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ON RAYLEIGH QUOTIENT MATRICES: THEORY AND APPLICATIONS*

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Abstract

Many authors have studied the Rayleigh quotient and Rayleigh quotient matrix. This paper consists of two parts. First, generalizations of some results on the Rayleigh quotient are proved. Second, we give some applications of these theoretical results.

Key words: Rayleigh quotient Matrix, Eigenvalue, Approximation.

1. Introduction

Throughout this paper we shall use the following notation. $R^{m\times n}$ and $C^{m\times n}$ denote the sets of real and complex $m\times n$ matrices, respectively, R^n and C^n denote the sets of real and complex n-dimentional column vectors, respectively. The superscript H means the conjugate transpose of matrix. I_n is the $n\times n$ identity matrix, and 0 is the null matrix. $R(A)$ stands for the column space of a matrix A; $\lambda(A)$ denotes the set of the eigenvalues of matrix A. $\lambda(A,B)$ denotes the set of the generalized eigenvalues of a regular matrix-pair $\{A,B\}$. $\sigma(A)$ the set of the singular values of matrix A.$\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalue of Hermitian matrix A, respectively. $\sigma_{\min}(A)$ is the smallest singular value of matrix A. $\|\ \|$ refers to a uniformly generalized, unitarily invariant norm for matrices. $\|\ \|_2$ denotes the Euclidean norm for vectors and spectral norm for matrices, respectively. $\|\ \|_F$ is the Frobenius norm. For X_1, $Y_1\in C^{m\times p}$ with $X_1^HX_1=Y_1^HY_1=I_p$, the matrix $\theta(R(X_1),R(Y_1))$ is defined by

$$\theta(R(X_1),R(Y_1))=\arccos(X_1^HY_1Y_1^HX_1)^{1/2}\geq 0$$

Let $A\in C^{n\times n}$ be a Hermitian matrix, and $Y_1\in C^{n\times p}$ satisfy $Y_1^HY_1=I_p$. Then the matrix $H_1=Y_1^HAY_1$ is called the Rayleigh quotient matrix of A with respect to Y_1. If $p=1$, then $y_1^HAy_1$ is called the Rayleigh quotient of A respect to y_1.

First of all we cite some important results on the Rayleigh quotient. let A be $n\times n$ Hermitian matrix, and $\lambda(A)=\{\lambda_j\}_{j=1}^n$, moreover, let $y_1\in C^n$ with $\|y_1\|_2=1$, and let

$$Ax_1=\lambda_1X_1,\|X_1\|_2=1,\quad X_1\in C^n$$
$$\mu_1=y_1^HAy_1,r=Ay_1-y_1\mu_1$$

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$$\theta = \arccos|y_1^H X_1|, \quad 0 \le \theta \le \pi/2$$
$$\delta = \min_{2\le j\le n} |\lambda_j - \mu_1|, \quad \Delta = \max |\lambda_j - \mu_1|$$
$$d = \min_{2\le j\le n} |\lambda_j - \lambda_1|, \quad D = \max_{2\le j\le n} |\lambda_j - \lambda_1|, \quad 2 \le j \le n.$$

Some elementary results are given in the following theorem, which delineates the most important relations between $\sin\theta$, $\|r\|_2$ and $\lambda_1 - \mu_1$.

Theorem[11]**.**

$$\sin\theta \le \|r\|_2/\delta \quad (\text{if } \delta > 0) \tag{1.1}$$
$$\|r\|_2 \le \frac{\Delta \sin\theta}{\sqrt{1-\sin^2\theta}} \quad (\text{if } \sin\theta < 1) \tag{1.2}$$
$$|\lambda_1 - \mu_1| \le \|r\|_2^2/\delta \quad (\text{if} \delta > 0, |\lambda_1 - \mu_1| < |\lambda_j - \mu_1|), \tag{1.3}$$
$$|\lambda_1 - \mu_1| \le D\sin^2\theta \tag{1.4}$$
$$|\lambda_1 - \mu_1| \le \frac{\|r\|_2 \sin\theta}{\sqrt{1-\sin^2\theta}} \quad (\text{if} \sin\theta < 1) \tag{1.5}$$

The inequalities (1.1)–(1.5) *have been extended to the case $p > 1$ by Sun*[10,11]*, Li*[5]*, Liu & Xu*[6]*, and Liu*[7]*. In this paper, we shall give some further generalizations of the inequalities* (1.1)–(1.5) *and applications of these theoretical results.*

2. Generalizations of the Rayleigh Quotient Matrix Theory

In this section, some extentions of the inequalities (1.1)–(1.5) are given. We shall study the eigenproblem, generalized eigenvalue problem and singular value problem.

2.1. Eigenproblem: $p = 1$

Let $A \in C^{n\times n}$, $y_1 \in C^n$ with $\|y_1\|_2 = 1$, and let

$$\mu_1 = y_1^H A y_1, \quad r = Ay_1 - y_1 u_1, \quad r_0 = A^H y_1 - y_1\bar{\mu}_1$$

Let the Schur decomposition of A be

$$A = Q\begin{pmatrix} \lambda_1 & a^H \\ 0 & A_1 \end{pmatrix} Q^H, \ Q = [q_1, Q_1], \ Q^H Q = 1_n$$

Denote

$$\delta = \text{sep}(\mu_1, A_1), \theta = \arccos|y_1^H q_1|, \quad 0 \le \theta \le \pi/2$$
$$\begin{pmatrix} P \\ S \end{pmatrix} = Q^H y_1, \ \Delta = \|A_1 - \mu_1 I_{n-1}\|_2, \quad D = \|A_1 - \lambda_1 I_{n-1}\|_2$$

Theorem 1.

$$(1) \sin\theta \le \|r\|_2/\delta, \quad (\text{if } \delta > 0) \tag{2.1}$$
$$(2) \|r\|_2 \le \sqrt{D^2 + \|a\|_2^2}\sin\theta. \tag{2.2}$$

$$(3)|\lambda_1 - \mu_1| \le \|a\|_2 \sin\theta + D\sin^2\theta. \tag{2.3}$$

$$(4)|\lambda_1 - \mu_1| \le \frac{\sin\theta\|r_0\|_2}{\sqrt{1-\sin^2\theta}} \text{ if } \sin\theta < 1), \tag{2.4}$$

Proof. (2.1) is a consequence of Stewart's result[14]. From $y_1^H r = 0$, we have

$$(\lambda_1 - \mu_1)\|p\|^2 + \bar{p}a^H s + s^H(A_1 - \mu_1 I_{n-1})_s = 0. \tag{2.5}$$

$$\begin{aligned}\|r\|_2^2 &= \|(A_1-\mu_1)S\|_2^2 + |a^H S|^2 + \overline{a^H S}(\lambda_1-\mu_1)p + \overline{(\lambda_1-\mu_1)}[\bar{p}a^H s + (\lambda_1 - u_1)|p|^2] \\ &= (\overline{a^H s}, s^H(A_1 - \lambda_1 I_{n-1})^H)\begin{pmatrix}(\lambda_1-\mu_1)p + a^H S \\ (A_1 - \mu_1 I_{n-1})S\end{pmatrix} \le \sqrt{\|a\|_2^2 + D^2}\|s\|_2\|r\|_2\end{aligned}$$

Noticing that $\|s\|_2 = \sin\theta$, we obtain (2.2).

As

$$\mu_1 - \lambda_1 = y_1^H A y_1 - \lambda_1 = y_1^H(A-\lambda_1)y_1 = \bar{p}a^H S + S^H(A_1 - \lambda_1 I_{n-1})S$$

we have

$$|\mu_1 - \lambda_1| \le \|a\|_2\|S\|_2 + D\|S\|_2^2.$$

Finally, from (2.5) and $y_1^H r = 0$, we have

$$(\lambda_1 - \mu_1)|p|^2 = \bar{p}a^H S - S^H(A_1 - \mu_1 I_{n-1})S$$

therefore

$$|\lambda_1 - \mu_1||p|^2 \le \|\bar{p}a^H + S^H(A_1 - \mu_1 I_{n-1})\|_2\|S\|_2$$

Pay attention to

$$\begin{aligned}&\|r_0\|_2^2 = |\lambda_1 - \mu_1|^2|p|^2 + \|pa + (A_1 - \mu_1 I_{n-1})^H S\|_2^2 \\ &\|p\|_2^2 + \|s\|_2^2 = 1\end{aligned}$$

we obtain

$$(|\lambda_1 - \mu_1||p|^2)^2 \le \|s\|_2^2(\|r_0\|_2^2 - |\lambda_1 - \mu_1|^2|p|^2)$$

and from this we prove (2.4) directly.

Remark. If the normality departure of A with respect to norm $\|\ \|$ is denoted by $\Delta(A)$[12], then

$$\|a\|_2 \le \Delta(A).$$

If A is an normal matrix, then $A_1 = \text{diag}(\lambda_2, \cdots, \lambda_n)$, and $\|r_0\| = \|r\|_2$, so we have

Corollary 1. *Let A be a normal matrix, then*

$$(1). \sin\theta \le \|r\|_2/\delta, \quad (\textit{if } \delta > 0) \tag{2.1'}$$

$$(2). \|r\|_2 \le D\sin\theta. \tag{2.2'}$$

$$(3). |\lambda_1 - \mu_1| \le D\sin^2\theta. \tag{2.3'}$$

$$|\lambda_1 - \mu_1| \le \frac{\sin\theta\|r\|_2}{\sqrt{1-\sin^2\theta}} \ (\textit{if} \sin\theta < 1) \tag{2.4'}$$

We note that (2.1)′–(2.4)′ are generalizations of the inequalities (1.1), (1.2), (1.4) and (1.5).

2.2. Generalized Eigenvalue Problem

Let $A, B \in C^{n\times n}$ be Hermitian matrices. Suppose $\{A, B\}$ is a definite matrix-pair[12], i.e.

$$c(A,B) \equiv \min_{\|x\|_2=1} |x^H(A+iB)x| > 0$$

First of all we cite some lemmas.

Lemma 2.1.[12] *Let*

$$A_\varphi = A\cos\varphi - B\sin\varphi, \quad B_\varphi = A\sin\varphi + B\cos\phi$$

Then there exists $\phi \in [0, 2\pi)$ such that B_ϕ is positive definite, and

$$c(A,B) = \lambda_{\min}(B_\varphi).$$

Let

$$\lambda(A,B) = \{(\alpha_i,\beta_i)|\alpha_i^2+\beta_i^2 = 1,\ i = 1,\cdots,n\}$$
$$\lambda(A_\phi,B_\phi) = \{(\hat\alpha_i,\hat\beta_i)|\hat\alpha_i^2+\hat\beta_i^2 = 1,\ i = 1,\cdots,n\}$$

Lemma 2.2.

$$\begin{pmatrix}\hat\alpha_i\\ \hat\beta_i\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}, \quad i = 1,2,\cdots,n$$

Definition 2.1. *Let $\hat Q \in C^{n\times p}$ satisfy $\hat Q^H\hat Q = I_p$. The matrix-pair $\{\hat Q^HA\hat Q, \hat Q^HB\hat Q\}$ is called the Rayleigh quotient matrix-pair of $\{A, B\}$ with respect to $\hat Q$.*

Let

$$M = (\hat Q^HB_\varphi\hat Q)^{-1}(\hat Q^HA_\varphi\hat Q), \quad R_0 = A_\varphi\hat Q - B_\varphi\hat QM$$
$$\dot R = B_\varphi^{-1/2}R_0(\hat Q^HB_\varphi\hat Q)^{-1/2}, \quad A_0 = B_\varphi^{-1/2}A_\varphi B_\varphi^{-1/2}$$
$$Q_0 = B_\varphi^{1/2}\hat Q(\hat Q^HB_\varphi\hat Q)^{-1/2}$$
$$M_0 = (\hat Q^HB_\varphi\hat Q)^{-1}(\hat Q^HA_\varphi\hat Q)(\hat Q^HB_\varphi\hat Q)^{-1/2}$$

Then we can easily prove following facts

(1) $Q_0^HQ_0 = I_p$

(2) *M_0 is the Rayleigh quotient matrix of A_0 with respect to Q_0.*

(3) *For any unitarily invariant norm $\|\ \|$*

$$\|B_\varphi^{-1/2}R_0\| = \min_{H\in C^{p\times p}} \|B_\varphi^{-1/2}(A_\varphi\hat Q - B_\varphi\hat QH)\|$$

Lemma 2.3[12]**.** *There exists $Q = [\hat Q_1, \hat Q_2] \in C^{n\times n}$ such that*

$$Q^HB_\varphi Q = I_n,\ Q^HA_\varphi Q = \begin{pmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{pmatrix}$$

$$\Lambda_1 = diag\ (\lambda_1, \cdots, \lambda_p),\ \Lambda_2 = diag\ (\lambda_{p+1}, \cdots, \lambda_n)$$

Let

$$\begin{aligned}
&\lambda(M_0) = \{\mu_j\}_1^p, \quad \mu_1 \geq \cdots \geq \mu_p \\
&\lambda_1 \geq \cdots \geq \lambda_p,\ \Omega_1 = diag\ (\mu_1, \cdots, \mu_p) \\
&\delta = \min |y_1^H X_1|, \quad \Delta = \max |\lambda_j - \mu_i| \\
&D = \max |\lambda_j - \lambda_i|, \quad 1 + p \leq j \leq n; \quad 1 \leq i \leq p. \\
&k = \sqrt{\lambda_{\max}(B_\varphi)/\lambda_{\min}(B_\varphi)}, \\
&\sin\theta = \| \sin\theta(R(\hat{Q}), R(\hat{Q}_1)) \|_2
\end{aligned}$$

Following the approach described in [11], we can prove the next theorem.

Theorem 2.

(1) $\sin\theta \leq k\|R\|_F / c(A, B)\delta,\ \ (if\ \delta > 0)$

(2) $\|R\|_2 \leq \Delta \sin\theta / c(A, B)\sqrt{1 - \sin^2\theta}$

(3) *for any unitarily invariant norm* $\|\ \|$

$$\|\Lambda_1 - \Omega_1\| \leq \sin\theta \|R_0\| / c(A, B)\sqrt{1 - \sin^2\theta}.$$

Remark. Consider the chordal metric defined on Gauss plane $G_{1,2}$ [12]

$$\dot{\rho}((\alpha, \beta), (\gamma, \delta)) = |\alpha\delta - \beta\gamma|$$

and pay attention to

$$\rho((\hat{\alpha}, \hat{\beta}), (\hat{\gamma}, \hat{\delta})) = \rho((\alpha, \beta), (\gamma, \delta))$$

here

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$
$$\begin{pmatrix} \hat{\gamma} \\ \hat{\delta} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

and

$$\rho((\alpha, \beta), (\gamma, \delta)) \leq |\alpha/\beta - \gamma/\delta| \quad (\text{if } |\beta|, |\delta| > 0)$$

We can give results expressed by chordal metric, weaker than theorem 2, but without assuming B to be positive definite.

2.3. Singular Value Problem

Let $A \in C^{m\times n}$ have the singular value decomposition

$$A = U\Sigma V^H$$

without loss of generality, we suppose $m \geq n$ so that

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \\ 0 & 0 \end{pmatrix}, \quad \Sigma_1 = \text{diag}\ (\sigma_1, \cdots, \sigma_p)$$

$$\Sigma_2 = \text{diag } (\sigma_{p+1}, \cdots, \sigma_n), \quad \sigma_1 \geq \cdots \geq \sigma_p$$
$$U = [U_1, U_2], V = [V_1, V_2],\ U_1 \in C^{m\times p},\ V_1 \in C^{n\times p}$$

Definition 2.2. *Let $\hat{U}_1 \in C^{m\times p}$ and $\hat{V}_1 \in C^{n\times p}$ satisfy*

$$\hat{U}_1^H \hat{U}_1 = I_p, \quad \hat{V}_1^H \hat{V}_1 = I_p$$

Then the matrix $S = \hat{U}_1^H A \hat{V}_1$ is called the singular quotient matrix of A with respect to $\hat{U}_1$ and $\hat{V}_1$.

Let

$$R_1 = AV_1 - U_1 S,\ R_2 = A^H U_1 - V_1 S^H$$
$$\sigma(S) = \{\hat{\sigma}_1, \cdots, \hat{\sigma}_p\}, \quad \hat{\sigma}_1 \geq \cdots \geq \hat{\sigma}_p$$
$$\Omega_1 = diag\ (\hat{\sigma}_1, \cdots, \hat{\sigma}_p)$$
$$\theta_1 = \arccos(\hat{U}_1^H U_1 U_1^H \hat{U}_1)^{1/2}$$
$$\theta_2 = \arccos(\hat{V}_1^H V_1 V_1^H \hat{V}_1)^{1/2}$$
$$\Delta = \max|\sigma_j - \hat{\sigma}_i|, \quad \delta = \min|\sigma_j - \hat{\sigma}_i|$$
$$D = \max|\sigma_j - \sigma_i|; \quad 1 + p \leq j \leq n; \quad 1 \leq i \leq p$$

we notice that

$$\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^H & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{U}_1 \\ \hat{V}_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \hat{U}_1 \\ \hat{V}_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & S^H \\ S & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & \hat{V}_1^H \\ \hat{U}_1^H & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ A^H & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{U}_1 \\ \hat{V}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & S^H \\ S & 0 \end{pmatrix}.$$

Let

$$Q = \begin{pmatrix} 0 & \hat{U}_1 \\ \hat{V}_1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & S^H \\ S & 0 \end{pmatrix},$$

then H is the Rayleigh quotient matrix of $\begin{pmatrix} 0 & A \\ A^H & 0 \end{pmatrix}$ with respect to Q. From the results in [11], we directly obtain following theorem.

Theorem 3.

(1). *For any unitarily invariant norm $\| \ \|$*

$$\left\| \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \right\| = \min_{T, H_0 \in C^{p\times p}} \left\| \begin{pmatrix} A\hat{V}_1 - \hat{U}_1 T \\ A^H \hat{U}_1 - \hat{V}_1 H_0 \end{pmatrix} \right\|$$

(2) *If $\delta > 0$ then for any unitarily invariant norm $\| \ \|$*

$$\left\| \begin{pmatrix} \sin\theta_1 & 0 \\ 0 & \sin\theta_2 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \right\| / \delta$$

(3). *If $\sin\theta \equiv \max\{\|\sin\theta_1\|_2, \|\sin\theta_2\|_2\} < 1$, then*

$$\max\{\|R_1\|_2, \|R_2\|_2\} \leq \Delta \sin\theta / \sqrt{1 - \sin^2\theta}$$

$$\sqrt{\|R_1\|_F^2 + \|R_2\|_F^2} \le \frac{\Delta\sqrt{\|\sin\theta_1\|_F^2 + \|\sin\theta_2\|_F^2}}{\sqrt{1-\sin^2\theta}}$$

for any unitarily invariant norm $\|\ \|$

$$\left\| \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \right\| \le \frac{\Delta \left\| \begin{pmatrix} \sin\theta_1 & 0 \\ 0 & \sin\theta_2 \end{pmatrix} \right\|}{\sqrt{1-\sin\theta}}$$

(4) *If* $\sin\theta < 1$, *then for any unitarily invariant norm* $\|\ \|$

$$\left\| \begin{pmatrix} \Lambda_1 - \Omega_1 & 0 \\ 0 & \Lambda_1 - \Omega_1 \end{pmatrix} \right\| \le \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \left\| \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \right\|$$

Corollary 2. *If* $\rho \equiv \max\{\|R_1\|_2, \|R_2\|_2\}/\delta < 1$, *then for any unitarily invariant norm* $\|\ \|$.

$$\left\| \begin{pmatrix} \Lambda_1 - \Omega_1 & 0 \\ 0 & \Lambda_{-1} - \Omega_1 \end{pmatrix} \right\| \le \frac{\rho}{\sqrt{1-\rho^2}} \left\| \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \right\|$$

3. Applications

In this section, we deal with some applications of the theoetical results proved in the last section.

3.1 Deflation to Computing eigenvalue

In this subsection, we describe an approach to analyse the influence of deflation on the accuracy of computed eigenvalues.

Let $A \in R^{n\times n}$ be a symmetric matrix. The QL algorithm with suitable shifts (QL-s for short) for computing all eigenvalues of A states as following

Step 1. Construct an $n \times n$ orthogonal matrix Q such that $A_0 \equiv Q^T AQ$ is a symmetric tridiagonal matrix.

Step 2. By QL-s calculations form tridiagonal matrices

$$A_k = \begin{pmatrix} \alpha_1^{(k)} & \beta_1^{(k)} & & \\ \beta_1^{(k)} & \ddots & & \\ \ddots & \ddots & \beta_{n-1}^{(k)} \\ & \beta_{n-1}^{(k)} & \alpha_n^{(k)} \end{pmatrix} \quad k = 0, 1, \cdots$$

Step 3. Deflation. For suitable small positive number ε, if $|\beta_1^{(k)}| < \varepsilon$, then we accept $\alpha_1^{(k)}$ as an approximate eigenvalue of A, and the algorithm goes on with lower order matrix

$$A_k = \begin{pmatrix} \alpha_2^{(k)} & \beta_2^{(k)} & & \\ \beta_2^{(k)} & \ddots & & \\ \ddots & \ddots & \beta_{n-1}^{(k)} \\ & \beta_{n-1}^{(k)} & \alpha_n^{(k)} \end{pmatrix}$$

and take the eigenvalues $\mu_2, \cdots, \mu_n$ of $A_k^{(1)}$ as approximate eigenvalues of A.

About above-mentioned algorithm, a natural question is: how to estimate the accuracy of $\alpha_1^{(k)}, \mu_2, \cdots, \mu_n$ as approximate eigenvalues.

Let

$$I_n = [e_1, \cdots, e_n], \quad Q_n \equiv [e_2, \cdots, e_n]$$

Obviously, $\alpha_1^{(k)}$ and $A_k^{(1)}$ are the Rayleigh quotient and Rayleigh quotient matrix of A_k with respect to e_1 and Q_1, respectively. Let

$$\lambda(A) = \{\lambda_j\}_{j-1}^n$$

without loss of generality we suppose

$$\begin{aligned}
&\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n; \quad \mu_2 \geq \mu_3 \geq \cdots \geq \mu_n \\
&\delta \equiv \min |\lambda_j - \lambda_1| > |\beta_1^{(k)}| \\
&|\lambda_1 - \alpha_1^{(k)}| = \min_{1 \leq i \leq n} |\lambda_i - \alpha_1^{(k)}|
\end{aligned}$$

Theorem 4. *If* $\rho_1 \equiv |\beta_1^{(k)}|/(\delta - |\beta_1^{(k)}|) < 1$ *then*

$$\begin{aligned}
&|\alpha_1^{(k)} - \lambda_1| \leq \rho_1 |\beta_1^{(k)}| \\
&|\lambda_j - \mu_j| \leq \frac{\rho_1 |\beta_1^{(k)}|}{\sqrt{1-\rho_1^2}}, \quad j = 2, \cdots, n
\end{aligned}$$

Proof. From

$$\begin{aligned}
&\beta_1^{(k)} e_2 = A_k e_1 - e_1 \alpha_1^{(k)}, \\
&[\beta_1^{(k)} e_1, 0] = A_k Q_1 - Q_1 A_k^{(1)},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\|A_k e_1 - e_1 \alpha_1^{(k)}\|_2 = |\beta_1^{(k)}|, \\
&\|A_k Q_1 - Q_1 A_k^{(1)}\|_F = |\beta_1^{(k)}|.
\end{aligned}$$

By Bauer-Fike theorem[12] and (1.3), we have

$$|\lambda_1 - \alpha_1^{(k)}| \leq \rho_1 \|\beta_1^{(k)}|$$

On the other hand, as $\rho_1 < 1$, for $j \geq 2$ we have

$$|\lambda_j - \alpha_1^{(k)}| \geq |\lambda_j - \lambda_1| - |\lambda_j - \alpha_1^{(k)}| \geq \delta - p_1 |\beta_1^{(k)}| > \rho_1 |\beta_1^{(k)}|$$

By Davis-Kahan theory[12] and Sun's results[11] we know

$$|\lambda_j - \mu_j| \leq \frac{\rho_1 |\beta_1^{(k)}|}{\sqrt{1-\rho_1^2}}, \quad j = 2, \cdots, n$$

This result means that, generally speaking, the influence of deflation on the accuracy of computed approximate eigenvalues is very small.

Remark. Above techniques can also be used to Lanczos algorithm and Jacobi algorithm.

3.2 Singular Value Problem

Let

$$M = \begin{pmatrix} A & H \\ G & E \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix}$$
$$\sigma(M) = \{\sigma_j\}_{j=1}^n, \quad \sigma(A) = \{\sigma_i(A)\}_{i=1}^p$$
$$\sigma(E) = \{\sigma_i(E)\}_{i=p+1}^n$$
$$\sigma_o = \min_{\substack{1\le j\le p \\ p+1\le i\le n}} |\sigma_j(A) - \sigma_i(E)| > 0$$

Suppose

$$\rho \equiv \max\{\|G\|_2, \|H\|_2\}/\rho_0(1/2)$$

According to Weyl-Mirsky theorem ([12] p.134, th.3.10), there exists a permutation $\pi(1), \cdots \pi(n)$ of $\{1, 2, \cdots, n\}$ such that

$$|\sigma_j(A) - \sigma_{\pi(j)}| \le \sigma_0\rho, \quad j = 1, \cdots, p$$
$$\sigma_k(E) - \sigma_{n(k)} \le \rho_0\rho, \quad k = p+1, \cdots, n$$

and therefore

$$|\sigma_j(A) - \sigma_{\pi(k)}| \ge \rho_0 \cdot (1-\rho), \quad j = 1, \cdots, p$$
$$|\sigma_k(E) - \sigma_{\pi(j)}| \ge \rho_0 \cdot (1-\rho), \quad k = p+1, \cdots, n$$

Let

$$\rho_1 \equiv \max\{\|G\|_2, \|H\|_2\}/[\rho_0(1-\rho)]$$

then from corollary 2 we have

Theorme 5. *If* $\rho_1 < 1/2$*, then*

$$|\sigma_j(A) - \sigma_{\pi(j)}| \le \frac{\rho_1 \max\{\|G\|_2, \|H\|_2\}}{\sqrt{1-\rho_1^2}}$$
$$|\sigma_k(A) - \sigma_{\pi(k)}| \le \frac{\rho_1 \max\{\|G\|_2, \|H\|_2\}}{\sqrt{1-\rho_1^2}} \quad j = 1, \cdots, p;\ k = p+1, \cdots, n$$

Remark. R. Mathias and G.W. Stewart[8] have considered following case: $G = 0$. Under the assumption

$$\sigma_{\min}(A) \gg \|E\|_2$$

they proved results slightly better than ours in this case.

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