the old variables of p and the new variables of q, emerges in the Hamiltonian. The following (P,q) method has the similar meaning.

for contact systems, as was done for Hamiltonian systems, by suitably composing the $\widetilde{Q}$, $\widetilde{P}$ or $\widetilde{C}$ method and the respective reversible counterpart. Another general method for the construction of contact difference schemes is based on the generating functions for phase flows of contact systems which will be developed in the next section.

5. Hamilton-Jacobi Equations for Contact Systems

We recall that a near identity contact map $g : (x, y, z) \to (\hat{x}, \hat{y}, \hat{z})$ can be generated from the so-called generating function $\psi(x, y, z)$, associated with a matrix $C_0 = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$, by the relations (3.12)-(3.14). Accordingly, to the phase flow e_K^t of a contact system with contact Hamiltonian K, there corresponds a time-dependent generating function $\psi^t(x, y, z)$ such that the map $e_K^t : (x, y, z) \to (\hat{x}, \hat{y}, \hat{z})$ is generated from ψ^t by the relations (3.12)-(3.14) in which ψ is replaced by ψ^t and C_0 is given in advance as above. The function ψ^t should be determined by K and C_0. Below we derive the relevant relations between them.

Let $H(p_0, p_1, q_0, q_1) = p_0 K(p_1/p_0, q_1, q_0)$ for $p_0 \neq 0$. With this conic Hamiltonian and with normal Darboux matrices $C = \begin{pmatrix} C_0 & 0 \\ 0 & I - C_0 \end{pmatrix}$ where $C_0 = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$, we get the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t}\phi^t(u) = H(u + (I - C)J\nabla\phi^t(u)) \quad \text{with} \quad u = (p_0, p_1, q_0, q_1)^T \tag{5.1}$$

satisfied by the generating function $\phi^t(u)$ of the phase flow g_H^t of the Hamiltonian system associated with the Hamiltonian H, while the phase flow g_H^t is generated from ϕ^t by the relation

$$g_H^t(u) - u = J\nabla\phi^t\Big(Cg_H^t(u) + (I - C)u\Big). \tag{5.2}$$

On the other hand, according to the discussions of Section 3, we have $\phi^t(p_0, p_1, q_0, q_1) = p_0\psi^t(x, y, z)$, with $x = p_1/p_0, y = q_1, z = q_0$. So, by simple calculations,

$$u + (I - C)J\nabla\phi^t(u) = \begin{pmatrix} p_0 - (1-\alpha)\phi_{q_0} \\ p_1 + \gamma\phi_{q_0} - (I - \delta)\phi_{q_1} \\ q_0 + \alpha\phi_{p_0} + \gamma^T\phi_{p_1} \\ q_1 + \delta^T\phi_{p_1} \end{pmatrix} = \begin{pmatrix} p_0(1 - (1-\alpha)\psi_z) \\ p_0(x + \gamma\psi_z - (I - \delta)\psi_y) \\ z + \alpha\psi_e + \gamma^T\psi_x \\ y + \delta^T\psi_x \end{pmatrix},$$

and

$$\begin{aligned} H(u + (I - C)J\nabla\phi^t(u)) =& p_0(1 - (1-\alpha)\psi_z) \\ & K\Big(\frac{x - (I-\delta)\psi_y + \gamma\psi_z}{1 - (1-\alpha)\psi_z}, y + \delta^T\psi_x, z + \alpha\psi_e + \gamma^T\psi_x\Big). \end{aligned}$$

Therefore, from Eq. (5.1), $\psi^t(x, y, z)$ satisfies

$$\frac{\partial}{\partial t}\psi^t = (1 - (1-\alpha)\psi_z)K\Big(\frac{x - (I-\delta)\psi_y + \gamma\psi_z}{1 - (1-\alpha)\psi_z}, y + \delta^T\psi_x, z + \alpha\psi_e + \gamma^T\psi_x\Big). \tag{5.3}$$

Now we claim[1)]: for all u, the following equality is valid

$$H(u+(I-C)J\nabla\phi^t(u)) = H(u-CJ\nabla\phi^t(u)). \tag{5.4}$$

So, replacing C by $C-I$ in above discussions or, equivalently, replacing α and δ by $\alpha-1$ and $\delta-I$ with γ unchanging in (5.3), we obtain another equation satisfied by the ψ^t

$$\frac{\partial}{\partial t}\psi^t = (1+\alpha\psi_z)K\Big(\frac{x+\delta\psi_y+\gamma\psi_z}{1+\alpha\psi_z}, y+(\delta^T-I)\psi_x, z+(\alpha-1)\psi_e+\gamma^T\psi_x\Big). \tag{5.5}$$

(5.3) and (5.5) define the same function ψ^t. When $t=0$, e_K^t is identity, so we should impose the initial condition

$$\psi^0(x,y,z)=0 \tag{5.6}$$

for solving the first order partial differential equation (5.3) or (5.5). We call the both equations the Hamilton-Jacobi equations of the contact system associated with the contact Hamiltonian K and the matrix $C_0 = \begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}$.

Specifically, we have Hamilton-Jacobi equations for particular cases:

(E1) $\alpha=0$, $\delta=0$, $\gamma=0$.

$$\frac{\partial}{\partial t}\psi^t = (1-\psi_z^t)K\Big(\frac{x-\psi_y^t}{1-\psi_z^t}, y, z\Big) = K(x, y-\psi_x^t, z-\psi_e^t); \tag{5.7}$$

(E2) $\alpha=1, \delta=I_n, \gamma=0$.

$$\frac{\partial}{\partial t}\psi^t = K(x, y+\psi_x^t, z+\psi_e^t) = (1+\psi_z^t)K\Big(\frac{x+\psi_y^t}{1+\psi_z^t}, y, z\Big); \tag{5.8}$$

(E3) $\alpha=\frac{1}{2}$, $\delta=\frac{1}{2}I_n$, $\gamma=0$.

$$\begin{aligned}\frac{\partial}{\partial t}\psi^t &= \Big(1-\frac{1}{2}\psi_z^t\Big)K\Big(\frac{x-\frac{1}{2}\psi_y^t}{1-\frac{1}{2}\psi_z^t}, y+\frac{1}{2}\psi_x^t, z+\frac{1}{2}\psi_e^t\Big)\\ &= \Big(1+\frac{1}{2}\psi_z^t\Big)K\Big(\frac{x+\frac{1}{2}\psi_y^t}{1+\frac{1}{2}\psi_z^t}, y-\frac{1}{2}\psi_x^t, z-\frac{1}{2}\psi_e^t\Big).\end{aligned} \tag{5.9}$$

Remark on the construction of high order contact difference schemes. If K is analytic, then one can solve $\psi^t(x,y,z)$ from the above Hamilton-Jacobi equations in the forms of power series in time t. Its coefficients are recursively determined by the K and the related matrix C_0. The power series are simply given from the corresponding conic Hamiltonian generating functions $\phi^t(p_0,p_1,q_0,q_1)$ by $\psi^t(x,y,z)=\phi^t(1,x,z,y)$,

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[1)]Proof of the claim: let $\hat{u}=u+(I-C)J\nabla\phi^t(u)$ and $\bar{u}=u-CJ\nabla\phi^t(u)$. Then we have $u = C\hat{u}+(I-C)\bar{u}$. From (5.2), it follows that $\hat{u}=g_H^t(\bar{u})$. The claim is then proved since $H(g_H^t(\bar{u}))=H(\bar{u})$.

since the power series expressions of ϕ^t with respect to t from the conic Hamiltonian $H(p_0, p_1, q_0, q_1) = p_0 K(p_1/p_0, q_1, q_0)$ have been well given in [7]. Taking a finite truncation of the power series up to order m, an arbitrary integer, with respect to the time t and replacing by the truncation the generating function ψ in (3.12)–(3.14), then one obtain a contact difference scheme of order m for the contact system defined by the contact Hamiltonian K. The proofs of these assertions are similar to those in the Hamiltonian system case and are omitted here.

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