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ORDER RESULTS FOR ALGEBRAICALLY STABLE MONO-IMPLICIT RUNGE-KUTTA METHODS*1)

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Abstract

It is well known that mono-implicit Runge-Kutta methods have been applied in the efficient numerical solution of initial or boundary value problems of ordinary differential equations. Burrage(1994) has shown that the order of an s-stage mono-implicit Runge-Kutta method is at most s+1 and the stage order is at most 3. In this paper, it is shown that the order of an s-stage mono-implicit Runge-Kutta method being algebraically stable is at most min$(\tilde{s}, 4)$, and the stage order together with the optimal B-convergence order is at most min(s,2), where

$$\tilde{s} = \begin{cases} s+1 & if \quad s = 1, 2, \\ s & if \quad s \geq 3. \end{cases}$$

Key words: Ordinary differential equations, Mono-implicit Runge-Kutta methods, Order, Algebraical stability.

1. Introduction

Consider the initial value problem

$$\begin{cases} y'(t) &= f(t, y(t)) \quad t \geq 0, \qquad f : [0, +\infty) \times R^N \to R^N, \\ y(0) &= y_0 \in R^N \end{cases} \tag{1.1}$$

which is assumed to have a unique solution $y(t)$ on the interval $[0, +\infty)$.

For solving (1.1), consider the s-stage implicit Runge-Kutta (IRK) method

$$\begin{cases} y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i f(t_n + c_i h, Y_i) \\ Y_i &= y_n + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_j), \qquad 1 \leq i \leq s \end{cases} \tag{1.2}$$

and the s-stage mono-implicit Runge-Kutta (MIRK) method[2,5]

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$$\begin{cases} y_{n+1} &= y_n + h\sum\limits_{i=1}^{s} b_i f(t_n + c_i h, Y_i) \\ Y_i &= (1-\nu_i)y_n + \nu_i y_{n+1} + h\sum\limits_{j=1}^{i-1} x_{ij} f(t_n + c_j h, Y_j), \quad 1\leq i\leq s \end{cases} \tag{1.3}$$

where $h\rangle 0$ is the stepsize, b_i, c_i, ν_i, x_{ij} and a_{ij} are real constants, $b_i\neq 0$, $\sum\limits_{i=1}^{s} b_i = 1, c_i\neq c_j$ when $i\neq j$, Y_i and y_n approximate $y(t_n + c_i h)$ and $y(t_n)$ respectively, $t_n = nh$ $(n\geq 0)$. The methods (1.2) and (1.3) can be given in the tableau forms respectively:

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \tag{1.4}$$

and

$$\begin{array}{c|c|c} c & \nu & X \\ \hline & & b^T \end{array} \tag{1.5}$$

where $c = (c_1, c_2, \cdots, c_s)^T, b = (b_1, b_2, \cdots, b_s)^T, \nu = (\nu_1, \nu_2, \cdots, \nu_s)^T, A = [a_{ij}]$ is an $s\times s$ matrix, $X = [x_{ij}]$ is an $s\times s$ matrix with $x_{ij} = 0$, when $i\leq j$. The method (1.5) is equivalent to the IRK method (1.4) with the coefficient matrix $A = X + \nu b^T$. The method (1.4) is said to be algebraically stable[4,7], if the matrixes $M = BA + A^T B - bb^T$ and $B = diag(b)$ are nonnegative definite.

A number of interesting subclasses of the IRK methods have recently been identified and investigated in the references. These methods represent attempts to trade-off the higher accuracy of the IRK methods for methods which can be implemented more efficiently. These methods include singly-implicit Runge-kutta (SIRK) methods[1,6,7], diagonally implicit Runge-Kutta (DIRK) methods[1,6,7],and MIRK methods[2,5]. Burrage[5] has shown that the order of an s-stage MIRK method is at most s+1 and the stage order is at most 3. In this paper, it is shown that the order of an s-stage MIRK method being algebraically stable is at most min($\tilde{s}$,4) and the stage order together with the optimal B-convergence order is at most min(s,2), here and in the following sections,

$$\tilde{s} = \begin{cases} s+1 & if \quad s = 1, 2, \\ s & if \quad s\geq 3. \end{cases}$$

2. Main Results and Proofs

For the method (1.4) or (1.5), we introduce the simplifying conditions[1,7]:

$$\begin{array}{lll} B(p): & b^T c^{k-1} = \frac{1}{k}, & k = 1, 2, \cdots, p \\ C(p): & Ac^{k-1} = \frac{c^k}{k}, & k = 1, 2, \cdots, p \\ D(p): & b^T C^{k-1} A = \frac{b^T - b^T C^k}{k}, & k = 1, 2, \cdots, p \end{array}$$

where $c^k = (c_1^k, c_2^k, \cdots, c_s^k)^T, C^k = diag(c^k)$. max$\{p : B(p)$ and $C(p)$ hold at the same time$\}$ is said to be the stage order of the method (1.4). Since the MIRK method (1.5) can be expressed in the standard IRK form (1.4) with $A = X + \nu b^T$, the conditions $C(p)$ and $D(p)$ have the following forms when $B(p)$ holds:

$$C(p): \quad \nu + kXc^{k-1} = c^k, \qquad k = 1, 2, \cdots, p$$
$$D(p): \quad b^T C^{k-1} X = \frac{b^T - b^T C^k}{k} - \Delta(k) b^T, \quad k = 1, 2, \cdots, p$$

where

$$\Delta(k) = b^T C^{k-1} \nu = \sum_{i=1}^{s} b_i \nu_i c_i^{k-1}.$$

To investigate the order of algebraically stable MIRK methods, we introduce the following Lemmas:

Lemma 2.1[1]. *$C(\eta), D(\xi), B(p)$, where $p \le \xi + \eta + 1, p \le 2\eta + 2$ implies that the method (1.4) is of order p.*

Lemma 2.2[6]. *If the method (1.4) of order $p(\ge 3)$ is algebraically stable with positive weights $b_i (1 \le i \le s)$, then $C([\frac{p-1}{2}]), B(p)$, and $D([\frac{p-1}{2}])$ must hold.*

Lemma 2.3[5]. *(1) The method (1.5) having stage order 2 must have $\nu_1 = c_1$ and either $c_1 = 0$ or $c_1 = 1$; (2) the method (1.5) having stage order 3 must have $x_{21} = 0, \nu_1 = c_1, \nu_2 = c_2$ and either $c_1 = 0, c_2 = 1$ or(equivalently) $c_1 = 1, c_2 = 0$; (3) the maximum satge order of the method (1.5) is min(s,3); (4) the maximum order of the method (1.5) cannot exceed s+1.*

Lemma 2.4[10]. *The method (1.4) with stage order $w \ge 1$ is algebraically stable if and only if (1) the conditions $B(2w-1)$ and $D(w-1)$ hold and the matrix B is positive definite; (2) the matrix H is nonnegative definite, where H is the $(s-w+1) \times (s-w+1)$ matrix whose (i,j) element is $\psi(i,j) + \psi(j,i) - \bar{B}(i)\bar{B}(j), i, j = w, w+1, \cdots, s$. where*

$$\psi(i,j) = ij \sum_{k,l=1}^{s} b_k c_k^{i-1} a_{kl} c_l^{j-1}, \qquad \bar{B}(i) = i \sum_{k=1}^{s} b_k c_k^{i-1}.$$

Lemma 2.5. *(i) The method (1.5) being algebraically stable must have $\nu_i \ge 1/2 (1 \le i \le s)$; (ii) the method (1.5) must satisfy $c_s \le 1/2$ if it is algebraically stable and satisfies the condition D(1); (iii) the method (1.5) with $s \ge 2$ can not satisfy the condition D(2).*

Proofs. For (i), the conclusion can be directly obtained from the definition of algebraic stability. For (ii), because the condition D(1) leads to

$$(1 - c_s - \sum_{i=1}^{s} b_i \nu_i) b_s = 0,$$

the conclusion follows from (i) together with $\sum_{i=1}^{s} b_i = 1, b_i > 0, 1 \le i \le s$. For (iii), if the method (1.5) with $s \ge 2$ satisfies the condition D(2), then

$$b_s(1 - c_s - \Delta(1)) = 0, \quad b_s x_{s,s-1} = b_{s-1}(1 - c_{s-1} - \Delta(1)), \tag{2.1a}$$

$$b_s(\frac{1-c_s^2}{2} - \Delta(2)) = 0, \quad b_s c_s x_{s,s-1} = b_{s-1}(\frac{1-c_{s-1}^2}{2} - \Delta(2)). \tag{2.1b}$$

Because of $b_{s-1}, b_s \neq 0$, (2.1a) and (2.1b) lead to $c_s = c_{s-1}$. But this is impossible for the method (1.5).

Theorem 2.1. (*i*) *The unique one-stage MIRK method being algebraically stable and having order 2 and stage order 1 is the midpoint rule*

$$\begin{array}{c|c|c} 1/2 & 1/2 & 0 \\ \hline & & 1 \end{array} = \begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array} \tag{2.2a}$$

(*ii*) *The unique two-stage MIRK method being algebraically stable and having order 3 and stage order 2 is the two-stage Radau IIA method*

$$\begin{array}{c|c|cc} 1 & 1 & 0 & 0 \\ 1/3 & 5/9 & -2/9 & 0 \\ \hline & & 1/4 & 3/4 \end{array} = \begin{array}{c|cc} 1 & 1/4 & 3/4 \\ 1/3 & -1/12 & 5/12 \\ \hline & 1/4 & 3/4 \end{array} \tag{2.2b}$$

Theorem 2.1 can be directly identified. From Lemma 2.3 and Theorem 2.1 together with (i) in Lemma 2.5, we easily obtain

Theorem 2.2. *The stage order of the method (1.5) being algebraically stable is at most min(s,2).*

Theorem 2.3. *The order of the method (1.5) being algebraically stable is at most min($\tilde{s}$,4).*

Proof. When s=1,2, Theorem 2.3 obviously holds from Lemma 2.3 and Theorem 2.1.

When s=3, Theorem 2.3 obviously holds from Lemma 2.1 and (iii) in Lemma 2.5 for the method (1.5) with stage order 1. Now we assume the method (1.5) is of stage order 2 and of order 4, in accordance with Lemma 2.2, 2.3 and 2.5, this method can be characterized by an one-parameter family of methods (where the parameter is a real number c_2, satisfying $c_2 \neq 1, 1/4, 1/3$)

$$\begin{array}{c|c|ccc} 1 & 1 & 0 & 0 & 0 \\ c_2 & c_2(2-c_2) & c_2(c_2-1) & 0 & 0 \\ \frac{2c_2-1}{6c_2-2} & \frac{\delta}{\gamma} & \frac{(1-2c_2)(1-4c_2)\beta}{2(c_2-1)\gamma} & \frac{(4c_2-1)\alpha}{2(c_2-1)\gamma} & 0 \\ \hline & & \frac{6c_2^2-6c_2+1)}{6(4c_2-1)(c_2-1)} & \frac{-1}{6(c_2-1)\alpha} & \frac{2(3c_2-1)^3}{3(4c_2-1)\alpha} \end{array} \tag{2.3}$$

where

$$\delta = 180c_2^4 - 240c_2^3 + 121c_2^2 - 26c_2 + 2, \qquad \gamma = 4(3c_2-1)^4,$$

$$\beta = 1 - 10c_2 + 24c_2^2 - 18c_2^3, \qquad \alpha = 6c_2^2 - 4c_2 + 1, \qquad \frac{2}{3} \le c_2 \le \frac{3+\sqrt{3}}{6},$$

$$2\psi(3,3) - 1 = -108c_2^4 - 240c_2^3 + 306c_2^2 - 105c_2 + 11 < 0. \tag{2.4}$$

But from Lemma 2.4 we must have $2\psi(3,3)-1\geq 0$. This is in contradicition with (2.4).

When $s\geq 4$, Theorem 2.3 follows directly from Theorem 2.2 and Lemma 2.1 together with (iii) in Lemma 2.5.

For reason given above, we have proved Theorem 2.3.

3. B-convergence Order

Frank, Schneid and Ueberhuber[8] showed algebraically and diagonally stable IRK methods with stage order q have optimal B-convergence order q. Burrage and Hundsdorfer[3] and the author of this paper[9] estabilished the conditions making an IRK method with optimal B-convergence order one higher than stage order for nonlinear stiff problems. From Theorem 2.2, we have known the stage order of the MIRK method (1.5) being algebraically stable is at most min(s,2). We cannot help asking whether the optimal B-convergence order of this method can reach 3. In fact, suppose that the method (1.5) being algebraically stable is of optimal B-convergence order 3 and of stage order 2. Then in accordance with the conditions estabilished in [3,9], we have

$$\nu + 3Xc^2 - c^3 = \lambda e,$$

where $e=(1,1,\cdots,1)^T\in R^s, \lambda$ is a constant. In view of Lemma 2.3 and 2.5 together with the above formula, we have $\nu_1 = c_1 = 1$ such that $\lambda = 0$, thus the condition C(3) holds. This contradict Theorem 2.2. Therefore, the optimal B-convergence order of the algebraically stable method (1.5) with stage order 2 is at most 2.

4. Conclusions

In this paper, we have extended the knowledge of the class of MIRK methods. This paper shows that the order of an s-stage MIRK method being algebraically stable is at most min($\tilde{s}$,4) and the stage order together with the optimal B-convergence order is at most min(s,2). Some characterizations of an s-stage MIRK method being algebraically stable are given. These results of this paper will be useful in an analysis for determination of new methods for solving stiff problems. Future work in this area could include a systematic construction of the various families of s-stage MIRK methods being algebraically stable and having optimal stage order 2 and optimal order s for s=3,4, and how to modify slightly a MIRK method to increase its stage order and its order without loss of its advantage.

References

[1] J.C. Butcher, The Numerical Analysis of Ordinary Differential Equations, John Wiley and Sons, Toronto, 1987.

[2] J.R. Cash, A. Singhal, Mono-implicit Runge-Kutta formulae for the numerical integration of stiff differential systems, *IMA J. Numer. Anal.,* **2** (1982), 211-227.

[3] K. Burrage, W.H. Hundsdorfer, The order of B-convergence of algebraically stable Runge-Kutta methods, *BIT,* **27** (1987), 62-71.

[4] K. Burrage, J.C. Butcher, Stability criteria for implicit Runge-Kutta methods, *SIAM J. Numer. Anal.,* **16** (1979), 46-57.

[5] K. Burrage, F.H. Chipman and P.H. Muir, Order results for mono-implicit Runge-Kutta methods, *SIAM J. Numer. Anal.,* **3** (1994), 876-891.

[6] K. Burrage, Efficiently implementable algebraically stable Runge-Kutta methods, *SIAM J. Numer. Anal.,* **2** (1982), 245-258.

[7] K. Dekker and J.G. Verwer, Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations, North-Holland, Amsterdam, 1984.

[8] R. Frank, J.Schneid and C.W. Ueberhuber, Order results for implicit Runge-Kutta methods applied to stiff systems, *SIAM J. Numer. Anal.,* **22** (1985), 515-534.

[9] Ai-guo Xiao, On the order of B-convergence of Runge-Kutta methods, *Natural Science Joural of Xiangtan University,* **2** (1992), 16-19.

[10] Ai-guo Xiao, $(\vec{\theta}, p, q)$-Algebric stability of Runge-Kutta methods, *Mathematica Numerica Sinica,* **4** (1993), 440-448.