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COMBINED LEGENDRE SPECTRAL-FINITE ELEMENT METHOD FOR THE TWO-DIMENSIONAL UNSTEADY NAVIER-STOKES EQUATION*

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Abstract

A combined Legendre spectral-finite element approximation is proposed for solving two-dimensional unsteady Navier-Stokes equation. The artificial compressibility is used. The generalized stability and convergence are proved strictly. Some numerical results show the advantages of this method.

Key words: Navier-Stokes equation, Combined Legendre spectral-finite element approximation.

1. Introduction

There is much literature concerning numerical solutions of Navier-Stokes equations, e.g., see [1-4]. For semi-periodic problems, some author used combined Fourier spectral-finite difference and Fourier spectral-finite element approximations (see[5-8]). In fluid dynamics, most of practical problems are fully non-periodic. But the sections of domain might be rectangular in certain directions. For example, the fluid flow in a cylindrical container. In this paper, we consider combined Legendre spectral-finite element approximation for the two-dimensional, non-periodic, unsteady Navier-Stokes equation. The method in this paper can raise the accuracy by Legendre spectral approximation in some directions and so saves work. On the other hand, such approximation is suitable for complex geometry in the remaining directions. Surely it is not necessary to use this approach for such two-dimensional problem. But it is easy to generalize it to three-dimensional problems with complex geometry.

2. The Scheme

Let $I_x = \{x\,/\,0 < x < 1\}$, $I_y = \{y\,/\,-1 < y < 1\}$ and $\Omega = I_x \times I_y$ with the boundary $\partial\Omega$. The speed vector and the pressure are denoted by $U(x,y,t)$ and $P(x,y,t)$ respectively. $\nu > 0$ is the kinetic viscosity. $U_0(x,y)$, $P_0(x,y)$ and $f(x,y,t)$ are given functions. Let $T > 0, \partial_t = \frac{\partial}{\partial t}, \partial_x = \frac{\partial}{\partial x}$, and $\partial_y = \frac{\partial}{\partial y}$. The Navier-Stokes equation is as follows

$$\begin{cases} \partial_t U + (U\cdot\nabla)U + \nabla P - \nu\nabla^2 U = f, & \text{in } \Omega\times(0,T], \\ \nabla\cdot U = 0, & \text{in } \Omega\times(0,T], \\ U(x,y,0) = U_0(x,y), \quad P(x,y,0) = P_0(x,y), & \text{in } \Omega \end{cases} \tag{2.1}$$

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Suppose that the boundary is a non-slip wall and so $U = 0$ on $\partial\Omega$. In addition, P satisfies the following normalizing condition:

$$\int_\Omega P(x,y,t)\,\mathrm{d}x\mathrm{d}y = 0, \qquad \forall t \in [0,T].$$

Let $\mathcal{D}$ be an interval (or a domain) in R^1(or R^2). We denote by $(\cdot,\cdot)_\mathcal{D}$ and $\|\cdot\|_\mathcal{D}$ the usual inner product and norm of $L^2(\mathcal{D})$. For simplicity, $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$ are replaced by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. $H^r(\mathcal{D})$ and $H_0^r(\mathcal{D})$ denote the usual Hilbert spaces with the usual inner products and norms. We also define

$$L_0^2(\mathcal{D}) = \{\eta \in L^2(\mathcal{D}) \;/\; \int_\mathcal{D} \eta\,\mathrm{d}\mathcal{D} = 0\;\}.$$

To tackle the incompressible constraint (i.e., the second equation of (2.1)), we adopt the idea of artificial compression, that is, to approximate the incompressible condition by the equation

$$\beta\frac{\partial P}{\partial t} + \nabla\cdot U = 0$$

where $\beta > 0$ is a small parameter.

In order to approximate the nonlinear term, we introduce a trilinear form $J(\cdot,\cdot,\cdot)$: $[(H^1(\Omega))^2]^3 \to R^1$ as follows:

$$J(\eta,\varphi,\xi) = \frac{1}{2}[((\varphi\cdot\nabla)\eta,\xi) - ((\varphi\cdot\nabla)\xi,\eta)].$$

Clearly, we have

$$J(\eta,\varphi,\xi) + J(\xi,\varphi,\eta) = 0, \tag{2.2}$$

and if $\nabla\cdot\varphi = 0$, then

$$J(\eta,\varphi,\xi) = ((\varphi\cdot\nabla)\eta,\xi).$$

Now we construct the scheme. For any integer $k \geq 0$, we denote by $\mathcal{P}_k$ the set of all polynomials of degree $\leq k$, defined on R^1. Suppose N is a positive integer, we define

$$V_N(I_y) = \{v(y) \in \mathcal{P}_N \;/\quad v(-1) = v(1) = 0\}.$$

Next, we divide I_x into M_h subintervals with the nodes $0 = x_0 < x_1 < \cdots < x_{M_h} = 1$. Let $I_l = (x_{l-1}, x_l), h_l = x_l - x_{l-1}$, $h = \max\limits_{1\leq l\leq M_h} h_l$ and $h' = \min\limits_{1\leq l\leq M_h} h_l$. Assume that there exists a positive constant d independent of the divisions of I_x, such that $h/h' \leq d$. Let

$$\tilde{S}_h^k(I_x) = \{v(x)\,/\,v(x)\,|_{I_l}\in \mathcal{P}_k, 1 \leq l \leq M_h\},\quad S_h^k(I_x) = \tilde{S}_h^k(I_x)\bigcap H_0^1(I_x).$$

The trial function spaces for the speed and the the pressure are defined respectively as follows

$$X_{h,N}^k(\Omega) = \{S_h^{k+1}(I_x)\otimes V_N(I_y)\}\times\{S_h^{k+2}(I_x)\otimes V_N(I_y)\},$$

$$Y_{h,N}^k(\Omega) = \{\tilde{S}_h^k(I_x)\otimes \mathcal{P}_{N-2}(I_y)\}\bigcap L_0^2(\Omega).$$

We denote by $\prod_h^{k+1}$ the piecewise Lagrange interpolation of order $k+1$ from $C(\bar{I}_x)$ onto $\tilde{S}_h^{k+1}(I_x)\bigcap H^1(I_x)$, $\tilde{P}_{h,N} : L_0^2(\Omega) \to Y_{h,N}^k(\Omega)$ be the L^2-orthogonal projection. We also introduce the operator $Q_N : C(\bar{I}_y) \to \mathcal{P}_N(I_y)$ such that for any $v \in C(\bar{I}_y)$,

$$\begin{cases} Q_N v(\pm 1) = v(\pm 1), \\ \int_{-1}^1 (v - Q_N v) q \,\mathrm{d}y = 0, \quad \forall q \in \mathcal{P}_{N-2}(I_y). \end{cases}$$

Let τ be the mesh size in time t and $S_\tau = \{t = l\tau \ / \ 0 \le l \le [\frac{T}{\tau}]\}$. For simplicity, $u(x,y,t)$ is denoted by $u(t)$ or u usually. Let

$$u_t(t) = \frac{1}{\tau}(u(t+\tau) - u(t)).$$

A fully discrete Legendre spectral-finite element scheme for (2.1) is to find the pair $(u(t), p(t)) \in X_{h,N}^k(\Omega) \times Y_{h,N}^k(\Omega)$ for all $t \in S_\tau$ such that

$$\begin{cases} (u_t, v) + J(u + \delta\tau u_t, u, v) + \nu(\nabla(u + \sigma\tau u_t), \nabla v) & \\ +(\nabla(p + \theta\tau p_t), v) = (f, v), & \forall v \in X_{h,N}^k(\Omega), \\ (\beta p_t, v) + (\nabla \cdot (u + \theta\tau u_t), v) = 0, & \forall v \in Y_{h,N}^k(\Omega), \\ u(0) = \Pi_h^{k+1} Q_N U_0, \quad p(0) = \tilde{P}_{h,N} P_0, & \end{cases} \tag{2.3}$$

where $\delta, \sigma \ge 0$ and $\theta > \frac{1}{2}$ are parameters.

3. Some Lemmas

For error estimations, we need some notations. Let B be a Banach space with the norm $\|\cdot\|_B$. Define

$$\begin{aligned} L^2(\mathcal{D}, B) &= \left\{ v(z) : \mathcal{D} \to B \ / \ v \ \text{ is strongly measurable,} \ \|v\|_{L^2(\mathcal{D},B)} < \infty \right\}, \\ C(\mathcal{D}, B) &= \{ v(z) : \mathcal{D} \to B \ / \ v \ \text{ is strongly continuous,} \ |||v|||_B < \infty \} \end{aligned}$$

where

$$\|v\|_{L^2(\mathcal{D},B)} = \left(\int_{\mathcal{D}} \|v(z)\|_B^2 \,\mathrm{d}z \right)^{\frac{1}{2}}, \qquad |||v|||_B = \max_{z \in \mathcal{D}} \|v(z)\|_B.$$

Moreover for all integer $\mu \ge 0$, define

$$H^\mu(\mathcal{D}, B) = \left\{ v(z) \in L^2(\mathcal{D}, B) \ / \ \|v\|_{H^\mu(\mathcal{D},B)} < \infty \right\}$$

with the norm

$$\|v\|_{H^\mu(\mathcal{D},B)} = \left(\sum_{k=0}^{\mu} \|\frac{\partial^k v}{\partial z^k}\|_{L^2(D,B)}^2 \right)^{\frac{1}{2}}.$$

For real $\mu \ge 0$, we define the space $H^\mu(\mathcal{D}, B)$ by the complex interpolation between $H^{[\mu]}(\mathcal{D}, B)$ and $H^{[\mu+1]}(\mathcal{D}, B)$.

We also introduce some non-isotropic Sobolev spaces. Let

$$H^{r,s}(\Omega)=L^2(I_y,H^r(I_x))\bigcap H^s(I_y,L^2(I_x)),\qquad r,s\geq 0$$

equipped with the norm

$$\|v\|_{H^{r,s}(\Omega)}=\left(\|v\|^2_{L^2(I_y,H^r(I_x))}+\|v\|^2_{H^s(I_y,L^2(I_x))}\right)^{\frac{1}{2}}.$$

Also let

$$\begin{aligned} M^{r,s}(\Omega) &= H^1(I_y,H^r(I_x))\bigcap H^s(I_y,H^1(I_x)),\quad r,\ s\geq 0,\\ A^{r,s}(\Omega) &= H^s(I_y,H^{r+1}(I_x))\bigcap H^{s+1}(I_y,H^r(I_x))\quad r,s\geq 0. \end{aligned}$$

Their norms are defined in the way similar to $\|\cdot\|_{H^{r,s}(\Omega)}$.

Besides, we denote by $L^\infty(I_x), L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$ the usual Sobolev spaces with the usual norms $\|\cdot\|_{\infty,I_x}, \|\cdot\|_\infty$ and $\|\cdot\|_{1,\infty}$ respectively. The corresponding semi-norms are denoted by $|\cdot|_{\infty,I_x}, |\cdot|_\infty$ and $|\cdot|_{1,\infty}$, etc..

For simplicity, we denote throughout this paper by C a positive constant independent of h,N,τ and any function, which may be different in different cases. Let $\bar{r}=\min(r,k+2)$ and $\hat{r}=\min(r,k+1)$.

Lemma 3.1. *There exists a positive constant c_d depending only on the value of d, such that for all $u\in(H^1(I_x)\bigcap\tilde{S}_h^{k+m}(I_x))\otimes\mathcal{P}_N(I_y)$, $m=1,2$,*

$$|v|_1^2\leq(c_dh^{-2}+\frac{4}{3}N^4)\|v\|^2.$$

Proof. Let u_q and $u_q^{(1)}$ be the coefficients of Legendre expansions of $u\in\mathcal{P}_N$ and $\dfrac{\mathrm{d}u}{\mathrm{d}y}$ respectively. By (2.3.15) of [9],

$$u_q^{(1)}=(2q+1)\sum_{\substack{l=q+1\\ l+q\mathrm{odd}}}^{N}u_l,\qquad q=0,1,\cdots,N-1.$$

Thus

$$(u_q^{(1)})^2\leq\frac{1}{2}(N+1)(2q+1)^2\sum_{l=0}^{N}u_l^2$$

and

$$\|\frac{\mathrm{d}u}{\mathrm{d}y}\|^2_{I_y}\leq\frac{1}{2}(N{+}1)\sum_{q=0}^{N-1}(2q{+}1)^2\|u\|^2_{I_y}=\frac{N(N+1)(2N+1)(2N-1)}{6}\|u\|^2_{I_y}\leq\frac{4}{3}N^4\|u\|^2_{I_y}.$$

Therefore for any $v\in(H^1(I_x)\cap\tilde{S}_h^{k+m}(I_x))\otimes\mathcal{P}_N(I_y)$, $m=1,2$, we have

$$\|\partial_yv\|^2\leq\frac{4}{3}N^4\|v\|^2.$$

On the other hand, we have from the inverse inequality (3.2.30) of [10] that

$$\|\partial_x v\|^2 \le c_d h^{-2} \int_{-1}^{1} \|v\|_{I_x}^2 \,\mathrm{d}y \le c_d h^{-2} \|v\|^2.$$

Then the conclusion follows.

Lemma 3.2. *For any* $v \in \tilde{S}_h^{k+m}(I_x) \otimes \mathcal{P}_N(I_y), \quad m = 1, 2,$

$$\|v\|_\infty \le \frac{CN}{\sqrt{h}} \|v\|.$$

Proof. By the inverse inequality (3.2.30) of [10] and (9.4.3) of [9], we have

$$\|v\|_\infty = \sup_{y \in I_y} \|v\|_{\infty, I_x} \le \frac{C}{\sqrt{h}} \sup_{y \in I_y} \|v\|_{I_x} \le \frac{C}{\sqrt{h}} (\int_0^1 \sup_{y \in I_y} v^2 \,\mathrm{d}y)^{\frac{1}{2}} \le \frac{CN}{\sqrt{h}} \|v\|.$$

Let $P_N : L^2(I_y) \to \mathcal{P}_N(I_y)$ and $\mathcal{L}_h : L^2(I_x) \to \tilde{S}_h^k(I_x)$ be the L^2- orthogonal projection operators. We know from (9.4.6) of [9] that for any $u \in H^s(I_y)$ with $s \ge 0$,

$$\|u - P_N u\|_{I_y} \le C N^{-s} \|u\|_{s, I_y}. \tag{3.1}$$

Also by Lemma A.5 of [1], we get that for any $v \in H^r(I_x)$ with $r \ge 0$,

$$\|v - \mathcal{L}_h v\|_{I_x} \le C h^{\hat{r}} |v|_{\hat{r}, I_x}. \tag{3.2}$$

Clearly, if $v \in L_0^2(\Omega)$, then

$$\mathcal{L}_h P_{N-2} v \in L_0^2(\Omega). \tag{3.3}$$

Lemma 3.3. *For any* $v \in H^{r,s}(\Omega) \cap L_0^2(\Omega)$ *with* $r, s \ge 0$,

$$\|v - \tilde{P}_{h,N} v\| \le C(h^{\hat{r}} + N^{-s}) \|v\|_{H^{\hat{r},s}(\Omega)}.$$

Proof. The combination of (3.1)–(3.3) leads to the conclusion.

Lemma 3.4. *For any* $v \in H_0^1(I_y) \cap H^s(I_y)$ *with* $s \ge 1$,

$$\|v - Q_N v\|_{\mu, I_y} \le C N^{\mu - s} \|v\|_{s, I_y}, \qquad \mu = 0, 1.$$

Proof. Firstly, the definition of Q_N implies that for any $w \in \mathcal{P}_N(I)$,

$$\int_{-1}^{1} (v - Q_N v)' w' \,\mathrm{d}y = -\int_{-1}^{1} (v - Q_N v) w'' \,\mathrm{d}y = 0.$$

Thus

$$\begin{aligned} |v - Q_N v|_{1, I_y}^2 &= \int_{-1}^{1} (v - Q_N v)' v' \,\mathrm{d}y = \int_{-1}^{1} (v - Q_N v)' (v' - P_{N-1}(v')) \,\mathrm{d}y \\ &\le |v - Q_N v|_{1, I_y} \|v' - P_{N-1}(v')\|_{I_y}. \end{aligned}$$

Furthermore, we have from (3.1) that

$$|v - Q_N v|_{1, I_y} \le \|v' - P_{N-1} v'\|_{I_y} \le C N^{1-s} \|v\|_{s, I_y}.$$

Next, by means of the duality, we can show the conclusion for the case $\mu = 0$.

To analyze the convergence of Scheme (2.3), we introduce the operator $Q_h : (H_0^1(I_x))^2 \to S_h^{k+1}(I_x) \times S_h^{k+2}(I_x)$ such that for any $\eta = (\eta^{(1)}, \eta^{(2)}) \in (H_0^1(I_x))^2$,

$$Q_h\eta = (Q_h^{(1)}\eta^{(1)}, Q_h^{(2)}\eta^{(2)}) \in S_h^{k+1}(I_x) \times S_h^{k+2}(I_x)$$

where $Q_h^{(m)}\eta^{(m)} \in S_h^{k+m}$, $m = 1, 2$, are defined by

$$\begin{aligned} &Q_h^{(m)}\eta^{(m)}(x_l) = \eta^{(m)}(x_l), \quad 0 \le l \le M_h, \\ &\textstyle\int_{I_l}[Q_h^{(m)}\eta^{(m)}(x) - \eta^{(m)}(x)]w(x)\,\mathrm{d}x = 0, \quad \forall w \in \mathcal{P}_{k+m-2}(I_x),\ 1 \le l \le M_h. \end{aligned}$$

Lemma 3.5. (Lemma 3 of [7]) *Let $\bar{r}_m = \min(r, k+m+1)$, $m = 1, 2$. Then for any $\eta \in H_0^1(I_x) \cap H^r(I_x)$ with $r \ge 1$,*

$$\|\eta - Q_h^{(m)}\eta\|_{\mu,I_x} \le Ch^{\bar{r}_m-\mu}|\eta|_{\bar{r}_m,I_x}, \quad \mu = 0, 1.$$

Lemma 3.6. *There exists a linear operator $Q_{h,N} : (H_0^1(\Omega))^2 \to X_{h,N}^k(\Omega)$ such that*
(i) $(\nabla\cdot(\eta - Q_{h,N}\eta), w) = 0$, $\forall w \in Y_{h,N}^k(\Omega)$, $\forall \eta \in (H_0^1(\Omega))^2$. (3.4)
(ii) If $r, s \ge 1$, then we have for all $\eta \in (M^{r,s}(\Omega))^2$ that

$$|\eta - Q_{h,N}\eta|_1 \le C(h^{\bar{r}-1} + N^{1-s})\|\eta\|_{M^{\bar{r},s}(\Omega)}, \tag{3.5}$$

$$\|\eta - Q_{h,N}\eta\| \le C(h^{\bar{r}} + N^{-s})\|\eta\|_{M^{\bar{r},s}(\Omega)}, \tag{3.6}$$

Proof. Define $Q_{h,N} = Q_hQ_N$. Hence (3.4) follows from the definitions of Q_h and Q_N. By using Lemma 3.4 and Lemma 3.5, we can prove (3.5) and (3.6).

Lemma 3.7. *Let $Nh \le C$. Then for any $v \in H^1(I_y, H^1(I_x))$,*

$$\|Q_{h,N}v\|_\infty \le C\|v\|_{H^1(I_y,H^1(I_x))}.$$

If in addition $v \in A^{r,1}(\Omega)$ with $r > \frac{1}{2}$, then

$$\|Q_{h,N}v\|_{1,\infty} \le C\|v\|_{A^{r,1}(\Omega)}.$$

Proof. We have from Sobolev inequality together with Lemma 3.4 and Lemma 3.5 that

$$\|Q_Nu\|_{\infty,I_y} \le C\|u\|_{1,I_y}, \quad \forall u \in H^1(I_y), \tag{3.7}$$

$$\|Q_hu\|_{\infty,I_x} \le C\|u\|_{1,I_x}, \quad \forall u \in H^1(I_x). \tag{3.8}$$

Then the first conclusion follows from (3.7) and (3.8). We now prove the second one. Observing that $\Pi_h^{k+1}Q_N = Q_N\Pi_h^{k+1}$, by Lemma 3.2, Theorem 3.1.5, Theorem 3.2.1 and Theorem 3.2.6 of [10], (9.4.3) of [9] and previous estimates of Q_N and Q_h, we obtain that

$$\begin{aligned} &\|\partial_xQ_{h,N}v\|_\infty \le \|\partial_xQ_hQ_Nv - \partial_x\Pi_h^{k+1}Q_Nv\|_\infty + \|Q_N(\partial_x\Pi_h^{k+1}v \\ &-\Pi_h^{k+1}\partial_xv)\|_\infty + \|\Pi_h^{k+1}Q_N\partial_xv\|_\infty \\ &\le \tfrac{C}{\sqrt{h}}\|\partial_x(Q_h - \Pi_h^{k+1})(Q_Nv)\|_{C(I_y,L^2(I_x))} \\ &+\tfrac{C}{\sqrt{h}}\|\partial_x\Pi_h^{k+1}v - \Pi_h^{k+1}\partial_xv\|_{H^1(I_y,L^2(I_X))} + C\|v\|_{H^1(I_y,H^{r+1}(I_x))} \\ &\le C\|v\|_{H^1(I_y,H^{\frac{3}{2}}(I_X))} + C\|v\|_{H^1(I_y,H^{r+1}(I_x))} + C\|v\|_{H^1(I_y,H^{r+1}(I_x))} \\ &\le C\|v\|_{H^1(I_y,H^{r+1}(I_x))}, \end{aligned}$$

and

$$\begin{aligned}
&\|\partial_y Q_{h,N}v\|_\infty \le \|\partial_y Q_h Q_N v - \partial_y \Pi_h^{k+1} Q_N v\|_\infty \\
&+\|\partial_y \Pi_h^{k+1} Q_N v - \Pi_h^{k+1} Q_N \partial_y v\|_\infty + \|\Pi_h^{k+1} Q_N \partial_y v\|_\infty \\
&\le \tfrac{CN}{\sqrt{h}} \|(Q_h - \Pi_h^{k+1})(\partial_y Q_N v)\| + CN\|\partial_y Q_N v - Q_N \partial_y v\|_{L^2(I_y,H^r(I_x))} + C\|v\|_{H^2(I_y,H^r(I_x))} \\
&\le CNh\|v\|_{H^1(I_y,H^{\frac{3}{2}}(I_x))} + C\|v\|_{H^2(I_y,H^r(I_x))} \\
&\le C\|v\|_{H^1(I_y,H^{\frac{3}{2}}(I_x))\cap H^2(I_y,H^r(I_x))}.
\end{aligned}$$

Lemma 3.8. (Lemma 4.16 of [2]). *Suppose that the following conditions are fulfilled*
(i) *Z(t) is a non-negative function defined on* S_τ, D_1, D_2 *and* ρ *are non-negative constants;*
(ii) $H(\xi)$ *is a real-valued function defined on* R^1, *such that* $H(\xi) \le 0$ *for* $\xi \le D_2$;
(iii) *for all* $t \in S_\tau$ *and* $t > 0$,

$$Z(t) \le \rho + \tau \sum_{t' \le t-\tau} (D_1 Z(t') + H(Z(t')));$$

(iv) $Z(0) \le \rho$ *and* $\rho e^{D_1 t_1} \le D_2$ *for some* $t_1 \in S_\tau$.
Then for all $t \in S_\tau$ and $t \le t_1$, we have

$$Z(t) \le \rho e^{D_1 t}.$$

4. Error Estimation

We first analyze the generalized stability of Scheme (2.3). Assume that the initial values and the right terms of Scheme (2.3) have the errors $\tilde{u}(0), \tilde{p}(0), \tilde{f}(t)$ and $\tilde{g}(t)$ respectively, which induce the errors of $u(t)$ and $p(t)$, denoted by $\tilde{u}(t)$ and $\tilde{p}(t)$ respectively. They satisfy

$$\begin{cases}
(\tilde{u}_t, v) + J(u + \delta\tau u_t, \tilde{u}, v) + J(\tilde{u} + \delta\tau\tilde{u}_t, u + \tilde{u}, v) + \nu(\nabla(\tilde{u} + \sigma\tau\tilde{u}_t), \nabla v) \\
\;+ (\nabla(\tilde{p} + \theta\tau\tilde{p}_t), v) = (\tilde{f}, v), \qquad \forall v \in X_{h,N}^k(\Omega), \\
(\beta\tilde{p}_t, v) + (\nabla\cdot(\tilde{u} + \theta\tau\tilde{u}_t), v) = (\tilde{g}, v), \qquad \forall v \in Y_{h,N}^k(\Omega).
\end{cases} \tag{4.1}$$

Let $\varepsilon > 0$, and m be an undetermined positive constant. By taking $v = 2\tilde{u}(t) + m\tau\tilde{u}_t(t)$ in the first formula of (4.1), we have from (2.2) that

$$\begin{aligned}
&(\|\tilde{u}\|^2)_t + \tau(m-1-\varepsilon)\|\tilde{u}_t\|^2 + 2\nu|\tilde{u}|_1^2 + \nu\tau(\sigma + \tfrac{m}{2})(|\tilde{u}|_1^2)_t + \nu\tau^2(\sigma m - \sigma - \tfrac{m}{2})|\tilde{u}_t|_1^2 \\
&+ (\nabla(\tilde{p} + \theta\tau\tilde{p}_t), 2\tilde{u} + m\tau\tilde{u}_t) + \sum_{j=1}^{3} F_j(t) \le \|\tilde{u}\|^2 + (1 + \tfrac{\tau m^2}{4\varepsilon})\|\tilde{f}\|^2
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
F_1(t) &= J(u + \delta\tau u_t, \tilde{u}, 2\tilde{u} + m\tau\tilde{u}_t), \\
F_2(t) &= \tau(m - 2\delta)J(\tilde{u}, u, \tilde{u}_t), \\
F_3(t) &= \tau(m - 2\delta)J(\tilde{u}, \tilde{u}, \tilde{u}_t).
\end{aligned}$$

Similarly, we get by taking $v = 2\tilde{p} + m\tau\tilde{p}_t$ in the second formula of (4.1) that

$$\begin{aligned}&\beta(\|\tilde{p}\|^2)_t + \beta\tau(m-1-\varepsilon)\|\tilde{p}_t\|^2 + (\nabla\cdot(\tilde{u}+\theta\tau\tilde{u}_t), 2\tilde{p}+m\tau\tilde{p}_t)\\ &\leq \beta\|\tilde{p}\|^2 + (\tfrac{1}{\beta} + \tfrac{\tau m^2}{4\beta\varepsilon})\|\tilde{g}\|^2.\end{aligned} \tag{4.3}$$

By (4.2) and (4.3), we obtain

$$\begin{aligned}&(\|\tilde{u}\|^2 + \beta\|\tilde{p}\|^2)_t + \tau(m-1-\varepsilon)(\|\tilde{u}_t\|^2 + \beta\|\tilde{p}_t\|^2) + 2\nu|\tilde{u}|_1^2\\ &+ \nu\tau(\sigma + \tfrac{m}{2})(|\tilde{u}|_1^2)_t + \nu\tau^2(\sigma m - \sigma - \tfrac{m}{2})|\tilde{u}_t|_1^2 + \sum_{j=1}^{3} F_j(t) + H(t)\\ &\leq \|\tilde{u}\|^2 + \beta\|\tilde{p}\|^2 + (1 + \tfrac{\tau m^2}{4\varepsilon})(\|\tilde{f}\|^2 + \tfrac{1}{\beta}\|\tilde{g}\|^2)\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}H(t) &= (\nabla(\tilde{p}+\theta\tau\tilde{p}_t), 2\tilde{u}+m\tau\tilde{u}_t) + (\nabla\cdot(\tilde{u}+\theta\tau\tilde{u}_t), 2\tilde{p}+m\tau\tilde{p}_t)\\ &= (m-2\theta)\tau[(\nabla\cdot\tilde{u}, \tilde{p}_t) + (\nabla\tilde{p}, \tilde{u}_t)].\end{aligned}$$

Let $|||u|||_{1,\infty} = \max_{t\in S_\tau}\|u(t)\|_{1,\infty}$, etc.. We now estimate $|\ F_j(t)\ |$ $(j = 1,\ 2,\ 3)$. It is easy to show that,

$$\begin{aligned}|\ F_1(t)\ | &= C(1+\delta)|||u|||_{1,\infty}|2\tilde{u}+m\tau\tilde{u}_t|_1\|\tilde{u}\|\\ &\leq \varepsilon\nu|\tilde{u}|_1^2 + \varepsilon\nu\tau^2|\tilde{u}_t|_1^2 + \frac{C(1+\delta^2)(1+m^2)}{\varepsilon\nu}|||u|||_{1,\infty}^2\|\tilde{u}\|^2.\end{aligned}$$

By integrating by parts in $F_2(t)$, we have from $\|\tilde{u}_t\| \leq C|\tilde{u}_t|_1$ that

$$\begin{aligned}|\ F_2(t)\ | &\leq C\tau|m-2\delta|\ |||u|||_{1,\infty}|\tilde{u}_t|_1\|\tilde{u}\|\\ &\leq \varepsilon\nu\tau^2|\tilde{u}_t|_1^2 + \frac{C(m-2\delta)^2}{\varepsilon\nu}|||u|||_{1,\infty}^2\|\tilde{u}\|^2.\end{aligned}$$

By Lemma 3.2, we get furthermore that

$$\begin{aligned}|\ F_3(t)\ | &\leq C\tau|m-2\delta|\ \|\tilde{u}\|_\infty|\tilde{u}|_1\|\tilde{u}_t\|\\ &\leq \varepsilon\tau\|\tilde{u}_t\|^2 + \frac{C\tau(m-2\delta)^2N^2}{\varepsilon h}\|\tilde{u}\|^2|\tilde{u}|_1^2.\end{aligned}$$

By substituting the above estimations into (4.4), we have

$$\begin{aligned}&(\|\tilde{u}\|^2 + \beta\|\tilde{p}\|^2)_t + \tau(m-1-2\varepsilon)(\|\tilde{u}_t\|^2 + \beta\|\tilde{p}_t\|^2) + \nu(1-\varepsilon)|\tilde{u}|_1^2 + \nu\tau(\sigma+\tfrac{m}{2})(|\tilde{u}|_1^2)_t\\ &+ \nu\tau^2(\sigma m - \sigma - \tfrac{m}{2} - 2\varepsilon)|\tilde{u}_t|_1^2 + H(t)\\ &\leq M_1(\|\tilde{u}\|^2 + \beta\|\tilde{p}\|^2) + B(\|\tilde{u}\|)|\tilde{u}|_1^2 + G_1(t)\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}&M_1 = 1 + \frac{C[(m-2\delta)^2 + (1+\delta^2)(1+m^2)]}{\varepsilon\nu}|||u|||_{1,\infty}^2,\\ &B(\|\tilde{u}\|) = -\nu + \frac{C\tau(m-2\delta)^2N^2}{\varepsilon h}\|\tilde{u}\|^2,\\ &G_1(t) = (1 + \frac{\tau m^2}{4\varepsilon})(\|\tilde{f}\|^2 + \frac{1}{\beta}\|\tilde{g}\|^2).\end{aligned}$$

Now we choose the constants m and ε. Take $m = 2\theta$ and $r_0 \geq 0$ to be sufficiently small. If $\sigma > \frac{1}{2}$ and $\theta > \frac{\sigma}{2\sigma - 1}$, then we can take ε and r_0 to be so small that

$$2\theta \geq \max(1 + 2\varepsilon + r_0, \frac{2(\sigma + 2\varepsilon)}{2\sigma - 1}).$$

If $\sigma \leq \frac{\theta}{2\theta - 1}$, and

$$\nu\tau(c_d h^{-2} + \frac{4}{3}N^4) < \frac{2\theta - 1}{\sigma + \theta(1 - 2\sigma)}, \tag{4.6}$$

then we take ε and r_0 to be so small that

$$2\theta - 1 - 2\varepsilon - r_0 \geq \nu\tau(\sigma + \theta - 2\theta\sigma + 2\varepsilon)(c_d h^{-2} + \frac{4}{3}N^4).$$

By Lemma 3.1, we have in both cases that

$$\tau(m - 1 - 2\varepsilon)(\|\tilde{u}_t\|^2 + \beta\|\tilde{p}_t\|^2) + \nu\tau^2(m\sigma - \sigma - \frac{m}{2})|\tilde{u}_t|_1^2 \geq r_0\tau(\|\tilde{u}_t\|^2 + \beta\|\tilde{p}_t\|^2).$$

Thus we obtain from (4.5) that

$$\begin{aligned}&(\|\tilde{u}\|^2 + \beta\|\tilde{p}\|^2)_t + r_0\tau(\|\tilde{u}_t\|^2 + \beta\|\tilde{p}_t\|^2) + \tfrac{\nu}{2}|\tilde{u}|_1^2 + \nu\tau(\sigma + \tfrac{m}{2})(|\tilde{u}|_1^2)_t \\ &\leq M_1(\|\tilde{u}\|^2 + \beta\|\tilde{p}\|^2) + B(\|\tilde{u}\|)|\tilde{u}|_1^2 + G_1(t)\end{aligned} \tag{4.7}$$

Let

$$\begin{aligned} E(\tilde{u}, \tilde{p}, t) &= \|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2 + \tau \sum_{t' \in S_\tau, t' < t} \{r_0\tau(\|\tilde{u}_t(t')\|^2 + \beta\|\tilde{p}_t(t')\|^2) + \frac{\nu}{2}|\tilde{u}(t')|_1^2\}, \\ \rho(t) &= \|\tilde{u}(0)\|^2 + \beta\|\tilde{p}(0)\|^2 + \nu\tau(\sigma + \frac{m}{2})|\tilde{u}(0)|_1^2 + \tau \sum_{t' \in S_\tau, t' < t} G_1(t'). \end{aligned}$$

By summing (4.7) for all $t' \in S_\tau$ and $t' < t$, we obtain

$$E(\tilde{u}, \tilde{p}, t) \leq \rho(t) + \tau \sum_{t' \in S_\tau, t' < t} \{M_1 E(\tilde{u}, \tilde{p}, t') + B(\|\tilde{u}(t')\|)|\tilde{u}(t')|_1^2\}.$$

By Lemma 3.8, we have the following result.

Theorem 4.1. *Assume that*

(i) $\rho(T)e^{M_1 T} \leq \frac{\varepsilon\nu h}{C\tau N^2(m - 2\delta)^2}$.

In addition, either of the following two conditions is satisfied:

(ii) $\sigma > \frac{\theta}{2\theta - 1}$;

(iii) $\sigma \leq \frac{\theta}{2\theta - 1}$ *and* $\nu\tau(c_d h^{-2} + \frac{4}{3}N^4) < \frac{2\theta - 1}{\sigma + \theta(1 - 2\sigma)}$.

Then for all $t \in S_\tau$,

$$E(\tilde{u}, \tilde{p}, t) \leq \rho(t)e^{M_1 t}.$$

Next, we consider the convergence of Scheme (2.3). Let the pair (U,P) be the solution of (2.1). Let

$$U^*(t)=Q_{h,N}U(t),\quad P^*(t)=\tilde{P}_{h,N}P(t).$$

Then we have from (2.1) and (3.4) that

$$\begin{cases}(U_t^*,v)+J(U^*+\delta\tau U_t^*,U^*,v)+(\nabla(P^*+\theta\tau P_t^*),v)\\ \ +\nu(\nabla(U^*+\sigma\tau U_t^*),\nabla v)=(f,v)+\sum\limits_{l=1}^{7}E_l(v), & \forall v\in X_{h,N}^k(\Omega),\\ \beta(P_t^*,v)+(\nabla\cdot(U^*+\theta\tau U_t^*),v)=E_8(v), & \forall v\in Y_{h,N}^k(\Omega),\\ U^*(0)=Q_{h,N}U_0,\quad P^*(0)=\tilde{P}_{h,N}P_0,\end{cases} \tag{4.8}$$

where

$$\begin{aligned}&E_1(v)=(U_t^*-\partial_tU,v), && E_2(v)=J(U^*,U^*,v)-J(U,U,v),\\ &E_3(v)=\delta\tau J(U_t^*,U^*,v), && E_4(v)=(\nabla(P^*-P),v),\\ &E_5(v)=\theta\tau(\nabla P_t^*,v), && E_6(v)=\nu(\nabla(U^*-U),\nabla v),\\ &E_7(v)=\nu\sigma\tau(\nabla U_t^*,\nabla v), && E_8(v)=\beta(P_t^*,v).\end{aligned}$$

Let the pair (u,p) be the solution of (2.3). Define

$$\tilde{U}=u-U^*,\quad \tilde{P}=p-P^*.$$

By subtracting (4.8) from (2.3), we obtain that

$$\begin{cases}(\tilde{U}_t,v)+J(U^*+\delta\tau U_t^*,\tilde{U},v)+J(\tilde{U}+\delta\tau\tilde{U}_t,U^*+\tilde{U},v)\\ \ +\nu(\nabla(\tilde{U}+\sigma\tau\tilde{U}_t),\nabla v)+(\nabla(\tilde{P}+\theta\tau\tilde{P}_t),v)=-\sum\limits_{l=1}^{7}E_l(v), & \forall v\in X_{h,N}^k(\Omega),\\ \beta(\tilde{P}_t,v)+(\nabla\cdot(\tilde{U}+\theta\tau\tilde{U}_t),v)=-E_8(v), & \forall v\in Y_{h,N}^k(\Omega),\\ \tilde{U}(0)=\Pi_h^{k+1}Q_NU_0-Q_{h,N}U_0,\qquad \tilde{P}(0)=0.\end{cases}$$

We estimate $|E_l(v)|,\ l=1,2,\cdots,8$. First we have

$$\begin{aligned}|E_1(2\tilde{U}+m\tau\tilde{U}_t)| &\le \|\tilde{U}\|^2+\varepsilon\tau\|\tilde{U}_t\|^2+(1+\tfrac{\tau m^2}{4\varepsilon})\|U_t^*-\partial_tU\|^2,\\ |E_2(2\tilde{U}+m\tau\tilde{U}_t)| &\le |J(U,U^*-U,2\tilde{U}+m\tau\tilde{U}_t)|+|J(U^*-U,U^*,2\tilde{U}+m\tau\tilde{U}_t)|\\ &\le C(\|U\|_{1,\infty}+\|U^*\|_{1,\infty})|2\tilde{U}+m\tau\tilde{U}_t|_1\|U^*-U\|\\ &\le \varepsilon\nu|\tilde{U}|_1^2+\varepsilon\nu\tau^2|\tilde{U}_t|_1^2+\tfrac{C(1+m^2)}{\varepsilon\nu}(\|U\|_{1,\infty}^2+\|U^*\|_{1,\infty}^2)\|U^*-U\|^2.\end{aligned}$$

Next, it is not difficult to show that

$$\begin{aligned}\sum_{l=3}^{5}|E_l(2\tilde{U}+m\tau\tilde{U}_t)| &\le \varepsilon\nu|\tilde{U}|_1^2+\varepsilon\nu\tau^2|\tilde{U}_t|_1^2+\tfrac{C}{\varepsilon\nu}(1+\tfrac{m^2}{4})\{\delta^2\tau^2|U_t^*|_1^2\|U^*\|_\infty^2\\ &\quad +\|P^*-P\|^2+\theta^2\tau^2\|P_t^*\|^2\},\\ |E_7(2\tilde{U}+m\tau\tilde{U}_t)| &\le \nu\sigma\tau|U_t^*|_1|2\tilde{U}+m\tau\tilde{U}_t|_1\\ &\le \varepsilon\nu|\tilde{U}|_1^2+\varepsilon\nu\tau^2|\tilde{U}_t|_1^2+\tfrac{C\nu\sigma^2\tau^2}{\varepsilon}(1+\tfrac{m^2}{4})|U_t^*|_1^2,\end{aligned}$$

and

$$|E_8(2\tilde{P}+m\tau\tilde{P}_t)| \le \beta\|P_t^*\|\,\|2\tilde{P}+m\tau\tilde{P}_t\| \le \beta\|\tilde{P}\|^2+\varepsilon\beta\tau\|\tilde{P}_t\|^2+\beta(1+\tfrac{\tau m^2}{4\varepsilon})\|P_t^*\|^2.$$

The definitions of Q_N and Q_h imply that for any $v \in X_{h,N}^k(\Omega)$,

$$\begin{aligned}
&(\nabla(U^*-U),\nabla v) = \textstyle\int_\Omega \partial_x(Q_hQ_NU-U)\partial_x v\,\mathrm{d}x\mathrm{d}y + \int_\Omega \partial_y(Q_hQ_NU-U)\partial_y v\,\mathrm{d}x\mathrm{d}y\\
&= \textstyle\int_\Omega \partial_x(Q_h(Q_NU)-Q_NU)\partial_x v\,\mathrm{d}x\mathrm{d}y + \int_\Omega (Q_N(\partial_xU)-\partial_xU)\partial_x v\,\mathrm{d}x\mathrm{d}y\\
&\quad + \textstyle\int_\Omega \partial_y[Q_N(Q_hU)-Q_hU]\partial_y v\,\mathrm{d}x\mathrm{d}y + \int_\Omega [Q_h(\partial_yU)-\partial_yU]\partial_y v\,\mathrm{d}x\mathrm{d}y\\
&= -\sum_{l=1}^{M_h}\textstyle\int_{I_y}\{\int_{I_l}[Q_h(Q_NU)-Q_NU]\partial_{xx}v\,\mathrm{d}x\}\mathrm{d}y + \int_\Omega[Q_N(\partial_xU)-\partial_xU]\partial_x v\,\mathrm{d}x\mathrm{d}y\\
&\quad - \textstyle\int_{I_x}\{\int_{I_y}[Q_N(Q_hU)-Q_hU]\partial_{yy}v\,\mathrm{d}y\}\mathrm{d}x + \int_\Omega[Q_h(\partial_yU)-\partial_yU]\partial_y v\,\mathrm{d}x\mathrm{d}y\\
&= \textstyle\int_\Omega(Q_N(\partial_xU)-\partial_xU)\partial_x v\,\mathrm{d}x\mathrm{d}y + \int_\Omega(Q_h(\partial_yU)-\partial_yU)\partial_y v\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Finally

$$\begin{aligned}
&|E_6(2\tilde{U}+m\tau\tilde{U}_t)| \le \nu(\|Q_N(\partial_xU)-\partial_xU\|+\|Q_h(\partial_yU)-\partial_yU\|)|2\tilde{U}+m\tau\tilde{U}_t|_1\\
&\le \varepsilon\nu|\tilde{U}|_1^2+\varepsilon\nu\tau^2|\tilde{U}_t|_1^2+\tfrac{C\nu}{\varepsilon}(1+\tfrac{m^2}{4})[\|Q_N(\partial_xU_1)-\partial_xU_1\|+\|Q_h(\partial_yU_2)-\partial_yU_2\|]^2.
\end{aligned}$$

So far, we can obtain a conclusion similar to Theorem 4.1, but with

$$\tilde{\rho}(t) = \|\tilde{U}(0)\|^2+\nu\tau(\sigma+\frac{m}{2})|\tilde{U}(0)|_1^2+\tau\sum_{t'\in S_\tau,t'<t}G_2(t'),$$

where

$$\begin{aligned}
G_2(t) &= (1+\tfrac{\tau m^2}{4\varepsilon})(\|U_t^*-\partial_tU\|^2+\beta\|P_t^*\|^2)\\
&+\tfrac{C}{\varepsilon\nu}(1+\tfrac{m^2}{4})\{(\|U\|_{1,\infty}^2+\|U^*\|_{1,\infty}^2)\|U^*-U\|^2+\delta^2\tau^2|U_t^*|_1^2\|U^*\|_\infty^2\\
&+\|P^*-P\|^2+\theta^2\tau^2\|P_t^*\|^2+\nu^2\sigma^2\tau^2|U_t^*|_1^2\\
&+\nu^2(\|Q_N(\partial_xU)-\partial_xU\|+\|Q_h(\partial_yU)-\partial_yU\|)^2\}.
\end{aligned}$$

It means that if

$$\tilde{\rho}(T) = O\left(\frac{h}{\tau N^2}\right), \tag{4.9}$$

then we have for all $t \in S_\tau$

$$E(\tilde{U},\tilde{P},t) = O(\tilde{\rho}(t)).$$

Thus, in order to obtain the convergence, we only need to estimate the order of $\tilde{\rho}(t)$ and verify (4.9).

By Lemma 3.4–3.6 and Theorem 3.2.1 of [10], we have

$$\begin{aligned}
&\|\tilde{U}(0)\| = \|Q_N(\Pi_h^{k+1}-Q_h)U(0)\|\\
&\le \|(\Pi_h^{k+1}U(0)-U(0))+(U(0)-Q_hU(0))\|_{H^1(I_y,L^2(I_x))} \le Ch^{\bar{r}}\|U(0)\|_{H^1(I_y,H^{\bar{r}}(I_x))},\\
&|\tilde{U}(0)|_1 \le \|\partial_yQ_N(\Pi_h^{k+1}-Q_h)U(0)\|+\|Q_N\partial_x(\Pi_h^{k+1}-Q_h)U(0)\|\\
&\le Ch^{\bar{r}-1}\|U\|_{H^1(I_y,H^{\bar{r}}(I_x))},\\
&\|U^*-U\| \le C(h^{\bar{r}}+N^{-s})\|U\|_{M^{\bar{r},s}(\Omega)},
\end{aligned}$$

and

$$\|Q_N(\partial_x U_1) - \partial_x U_1\| + \|Q_h(\partial_y U_2) - \partial_y U_2\| \le C(h^{\bar{r}} + N^{-s})\|U\|_{M^{\bar{r},s}(\Omega)}.$$

Besides, we have from Lemma 3.3 that

$$\|P^* - P\| \le C(h^{\hat{r}} + N^{-s})\|P\|_{H^{\hat{r},s}(\Omega)}.$$

By using Lemma 3.7, we obtain

$$\|U^*\|_{1,\infty} \le C\|U\|_{A^{\alpha,1}(\Omega)}, \qquad \alpha > \frac{1}{2}.$$

On the other hand, we have from Lemma 3.6 that

$$\begin{aligned} |U_t^*|_1 &= \frac{1}{\tau}\Big|\int_t^{t+\tau} \partial_t U^*(t')\,\mathrm{d}t'\Big|_1 \le \tau^{-\frac{1}{2}}\Big(\int_t^{t+\tau} |\partial_t U^*(t')|_1^2\,\mathrm{d}t'\Big)^{\frac{1}{2}} \\ &\le \tau^{-\frac{1}{2}}\Big(\int_t^{t+\tau} |\partial_t U(t')|^2_{H^1(I_y,H^1(I_x))}\,\mathrm{d}t'\Big)^{\frac{1}{2}}. \\ \|P_t^*\| &\le \|P_t\| \le \tau^{-\frac{1}{2}}\Big(\int_t^{t+\tau} \|\partial_t P(t')\|^2\,\mathrm{d}t'\Big)^{\frac{1}{2}} \end{aligned}$$

Since

$$\partial_t U(t) - U_t(t) = -\frac{1}{\tau}\int_t^{t+\tau} (t+\tau-\xi)\frac{\partial^2 U(\xi)}{\partial \xi^2}\,\mathrm{d}\xi.$$

Then

$$\begin{aligned} \|U_t^*(t) - \partial_t U(t)\| &\le \|U_t(t) - U_t^*(t)\| + \|U_t(t) - \partial_t U(t)\| \\ &\le C\tau^{-\frac{1}{2}}(h^{\bar{r}} + N^{-s})\Big[\int_t^{t+\tau} \|\partial_t U(t')\|^2_{M^{\bar{r},s}(\Omega)}\,\mathrm{d}t'\Big]^{\frac{1}{2}} \\ &\quad + C\tau^{\frac{1}{2}}\Big[\int_t^{t+\tau} \Big\|\frac{\partial^2 U}{\partial t^2}(t')\Big\|^2\,\mathrm{d}t'\Big]^{\frac{1}{2}}. \end{aligned}$$

Thus we have from the above estimates that

$$\tilde{\rho}(t) \le M_2(\tau^2 + \beta + \tau h^{2\bar{r}-2} + h^{2\hat{r}} + N^{-2s})$$

where M_2 is a positive constant depending only on ν and the norms of U and P in the spaces mentioned in the above. Finally by an argument similar to the proof of Theorem 1, we have the following result.

Theorem 4.2. *Assume that*
(i) $Nh \le C$, *and* $\tau N^2 h^{-1} \le C$;
(ii) *condition (ii) or (iii) of Theorem 4.1 holds;*
(iii) *For* $r, s \ge 1$ *and* $\alpha > \frac{1}{2}$,

$$\begin{aligned} U \in\ & C(0,T;(H_0^1(\Omega))^2 \textstyle\bigcap (M^{r,s}(\Omega))^2 \bigcap (A^{\alpha,1}(\Omega))^2 \bigcap (W^{1,\infty}(\Omega))^2), \\ & \frac{\partial U}{\partial t} \in L^2(0,T;(M^{r,s}(\Omega))^2), \\ & \frac{\partial^2 U}{\partial t^2} \in L^2(0,T;(L^2(\Omega))^2), \\ P \in\ & C(0,T;H^{r,s}(\Omega)), \quad \frac{\partial P}{\partial t} \in L^2(0,T;L^2(\Omega)). \end{aligned}$$

Then there exists a positive constant M_3 depending only on ν and the norms of U and P in the spaces mentioned in the above, such that for all $t \le T$;

$$||U(t) - u(t)||^2 \le M_3(\beta + \tau^2 + \tau h^{2\tilde{r}-2} + h^{2\hat{r}} + N^{-2s}).$$

5. The Numerical Results

In this section, we examine the numerical performances. We choose the function f in such a way that the solution of (2.1) is of the form

$$\begin{aligned} U_1 &= Ae^{Bt}x(x-1)(2x-1)(y^2-1)^2, \\ U_2 &= -2Ae^{Bt}x^2(x-1)^2(y^3-y), \\ P &= 4Ae^{2Bt}(2x^3-3x^2+0.5)(y^3-3y). \end{aligned}$$

We use the Legendre spectral-finite element scheme(LSFM) (2.3) with $k = \delta = 0$ and $\theta = \sigma = 1$. Besides, we take $\nu = 0.001$ and $\tau = 0.005$. I_x is subdivided uniformly. For comparison, we also solve (2.1) by finite element scheme(FEM), in which Ω is divided into MN congruent small rectangles, each with the length $h_x = 1/M$ and the width $h_y = 2/N$. The finite element scheme is constructed similarly to (2.3) by artificial compression and the trial spaces for u and p are piecewise biquadratic and piecewise constant separately.

For describing the errors of numerical solutions, let

$$\begin{aligned} \hat{I}_x &= \{x_j \ / \ x_j = jh_x \quad 0 \le j \le M\}, \\ \hat{I}_y &= \{y_j \ / \ y_j = -1 + jh_y, \quad 0 \le j \le N\}. \end{aligned}$$

and

$$E(U(t)) = \left(\frac{\sum_{i=1}^{2} \sum_{x\in\hat{I}_x} \sum_{y\in\hat{I}_y} | \, u_i(x,y,t) - U_i(x,y,t) \, |^2}{\sum_{i=1}^{2} \sum_{x\in\hat{I}_x} \sum_{y\in\hat{I}_y} | \, U_i(x,y,t) \, |^2} \right)^{\frac{1}{2}}$$

$$E(P(t)) = \left(\frac{\sum_{x\in\hat{I}_x} \sum_{y\in\hat{I}_y} | \, p(x,y,t) - P(x,y,t) \, |^2}{\sum_{x\in\hat{I}_x} \sum_{y\in\hat{I}_y} | \, P(x,y,t) \, |^2} \right)^{\frac{1}{2}}.$$

According to Theorem 4.2, we would choose the parameter $\beta = O(\tau^2)$ so that the convergence order is not be lowered. But if β is too small, then the stability may be affected. Indeed, we can see the fact from Theorem 4.1.

The numerical results are shown in Table I and Table II. We find that scheme LSFM gives better results than scheme FEM.

Table I. $A = 0.2, \quad B = 0.1, \quad \beta = 0.001$

	Scheme LSFM, $M = 10,\ N = 4$		Scheme FEM, $M = N = 10$	
t	$E(U(t))$	$E(P(t))$	$E(U(t))$	$E(P(t))$
0.5	0.1268E-2	0.1508E-2	0.2726E-2	0.8294E-2
1.0	0.2427E-2	0.1584E-2	0.5291E-2	0.8717E-2
1.5	0.3445E-2	0.1668E-2	0.7721E-2	0.9165E-2
2.0	0.4360E-2	0.1751E-2	0.1003E-1	0.1012E-1
2.5	0.5183E-2	0.1841E-2	0.1224E-1	0.1063E-1

Table II. $A = 0.2, \quad B = 0.1, \quad \beta = 0.0001$

	Scheme LSFM, $M = 10,\ N = 4$		Scheme FEM, $M = N = 10$	
t	$E(U(t))$	$E(P(t))$	$E(U(t))$	$E(P(t))$
0.5	0.1279E-2	0.1585E-2	0.2975E-2	0.8719E-2
1.0	0.2490E-2	0.1750E-2	0.5392E-2	0.9133E-2
1.5	0.3631E-2	0.1935E-2	0.7822E-2	0.9690E-2
2.0	0.4702E-2	0.2139E-2	0.1034E-1	0.1053E-1
2.5	0.5713E-2	0.2363E-2	0.1281E-1	0.1110E-1

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