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# THE NUMERICAL STABILITY OF THE $\theta$-METHOD FOR DELAY DIFFERENTIAL EQUATIONS WITH MANY VARIABLE DELAYS*

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**Abstract**

This paper deals with the asymptotic stability of theoretical solutions and numerical methods for the delay differential equations(DDEs)

$$\begin{cases} y'(t) &= ay(t) + \sum_{j=1}^{m} b_j y(\lambda_j t) \quad t \geq 0, \\ y(0) &= y_0, \end{cases}$$

where $a, b_1, b_2, \ldots, b_m$ and $y_0 \in C$, $0 < \lambda_m \leq \lambda_{m-1} \leq \ldots \leq \lambda_1 < 1$. A sufficient condition such that the differential equations are asymptotically stable is derived.And it is shown that the linear $\theta$-method is $\Lambda GP_m$-stable if and only if $\frac{1}{2} \leq \theta \leq 1$.

*Key words*: Delay differential equation, Variable delays, Numerical stability, $\theta$-methods.

## 1. Introduction

In this paper,we will investigate the numerical solutions of the following initial value problems for DDEs with many variable delays

$$\begin{cases} y'(t) &= ay(t) + \sum_{j=1}^{m} b_j y(\lambda_j t) \quad t \geq 0, \\ y(0) &= y_0, \end{cases} \tag{1.1}$$

where $a, b_1, b_2, \ldots, b_m$ and $y_0 \in C$, $0 < \lambda_m \leq \lambda_{m-1} \leq \ldots \leq \lambda_1 < 1$.It is difficult to investigate numerically the long time dynamical behaviour of the exact solution due to limited computer memory. To avoid this problem we transform (1.1) into the differential

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equations with constant time lags in the following way.(see [3]) Let $x(t) = y(e^t)$ for $t \geq \log \lambda_m$.Then $x(t)$ satisfies the following initial value problems

$$\begin{cases} x'(t) &= ae^t x(t) + \sum_{j=1}^{m} b_j e^t x(t + \log \lambda_j) \quad t \geq 0, \\ x(t) &= y(e^t) := \Phi(t) \qquad t \in [\log \lambda_m, 0], \end{cases} \tag{1.2}$$

where $y(t), 0 \leq t \leq e^0 = 1$,can be obtained numerically by using $\theta$ -method to (1.1). Then,let us consider the following linear test equations which were introduced in [4],

$$\begin{cases} y'(t) &= a(t)y(t) + b(t)y(t-\tau) \quad \tau > 0, t \geq 0, \\ y(t) &= \Phi(t) \quad -\tau \leq t \leq 0, \end{cases} \tag{1.3}$$

where $y : [-\tau, +\infty) \to C, \quad a, b : [0, +\infty) \to C$.

If a(t) and b(t) are continuous and satisfy

$$Re(a(t)) \leq -\beta < 0, \tag{1.4a}$$

$$|b(t)| \leq -q \cdot Re(a(t)), 0 \leq q < 1 \tag{1.4b}$$

and $\Phi(t)$ is continuous,then the solution $y(t)$ of (1.3) is asymptotically stable, namely, $y(t) \to 0$,as $t \to \infty$.

In [4],the authors introduced two definitions of stability based on the test equations (1.3) as follows.

**Definition 1.** *A numerical method for DDEs is called TP-stable if, under the condition (1.4),the numerical solution $y_n$ of (1.3) satisfies*

$$\lim_{n \to \infty} y_n = 0 \tag{1.5}$$

*for every stepsize h such that $h = \tau / l$ where $l \geq 1$ is a positive integer.*

**Definition 2.** *A numerical method for DDEs is called TGP-stable if, under the condition (1.4),the numerical solution $y_n$ of (1.3) satisfies (1.5) for every stepsize $h > 0$.*

It is the purpose of this paper to investigate the asymptotic stability behaviour of the theoretical solution and the numerical solution of (1.1).In Section 2,we derive a sufficient condition for (1.1) such that the solution of (1.1) is asymptotically stable. In Section 3,it is proven that the linear $\theta$-method is $\Lambda GP_m$-stable if and only if $\frac{1}{2} \leq \theta \leq 1$.

## 2. Asymptotic Stability Of The Theoretic Solution Of DDEs

Now we consider the following equations:

$$\begin{cases} x'(t) &= a(t)x(t) + b_2(t)x(t-\tau_2) + b_1(t)x(t-\tau_1) \quad t \geq 0, \tau_2 \geq \tau_1 > 0, \\ x(t) &= \Phi(t) \qquad t \leq 0, \end{cases} \tag{2.1}$$

where $x : R \to C, \quad a, b_1, b_2 : [0, +\infty) \to C$,and $\Phi : (-\infty, 0] \to C$.

**Theorem 2.1.** *Assume that the continuous functions* $a, b_1$ *and* $b_2 : [0, +\infty) \to R$, *and satisfy*

$$b_1(t), b_2(t) \geq 0, \Phi(t) \geq 0, \tag{2.2a}$$
$$a(t) \leq -\beta < 0, \tag{2.2b}$$
$$b_1(t) + b_2(t) \leq -q \cdot a(t) \quad 0 \leq q < 1, \tag{2.2c}$$

*then all exact solutions to (2.1) satisfy* $\lim_{t\to\infty} x(t) = 0$.

*Proof.* At first,let $\tau = \tau_1$ and if $\tau_2/\tau_1 = 2$,then when $t \in [0, \tau]$,(2.1) reads

$$\begin{cases} x'(t) &= a(t)x(t) + b_2(t)\Phi(t-2\tau) + b_1(t)\Phi(t-\tau) \quad t \in [0, \tau], \\ x(0) &= \Phi(0). \end{cases} \tag{2.3}$$

The solution of (2.3) is

$$x(t) = e^{A_0(t)} \cdot \Phi(0) + e^{A_0(t)} \int_0^t e^{-A_0(s)} \cdot [b_2(s)\Phi(s-2\tau) + b_1(s)\Phi(s-\tau)]ds, \tag{2.4}$$

where $A_i(t) = \int_{i\tau}^t a(s)ds, \qquad t \in [i\tau, (i+1)\tau], i = 1, 2, \ldots$. Since (2.2 a)$\sim$ (2.2 c) we have

$$\begin{aligned} x(t) &\leq [e^{-\beta t} + q(1 - e^{-\beta t})] \cdot M \\ &= G_0(t) \cdot M, \end{aligned} \tag{2.5}$$

where $M = \max_{-2\tau \leq t \leq 0} \Phi(t)$, $G_0(t) = e^{-\beta t} + q(1 - e^{-\beta t})$.

When $t \in [\tau, 2\tau]$,then (2.1) reads

$$\begin{cases} x'(t) &= a(t)x(t) + b_2(t)\Phi(t-2\tau) + b_1(t)x(t-\tau) \quad t \in [\tau, 2\tau], \\ x(\tau) &= x(\tau). \end{cases} \tag{2.6}$$

Then the solution of (2.6) is

$$\begin{aligned}
x(t) &= e^{A_1(t)} \cdot x(\tau) + e^{A_1(t)} \int_\tau^t e^{-A_1(s)}[b_2(s)\Phi(s-2\tau) + b_1(s)x(s-\tau)]ds \\
&\le \{e^{A_1(t)} \cdot G_0(\tau) + e^{A_1(t)} \int_\tau^t e^{-A_1(s)}[b_2(s) + b_1(s)G_0(s-\tau)]ds\} \cdot M \\
&= \{e^{A_1(t)} \cdot G_0(\tau) + e^{A_1(t)} \int_\tau^t e^{-A_1(s)} b_2(s)ds \\
&\quad + e^{A_1(t)} G_0(\xi_0) \int_\tau^t e^{-A_1(s)} b_1(s)ds\} \cdot M \quad (\xi_0 \in [0,\tau]) \\
&\le \{e^{A_1(t)} \cdot G_0(\tau) + e^{A_1(t)} \int_\tau^t e^{-A_1(s)} b_2(s)ds \\
&\quad + e^{A_1(t)} G_0(0) \int_\tau^t e^{-A_1(s)} b_1(s)ds\} \cdot M \\
&= \{e^{A_1(t)} \cdot G_0(\tau) + e^{A_1(t)} \int_\tau^t e^{-A_1(s)}[b_1(s) + b_2(s)]ds\} \cdot M \\
&\le \{e^{A_1(t)} \cdot G_0(\tau) + q e^{A_1(t)} \int_\tau^t e^{-A_1(s)}[-a(s)]ds\} \cdot M \\
&= \{e^{A_1(t)} \cdot G_0(\tau) + q(1 - e^{A_1(t)})\} \cdot M \\
&\le \{e^{-\beta(t-\tau)} \cdot G_0(\tau) + q(1 - e^{-\beta(t-\tau)})\} \cdot M \\
&= G_1(t-\tau) \cdot M,
\end{aligned} \tag{2.7}$$

where $G_1(t) = e^{-\beta t} \cdot G_0(\tau) + q(1 - e^{-\beta t})$.

When $t \in [2\tau, 3\tau]$,then (2.1) reads

$$\begin{cases} x'(t) &= a(t)x(t) + b_2(t)x(t-2\tau) + b_1(t)x(t-\tau), t \in [2\tau, 3\tau] \\ x(2\tau) &= x(2\tau). \end{cases} \tag{2.8}$$

Then we can get

$$\begin{aligned}
x(t) &= e^{A_2(t)} \cdot x(2\tau) + e^{A_2(t)} \int_{2\tau}^t e^{-A_2(s)}[b_2(s)x(s-2\tau) + b_1(s)x(s-\tau)]ds \\
&\le \{e^{A_2(t)} \cdot G_1(\tau) + e^{A_2(t)} G_0(\xi_1) \int_{2\tau}^t e^{-A_2(s)} b_2(s)ds \\
&\quad + e^{A_2(t)} G_1(\xi_2) \int_{2\tau}^t e^{-A_2(s)} b_1(s)ds\} \cdot M \quad (\xi_1, \xi_2 \in [0,\tau]) \\
&\le \{e^{A_2(t)} \cdot G_0(\tau) + e^{A_2(t)} G_0(\xi_1) \int_{2\tau}^t e^{-A_2(s)} b_2(s)ds \\
&\quad + e^{A_2(t)} G_0(\xi_2) \int_{2\tau}^t e^{-A_2(s)} b_1(s)ds\} \cdot M \\
&\le \{e^{A_2(t)} \cdot G_0(\tau) + e^{A_2(t)} \int_{2\tau}^t e^{-A_2(s)}[b_1(s) + b_2(s)]ds\} \cdot M \\
&\le \{e^{-\beta(t-2\tau)} \cdot G_0(\tau) + q(1 - e^{-\beta(t-2\tau)})\} \cdot M \\
&= G_1(t-2\tau) \cdot M.
\end{aligned} \tag{2.9}$$

When $t \in [3\tau, 4\tau]$,then we have

$$\begin{aligned} x(t) &\le e^{A_3(t)} \cdot x(3\tau) + e^{A_3(t)} \int_{3\tau}^{t} e^{-A_3(s)}[b_2(s)x(s-2\tau) + b_1(s)x(s-\tau)]ds \\ &\le \{e^{A_3(t)} \cdot G_1(\tau) + e^{A_3(t)} G_1(0) \int_{3\tau}^{t} e^{-A_3(s)}[b_1(s) + b_2(s)]ds\} \cdot M \\ &\le G_1(0)\{e^{A_3(t)} + q(1 - e^{A_3(t)})\} \cdot M \\ &\le G_0(\tau)\{e^{-\beta(t-3\tau)} + q[1 - e^{-\beta(t-3\tau)}]\} \cdot M \\ &= G_0(\tau) G_0(t-3\tau) \cdot M. \end{aligned} \tag{2.10}$$

When $t \in [4\tau, 5\tau]$,we have

$$x(t) \le G_0(\tau) G_0(t - 4\tau) \cdot M. \tag{2.11}$$

When $t \in [5\tau, 6\tau]$,we get

$$x(t) \le G_0(\tau) G_1(t - 5\tau) \cdot M. \tag{2.12}$$

By induction for $k = 0, 1, 2, \ldots$, we obtain

$$\begin{cases} x(t) \le [G_0(\tau)]^k G_0(t - 3k\tau) \cdot M & \text{for } t \in [3k\tau, (3k+1)\tau] \\ x(t) \le [G_0(\tau)]^k G_1(t - (3k+1)\tau) \cdot M & \text{for } t \in [(3k+1)\tau, (3k+2)\tau] \\ x(t) \le [G_0(\tau)]^k G_1(t - (3k+2)\tau) \cdot M & \text{for } t \in [(3k+2)\tau, (3k+3)\tau], \end{cases} \tag{2.13}$$

where $M = \max_{-2\tau \le t \le 0} \Phi(t), G_0(t) = e^{-\beta t} + q(1 - e^{-\beta t}), G_1(t) = e^{-\beta t} \cdot G_0(\tau) + q(1 - e^{-\beta t})$.

If $\tau_2/\tau_1 = s \in Z$,where $Z$ is the integeral set,then for $k = 0, 1, 2, \cdots$,we can obtain

$$\begin{cases} x(t) \le [G_0(\tau)]^k G_0(t - (s+1)k\tau) \cdot M \\ \qquad \text{for } t \in [(s+1)k\tau, ((s+1)k+1)\tau] \\ x(t) \le [G_0(\tau)]^k G_1(t - ((s+1)k+i)\tau) \cdot M \\ \qquad \text{for } t \in [((s+1)k+i)\tau, ((s+1)k+i+1)\tau] \\ \qquad (i = 1, 2, \ldots, s), \end{cases} \tag{2.14}$$

where $M = \max_{-s\tau \le t \le 0} \Phi(t)$.

If $\tau_2/\tau_1$ is not an integer,then there exists $s$, $s \in Z$,such that $s < \tau_2/\tau_1 < s + 1$. We let

$$\begin{aligned} &P_0 = 0, \\ &P_i = (k+1)\tau_2 - [(k+1)(s+1) - i]\tau_1 \\ &\qquad i = k(s+1)+1, k(s+1)+2, \ldots, (k+1)(s+1) \\ &\qquad (k = 0, 1, 2, \ldots). \end{aligned}$$

We can get

$$\begin{cases} x(t) \le [G_0(\tau_2 - s\tau_1)]^k G_0(t - P_{(s+2)k}) \cdot M \\ \qquad \text{for } t \in [P_{(s+2)k}, P_{(s+2)k+1}] \\ x(t) \le [G_0(\tau_2 - s\tau_1)]^k G_{\tau_2 - s\tau_1}(t - P_{(s+2)k+i}) \cdot M \\ \qquad \text{for } t \in [P_{(s+2)k+i}, P_{(s+2)k+i+1)}] \\ \qquad (i = 1, 2, \ldots, s+1), \end{cases} \tag{2.15}$$

where $G_{\tau_2 - s\tau_1}(t) = e^{-\beta t} \cdot G_0(\tau_2 - s\tau_1) + q(1 - e^{-\beta t}), M = \max_{-\tau_2 \le t \le 0} \Phi(t)$.

Since $G_0(\tau)$, $G_0(\tau_2 - s\tau_1) < 1$, from (2.13), (2.14), (2.15) we can obtain $\lim_{t\to\infty} x(t) = 0$. This completes the proof of this theorem.

Analogous to the proof of the previous theorem,we have the following Theorems.

**Theorem 2.2.** *Assume that* $Re(a(t))$, $|b_1(t)|$, $|b_2(t)|$ *are continuous and*

$$Re(a(t)) \le -\beta < 0, \tag{2.16a}$$

$$|b_1(t)| + |b_2(t)| \le -qRe(a(t)) \quad 0 \le q < 1, \tag{2.16b}$$

*then the solution to (2.1) is asymptotically stable.*

*Proof.* From (2.4),we can get

$$|x(t)| \le |e^{A_0(t)}|M + |e^{A_0(t)}|M \int_0^t |e^{-A_0(s)}|[|b_1(s)| + |b_2(s)|]ds,$$

since $|e^{-A_0(s)}| = e^{-\int_0^s Re(a(u))du}$,and $|b_1(s)| + |b_2(s)| \le -qRe(a(s))$, then we obtain

$$\begin{aligned} |x(t)| &\le [e^{-\beta t} + q(1 - e^{-\beta t})] \cdot M \\ &= G_0(t) \cdot M, \end{aligned}$$

where $M = \max_{-\tau_2 \le t \le 0} |\Phi(t)|, G_0(t) = e^{-\beta t} + q(1 - e^{-\beta t})$. The remaining parts can be proved analogously.

**Theorem 2.3.** *Assume that* $Re(a(t))$, $|b_1(t)|$, $|b_2(t)|, \ldots, |b_m(t)|$ *are continuous and*

$$Re(a(t)) \le -\beta < 0, \tag{2.17a}$$

$$\sum_{j=1}^{m} |b_j(t)| \le -qRe(a(t)) \quad 0 \le q < 1, \tag{2.17b}$$

*then the solution to*

$$\begin{cases} x'(t) &= a(t)x(t) + \sum_{j=1}^{m} b_j(t)x(t - \tau_j) \quad t \ge 0, \\ x(t) &= \Phi(t) \quad t \le 0, \end{cases} \tag{2.18}$$

*for any* $\tau_m \ge \tau_{m-1} \ge \cdots \ge \tau_1 > 0$ *is asymptotically stable.*

**Corollary 2.4.** *Assume that* $a$, $b_1, b_2, \ldots, b_m \in C$, *and*

$$Re(a) < 0, \tag{2.19a}$$

$$\sum_{j=1}^{m} |b_j| < -Re(a), \tag{2.19b}$$

*then the solution to (1.2) is asymptotically stable,i.e.,the solution to (1.1) is asymptotically stable for any* $0 < \lambda_m \le \lambda_{m-1} \le \ldots \le \lambda_1 < 1$.

*Proof.* Since (2.19a) and (2.19b) hold,we get

$$Re(ae^t) \le -\beta < 0, \quad t \ge 0,$$

$$\sum_{j=1}^{m} |b_j e^t| \le -qRe(ae^t) \quad t \ge 0, 0 \le q < 1,$$

(For instance,we can take $\beta = -Re(a)$,and $q = \sum_{j=1}^{m} |b_j|/(-Re(a))$.)

Then we use Theorem 2.3 to prove this corollary.

## 3. Numerical Stability Of The Linear $\theta$-method

For the initial problem

$$x'(t) = f(t, x(t), x(\alpha_1[t]), \ldots, x(\alpha_m[t])) \quad t \geq 0, \tag{3.1a}$$

$$x(t) = \Phi(t) \quad t \leq 0, \tag{3.1b}$$

we consider the following method called the linear $\theta$-method

$$\begin{aligned} x_{n+1} = x_n + h\theta f((n+1)h, x_{n+1}, x^h(\alpha_1[(n+1)h]), \ldots, x^h(\alpha_m[(n+1)h])) \\ + h(1-\theta) f(nh, x_n, x^h(\alpha_1[nh]), \ldots, x^h(\alpha_m[nh])), \end{aligned} \tag{3.2}$$

for $n = 0, 1, 2, \ldots$, here $\theta$ is a parameter with $0 \leq \theta \leq 1$, $h > 0$ is the stepsize. $x_0 = \Phi(0)$, $x^h(t) = \Phi(t)$, for $t \leq 0$,and $x^h(t)$ with $t \geq 0$ is defined by piecewise linear interpolation,i.e.

$$x^h(t) = \frac{t - nh}{h} x_{n+1} + \frac{(n+1)h - t}{h} x_n, \text{for } nh \leq t \leq (n+1)h, n = 0, 1, \ldots. \tag{3.3}$$

Applying (3.2) and (3.3) to (1.2),we arrive at the following recurrence relation

$$\begin{aligned} x_{n+1} = R_n \cdot x_n + \sum_{i=1}^{m} S_n^{(i)} \{(1-\theta)[(1-\delta_i)x_{n-l_i} + \delta_i x_{n+1-l_i}] \\ + \theta e^h [(1-\delta_i)x_{n+1-l_i} + \delta_i x_{n+2-l_i}]\}, \end{aligned} \tag{3.4}$$

where $R_n = (1 + (1-\theta)ahe^{t_n})/(1 - \theta ahe^{t_{n+1}})$, $S_n^{(i)} = (b_i he^{t_n})/(1 - \theta ahe^{t_{n+1}})$, $-\log \lambda_i = (l_i - \delta_i)h, \delta_i \in [0, 1), l_i \in Z$, $(i = 1, 2, \ldots, m)$. At once we can find one important observation is that

$$R := \lim_{n \to \infty} R_n = -\frac{1-\theta}{\theta e^h}, S^{(i)} := \lim_{n \to \infty} S_n^{(i)} = -\frac{b_i}{\theta e^h a} \quad (i = 1, 2, \ldots, m). \tag{3.5}$$

**Definition 3.** *Let*

$$a, b_i \in C (i = 1, 2, \ldots, m), \textit{ and } \delta_i \in [0, 1) \quad (i = 1, 2, \ldots, m), \textit{ which are defined in (*)}.$$

*Then a numerical method for DDEs is called $\Lambda(\delta_1, \delta_2, \ldots, \delta_m)$-stable at $(a, b_1, b_2, \ldots, b_m)$, if any application of the method to (1.1) or (1.2) yields approximation $x_n \to 0$ as $n \to \infty$, whenever $\lambda_i, (i = 1, 2, \cdots, m)$ and stepsize $h$ satisfy $0 < \lambda_i < 1$,$h > 0$, and*

$$\delta_i = l_i + h^{-1} \log \lambda_i, (i = 1, 2, \ldots, m). \tag{*}$$

The set consisting of all $(a, b_1, b_2, \ldots, b_m)$ at which the method is $\Lambda(\delta_1, \delta_2, \ldots, \delta_m)$-stable is called $\Lambda(\delta_1, \delta_2, \ldots, \delta_m)$-stability region. For the linear $\theta$-method we denote it by $S_{\theta,\delta_1,\delta_2,\ldots,\delta_m}$. The stability region $S_\theta$ of the $\theta$-method is defined by

$$S_\theta = \bigcap_{0 \leq \delta_1, \delta_2, \ldots, \delta_m < 1} S_{\theta,\delta_1,\delta_2,\ldots,\delta_m}.$$

Define

$$H = \{(a, b_1, b_2, \ldots, b_m) : (a, b_1, b_2, \ldots, b_m) \text{ satisfies } (2.19)\}.$$

**Definition 4.** *The linear $\theta$-method for DDEs (1.1) is called $\Lambda P_m$-stable if and only if $H \subset S_{\theta,0,0,\ldots,0}$.*

**Definition 5.** *The linear $\theta$-method for DDEs (1.1) is called $\Lambda GP_m$-stable if and only if $H \subset S_\theta$.*

A polynomial is said to be Schur type if all of its roots are less than 1 in modulus. Now we will prove the following lemma.

**Lemma 3.1.** *Under the condition (2.19),if $1 \geq \theta \geq \frac{1}{2}$,the characteristic polynomial of*

$$\begin{aligned}\tilde{x}_{n+1} =& R \cdot \tilde{x}_n + \sum_{i=1}^{m} S^{(i)} \{(1-\theta)[(1-\delta_i)\tilde{x}_{n-l_i} + \delta_i \tilde{x}_{n+1-l_i}] \\ & + \theta e^h [(1-\delta_i)\tilde{x}_{n+1-l_i} + \delta_i \tilde{x}_{n+2-l_i}]\Big\}\end{aligned} \tag{3.6}$$

*is a Schur polynomial.*

*Proof.* The characteristic polynomial of difference equation (3.6) is

$$P(z, \delta_1, \delta_2, \cdots, \delta_m) = Q_{m+1}(z) \cdot z^{l_m} - \sum_{j=1}^{m} Q_j(z, \delta_j) z^{l_m - l_j} \tag{3.7}$$

where

$$\begin{aligned}&Q_{m+1}(z) = z - R, \\ &Q_j(z, \delta_j) = S^{(j)} \cdot [\theta \epsilon^h z + (1-\theta)][\delta_j z + (1-\delta_j)] \quad j = 1, 2, \cdots, m.\end{aligned}$$

The following proof is always under the condition $1 \geq \theta \geq \frac{1}{2}$.

(i)we can easily get that $|R| < 1$,

(ii)we will show that $\sum_{j=1}^{m} |Q_j(z, \delta_j)| < |Q_{m+1}(z)|, \forall z \in c$.

Here c denotes the unit circle in the complex plane. Let $z = e^{i\phi}$.Since

$$\begin{aligned}|\delta_j e^{i\phi} + (1-\delta_j)| &\leq |\delta_j e^{i\phi}| + |1-\delta_j| \quad (\delta_j \in [0,1), j = 1, 2, \cdots, m) \\ &\leq \delta_j + 1 - \delta_j \\ &= 1,\end{aligned}$$

we can get

$$\begin{aligned}\sum_{j=1}^{m} |Q_j(z, \delta_j)| &\leq \sum_{j=1}^{m} |-\frac{b_j}{a}(e^{i\phi} + \frac{1-\theta}{\theta e^h})||\delta_j e^{i\phi} + (1-\delta_j)| \\ &\leq \sum_{j=1}^{m} \frac{|b_j|}{|-a|}|e^{i\phi} - (-\frac{1-\theta}{\theta e^h})| \\ &< |z - R| \\ &= |Q_{m+1}(z)|.\end{aligned}$$

From (i),(ii),we can get that the $p(z,\delta_1,\cdots,\delta_m)$ is a Schur polynomial (see [2] or [5]).

**Theorem 3.1.** *Suppose that $1\geq\theta\geq\frac{1}{2}$,then the linear $\theta$ -method is $\Lambda GP_m$-stable.*

*Proof.* Let

$$\begin{aligned}\mathcal{L}[x_n] =&x_{n+1}-R\cdot x_n-\sum_{i=1}^{m}S^{(i)}\left\{(1-\theta)[(1-\delta_i)x_{n-l_i}+\delta_i x_{n+1-l_i}]\right.\\ &\left.-\theta e^h[(1-\delta_i)x_{n+1-l_i}+\delta_i x_{n+2-l_i}]\right\}.\end{aligned}$$

Then the equation (3.4) can be written as

$$\mathcal{L}[x_n]=F_n\quad n\geq 0, \tag{3.8}$$

where

$$\begin{aligned}F_n =&(R_n-R)x_n+\sum_{j=1}^{m}(S_n^{(j)}-S^{(j)})\left\{(1-\theta)[(1-\delta_j)x_{n-l_j}+\delta_j x_{n+1-l_j}]\right.\\ &\left.+\theta e^h[(1-\delta_j)x_{n+1-l_j}+\delta_j x_{n+2-l_j}]\right\}\quad n\geq 0.\end{aligned} \tag{3.9}$$

Let

$$X_n=x_n+\sum_{j=1}^{m}\frac{b_j}{a}[(1-\delta_j)x_{n-l_j}+\delta_j x_{n+1-l_j}]. \tag{3.10}$$

It follows from (3.8) that

$$X_{n+1}=RX_n+F_n,\qquad n\geq 0,$$

from which we deduce by iteration that

$$X_{n+1}=\sum_{k=0}^{n}R^kF_{n-k}+R^{n+1}X_0,\qquad n\geq 0. \tag{3.11}$$

Let M be a positive constant such that

$$|F_k|\leq Me^{-kh}\cdot\max_{-l_m\leq j\leq k}|x_j|,\qquad k\geq 0.$$

If $-Re(a)>\sum_{j=1}^{m}|b_j|$,then we obtain from (3.11) that

$$|x_{n+1}|\leq(1+M(n+1)e^{-nh})\max_{-l_m\leq k\leq n}|x_k|+|R|^{n+1}|X_0|,n\geq 0$$

which implies that

$$|x_{n+1}|\leq\{\prod_{k=0}^{n}(1+M(k+1)e^{-kh}+|R|^{k+1})\}\max\{\max_{-l_m\leq i\leq 0}|x_i|,|X_0|\},n\geq 0.$$

Since the product in the previous inequality converges as $n \to \infty$,the solution sequence of (1.2) is bounded. If $-Re(a) > \sum_{j=1}^{m} |b_j|$,then $x_n$ tends to zero as $n \to \infty$.This is because that $\{x_n\}_{n=0}^{\infty}$ satisfies equation (3.8) whose right hand side term $F_n$ tends to zero with the exponential form as $n \to \infty$,and that the corresponding characteristic polynomial of (3.8) is of Schur type. This completes the proof of this theorem.

**Corollary 3.2.** *The linear $\theta$-method is $\Lambda GP_m$-stable if and only if* $1 \geq \theta \geq \frac{1}{2}$.

*Proof.* The "if" part is obtained from Theorem 3.1. The "only if" part can be reached by checking the case where $b_j = 0, (j = 1, 2, \cdots, m)$.

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