Journal of Computational Mathematics, Vol.17, No.5, 1999, 475–494.

INTERIOR ERROR ESTIMATES FOR NONCONFORMING FINITE ELEMENT METHODS OF THE STOKES EQUATIONS*

Xiao-bo Liu
(*Department of Mathematics and Computer Science Clarkson University Box 5815, Potsdam, NY 13699-5815, USA*)

Abstract

Interior error estimates are derived for nonconforming stable mixed finite element discretizations of the stationary Stokes equations. As an application, interior convergences of difference quotients of the finite element solution are obtained for the derivatives of the exact solution when the mesh satisfies some translation invariant condition. For the linear element, it is proved that the average of the gradients of the finite element solution at the midpoint of two interior adjacent triangles approximates the gradient of the exact solution quadratically.

Key words: Interior error estimates, Nonconforming element, Stokes equations.

1. Introduction

Interior error estimates for finite element discretizations (conforming) were first introduced by Nitsche and Schatz[14] for second order scalar elliptic equations in 1974. They proved that the local accuracy of the finite element approximation is bounded in terms of two factors: the local approximability of the exact solution by the finite element space and the global approximability measured in an arbitrarily weak Sobolev norm on a slightly larger domain. Since then, interior estimates of this nature have been obtained by Douglas, Jr. and Milner for mixed methods of the second order scalar elliptic equations[8], Douglas, Jr., Gupta, and Li for the hybrid method[7], by Gastaldi for a family of elements for the Reissner-Mindlin plate model[12], by Arnold and Liu for conforming finite element methods for the Stokes equations[1], and by Liu for nonconforming methods for the second order elliptic equations[13]. For a comprehensive review on this subject, see [17].

Recently, some quite interesting applications of interior estimates have been found in the areas of a posteriori error analysis and adaptive mesh refinement. In 1988 Eriksson and Johnson[11] introduced two a posteriori error estimators based on local difference quotients of the numerical solution. Their analysis was based on the interior convergence theory in [14] and [15]. In 1991, Babuška and Rodríguez[2] studied the estimators of Zhu and Zienkiewicz[19], [20] by using the interior estimate results of Bramble and Schatz[15]. For other applications in this direction, please refer to [9], [10]

* Received November 6, 1995.

and [3]. Through these investigations, it is now widely believed that the asymptotic exactness of a posteriori estimators essentially depends on some kind of superapproximation property of the finite element method. Interior error estimates, however, offer a standard approach to derive interior superconvergences.

The aim of this paper is to establish interior error estimates for nonconforming finite element approximations to solutions of the Stokes equations. Note that nonconforming methods are attractive for the Stokes problems for two reasons: (1) the inf-sup condition is easy to satisfy; (2) divergence-free nodal bases can be constructed. In addition, since the pressure can be eliminated first (when discontinuous functions are used to approximate the pressure), the velocity can be found through solving a positive system and thereafter, some preconditioned multigrid methods may be incorporated for constructing fast solvers.

The method used here and the structure of this paper closely follows that in [1]. Section 2 presents notations and preliminaries. Section 3 introduces hypotheses for the finite element spaces, which actually apply for both nonconforming and conforming methods. In Section 4, we introduce the interior equations and derive some basic properties of their solutions. Section 5 gives the precise statement of our main result and its proof. In Section 6 we prove interior convergences of difference quotients of the finite element solution to the derivatives of the exact solution when the finite element space is defined over meshes with certain translation invariant property. An interior superconvergence is obtained as an example application.

2. Notations and Preliminaries

Let Ω denote a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ its boundary. We shall use the usual standard L^2-based Sobolev spaces $H^m = H^m(\Omega)$, $m \in \mathbb{Z}$, with the norm $\|\cdot\|_{m,\Omega}$. Recall that for $m \in \mathbb{N}$, H^{-m} denotes the normed dual of $\overset{\circ}{H}{}^m$, the closure of $C_0^\infty(\Omega)$ in H^m. We use the notation $(\cdot,\cdot)$ for both the $L^2(\Omega)$-innerproduct and its extension to a pairing of $\overset{\circ}{H}{}^m$ and H^{-m}. If $\overline{\Omega} = \bigcup_j \overline{\Omega}_j$ for some disjoint open sets Ω_j, then let $H_h^m(\Omega) = \{u \in L^2(\Omega)$ and $u|_{\Omega_j} \in H^m(\Omega_j)$, for all $j\}$ with the norm $\|u\|_{m,\Omega}^h = \Big(\sum_j \|u\|_{m,\Omega_j}^2\Big)^{1/2}$. If X is any subspace of L^2, then $\hat{X}$ denotes the subspace of elements with average value zero. We use boldface type to denote 2-vector-valued functions, operators whose values are ector-valued or tensor-valued functions, and spaces of vector-valued functions. This is illustrated in the definitions of the following standard differential operators:

$$\operatorname{div}\phi = \partial\phi_1/\partial x + \partial\phi_2/\partial y,\ \mathbf{grad}\ p = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix},\ \mathbf{grad}\ \phi = \begin{pmatrix} \partial\phi_1/\partial x & \partial\phi_1/\partial y \\ \partial\phi_2/\partial x & \partial\phi_2/\partial y \end{pmatrix}.$$

For any function ϕ that is differentiable on each Ω_i where $\overline{\Omega} = \bigcup_i \overline{\Omega}_i$, a family of disjoint open sets Ω_i, we define the piecewise version (with notation div_h) of its divergence to be the function obtained by computing $\operatorname{div}\phi$ element-wise. The piecewise version of the gradient operator can be defined similarly and is denoted by grad_h.

The letter C denotes a generic constant, not necessarily the same in each occurrence, but always independent of the meshsize parameter h.

Let G be an open subset of Ω and s an integer. If $\phi \in H^s(G)$, $\psi \in H^{-s}(G)$, and $\omega \in C_0^\infty(G)$, then

$$|(\omega\phi, \psi)| \leq C\|\phi\|_{s,G}\|\psi\|_{-s,G},$$

with the constant C depending only on G, ω, and s. For $\Phi \in H^s(G)$, $\Psi \in H^{-s+1}(G)$ define

$$R(\omega, \Phi, \Psi) = (\Phi(\mathbf{grad}\,\omega)^t,\ \mathbf{grad}\,\Psi) - (\mathbf{grad}\,\Phi, \Psi(\mathbf{grad}\,\Omega)^t). \tag{2.1}$$

Then

$$|R(\omega, \Phi, \Psi)| \leq C\|\Phi\|_{s,G}\|\Psi\|_{-s+1,G}. \tag{2.2}$$

If, moreover, $\Psi \in H^{-s+2}$, we have the identity

$$(\mathbf{grad}\,(\omega\Phi),\ \mathbf{grad}\,\Psi) = (\mathbf{grad}\,\Phi, \mathbf{grad}\,(\omega\Psi)) + R(\omega, \Phi, \Psi).$$

The following lemma states the well-posedness and regularity of the Dirichlet problem for the generalized Stokes equations on smooth domains. (Because we are interested in interior estimates we really only need this results when the domain is a disk.) For the proof see [16, Chapter I, § 2].

Lemma 2.1. *Let G be a smoothly bounded plane domain and m a nonnegative integer. Then for any given functions $F \in H^{m-1}(G)$, $K \in H^m(G) \cap \hat{L}^2(G)$, there exist uniquely determined functions*

$$\phi \in H^{m+1}(G) \cap \overset{\circ}{\mathbf{H}}{}^1(G),\ p \in H^m(G) \cap \hat{L}^2(G),$$

such that

$$\begin{aligned} -\Delta\phi - \mathbf{grad}p &= \mathbf{F}, \\ div\,\phi &= K. \end{aligned}$$

Moreover,

$$\|\phi\|_{m+1,G} + \|p\|_{m,G} \leq C(\|\mathbf{F}\|_{m-1,G} + \|K\|_{m,G}),$$

where the constant C is independent of F and K.

3. Nonconforming Finite Element Spaces

In this section we collect assumptions on the nonconforming mixed finite element spaces that will be used in the paper.

Let $\Omega \subset \mathbb{R}^2$ be the bounded open set on which we solve the Stokes equations and let h denote a mesh size parameter. We denote by $\mathbf{V}_h$ the finite element space that is used to approximate the velocity, and by W_h the finite element subspace of $L^2(\Omega)$ that is for the pressure. Let Z_h denote the finite element space of continuous piecewise linear functions (for triangular elements) or bilinear functions (for rectangular elements), and

Y_h the space of piecewise constants or continuous linear functions. Remember that $\boldsymbol{V}_h = V_h \times V_h$ and $\boldsymbol{Z}_h = Z_h \times Z_h$. For $\Omega_0 \subseteq \Omega$, define

$$\boldsymbol{V}_h(\Omega_0) = \{\phi|_{\Omega_0} | \phi \in \boldsymbol{V}_h\}, \ \overset{\circ}{\boldsymbol{V}}_h(\Omega_0) = \{\phi \in \boldsymbol{V}_h | \mathrm{supp}\phi \subseteq \overline{\Omega}_0\}$$

Sets $W_h(\Omega_0)$, $\overset{\circ}{W}_h(\Omega_0)$, $\boldsymbol{Z}_h(\Omega_0)$, $\overset{\circ}{Z}_h(\Omega_0)$, and $Y_h(\Omega)_0$ can be defined similarly.

If $G_h \subseteq \Omega$ is a union of elements, let

$$\overline{\boldsymbol{V}}_h(G_h) = \{\phi | \phi \in \boldsymbol{V}_h(G_h) \text{ and vanishes at the nodes on } \partial G_h\}.$$

Let G_0 and G be concentric open disks with $G_0 \Subset G \Subset \Omega$, i.e, $\overline{G}_0 \subset G$ and $\overline{G} \subset \Omega$. We assume that there exists a positive real number h_0 such that for $h \in (0, h_0]$, the following properties hold.

A1. Approximation property. We will assume that $\boldsymbol{V}_h$ contains $\boldsymbol{Z}_h$ and W_h contains Y_h. Consequently,

(1) If $\phi \in \boldsymbol{H}^2(G)$, then there exists a continuous function $\phi^I \in \boldsymbol{V}_h$ such that

$$\|\phi - \phi^I\|_{1,G} \leq Ch|\phi|_{2,G}.$$

(2) If $p \in H^1(G)$, then there exists a $p^I \in W_h$, such that

$$\|p - p^I\|_{0,G} \leq Ch\|p\|_{1,G}.$$

Furthermore, if ϕ and p vanish on $G \backslash \overline{G}_0$, respectively, then ϕ^I and p^I can be chosen to vanish on $\Omega \backslash \overline{G}$.

A2. Superapproximation property. Let $\omega \in C_0^\infty(G)$, $\phi \in \boldsymbol{V}_h$, and $p \in W_h$. Then there exists $\psi \in \boldsymbol{V}_h(G)$ and $q \in \overset{\circ}{W}_h(G)$, such that

$$\|\omega\phi - \psi\|_{1,\Omega}^h \leq Ch\|\phi\|_{1,G}^h,$$
$$\|\omega p - q\|_{0,\Omega} \leq Ch\|p\|_{0,G},$$

where C depends only on G and ω.

A3. Stability property. There is a positive constant γ, such that for all $h \in (0, h_0]$ there is a domain G_h, $G_0 \Subset G_h \Subset G$ for which

$$\inf_{\substack{p \in \hat{W}_h(G_h) \\ p \neq 0}} \sup_{\substack{\phi \in \overline{\boldsymbol{V}}_h(G_h) \\ \phi \neq 0}} \frac{(\mathrm{div}_h \phi, p)_{G_h}}{\|\phi\|_{1,G_h}^h \|p\|_{0,G_h}} \geq \gamma$$

A4. Inverse property. For the set G_h in A3 and each nonnegative integer m ther is a constant C for which

$$\|\phi\|_{1,G_h}^h \leq Ch^{-1-m}\|\phi\|_{-m,G_h}, \quad \text{for all } \phi \in \boldsymbol{V}_h,$$
$$\|p\|_{0,G_h} \leq Ch^{-m}\|p\|_{-m,G_h}, \quad \text{for all } p \in W_h.$$

A5. Consistency property. Let G_h be the set in A3. Then

$$\sum_{T \subset G_h} \int_{\partial T} \psi \cdot \boldsymbol{n} ds = 0, \quad \text{for all} \psi \in \overline{\boldsymbol{V}}_h(G_h) \tag{3.1}$$

Moreover, if $\phi \in \overset{\circ}{H}{}^1(G_h)$ and $\psi \in V_h$ or $\phi \in H^1(G_h)$ and $\psi \in \overline{V}_h(G_h)$, then

$$\Big|\sum_{T\subset G_h}\int_{\partial T}\phi\cdot\mathbf{n}\psi ds\Big| \le Ch^\alpha(\|\phi\|_{0,G_h}+\|\mathrm{div}\,\phi\|_{0,G_h})\|\psi\|^h_{1,G_h}, \tag{3.2}$$

and

$$\Big|\sum_{T\subset G_h}\int_{\partial T}pn\cdot\psi ds\Big| \le Ch^\alpha\|p\|_{1,G_h}\|\psi\|^h_{1,G_h}$$

for $p \in \overset{\circ}{H}{}^1(G_h)$ and $\psi \in \boldsymbol{V}_h(G_h)$. Here α is a real positive number, the constant C depends only on the minimal angle of elements in G_h, and $\mathbf{n} = (n_1, n_2)^t$ is the outward normal direction of ∂T.

When $G_h = \Omega$, property A3 is the standard stability condition for Stokes elements. It will usually hold as long as G_h is chosen to be a union of elements. The following result will be used from time to time to construct local projections.

Lemma 3.1. *Let G_h be a subdomain for which the stability inequality in A3 holds. Then for $\phi \in \mathbf{H}^1_h(G_h)$ and $p \in L^2(G_h)$, there exist unique $\pi\phi \in \overline{\mathbf{V}}_h(G_h)$, and $\pi p \in \hat{W}_h(G_h)$ such that*

$$(\mathbf{grad}_h(\phi-\pi\phi),\mathbf{grad}_h\psi)-(div_h\psi,p-\pi p)=0,\quad \textit{for all } \psi\in\overline{\mathbf{V}}_h(G_h), \tag{3.3}$$

$$(div_h(\phi-\pi\phi),q)=0,\quad \textit{for all } q\in\hat{W}_h(G_h). \tag{3.4}$$

Moreover,

$$\|\phi-\pi\phi\|^h_{1,G_h}+\Big\|p-\pi p-\int_{G_h}pdx\Big\|_{0,G_h}\le C\Big(\int_{\psi\in\hat{\mathbf{V}}_h(G_h)}\|\phi-\psi\|^h_{1,G_h}+\int_{q\in W^h(G_h)}\|p-q\|_{0,G_h}\Big)$$

In addition, the function πp can be found with the property $\int_{G_h}(p-\pi p)dx = 0$ and the space $\hat{W}_h(G_h)$ in (3.4) can be replaced by $W_h(G_h)$ if the function ϕ is in $\overset{\circ}{\mathbf{H}}{}^1(G_h)$ and (3.1) in A5 holds.

Proof. The unique existence is guaranted by Prop 2.15 in [5]. The estimate can be obtained by using a similar argument as in Prop 2.16 of [5].

For conforming methods, $\sum_{T\in G_h}\int_{\partial T}\phi\psi n_i ds$ equals zero. So A5 can be considered as a measurement on the degree of continuity of the finite element space. It is well-known that the Crouzeix-Raviart family elements[6] satisfy (3.2) with $\alpha = 1$.

The superapproximation property is discussed as Assumptions 7.1 and 9.1 in [17] for conforming elements, but the arguments can be carried over to most nonconforming ones. The inverse inequality property for the continous elements is well-known[17]. A proof that holds for both conforming and nonconforming elements can be found in [13]. Many finite element spaces are known to have the superapproximation property. In particular, it was verified in [4] for Lagrange and Hermite elements.

We need two technical results for the analysis in Section 5.

Lemma 3.2. *Let G_h be a union of elements, $p \in L^2(G_h)$, and $\phi^I \in \bar{V}_h(G_h)$. Then there is a constant C, independent of h, p, and ϕ^I such that*

$$(p, \phi^I)_{G_h} \leq C(h^\alpha \|p\|_{0,G_h} + \|p\|_{-1,G_h})\|\phi^I\|^h_{1,G_h}, \tag{3.5}$$

where the constant α is as in A5.

Proof. If we can prove that for any $\phi^I \in \bar{V}_h(G_h)$, there exists a function $\phi \in \overset{\circ}{H}(G_h)$ with the properties:

$$\|\phi\|_{1,G_h} \leq C\|\phi^I\|^h_{1,G_h}, \tag{3.6}$$

$$\|\phi - \phi^I\|_{0,G)h} \leq Ch^\alpha \|\phi^I\|^h_{1,G_h}. \tag{3.7}$$

Then a straighforward computation yields (3.5):

$$\begin{aligned}|(p, \phi^I)_{G_h}| \leq& |(p, \phi^I - \phi)_{G_h}| + |(p, \phi)_{G_h}| \leq C(h^\alpha \|p\|_{0,G_h}\|\phi^I\|^h_{1,G_h} + \|p\|_{-1,G_h}\|\phi\|_{1,G_h}) \\ \leq& C(h^\alpha \|p\|_{0,G_h} + \|p\|_{-1,G_h})\|\phi^I\|^h_{1,G_h}.\end{aligned}$$

To prove the existence of such a function, consider a variational problem: find a function $\phi \in \overset{\circ}{H}{}^1(G_h)$ such that

$$(\mathbf{grad}\phi, \mathbf{grad}\psi) = (\mathbf{grad}_h\phi^I, \mathbf{grad}\psi), \quad \text{for all} \psi \in \overset{\circ}{H}{}^1(G_h). \tag{3.8}$$

Obviously this is uniquely solvable. Moreover,

$$\|\mathbf{grad}\phi\|_{0,G_h} \leq \|\mathbf{grad}_h\phi^I\|_{0,G_h}.$$

So (3.6) is satisfied. To prove (3.7), note that

$$\|\phi - \phi^I\|_{0,G_h} = \sup_{\substack{K \in L^2(G_h) \\ K \neq 0}} \frac{(\phi - \phi^I, K)_{G_h}}{\|K\|_{0,G_h}}.$$

For any $K \in L^2(G_h)$, consider the boundary value problem:

$$-\Delta\Phi = K, \quad \text{in } G_h, \quad \Phi = 0, \quad \text{on } \partial G_h.$$

It is easily seen that the solution $\Phi \in \overset{\circ}{H}{}^1(G_h)$ and $\Delta\Phi \in L^2(G_h)$ with the estimate

$$\|\Phi\|_{1,G_h} + \|\Delta\Phi\|_{0,G_h} \leq C\|K\|_{0,G_h}. \tag{3.9}$$

Applying integration by parts yields

$$\begin{aligned}(\phi - \phi^I, K)_{G_h} =& \sum_{T \subset G_h} \left\{(\mathbf{grad}(\phi - \phi^I), \mathbf{grad}\Phi)_T - \int_{\partial T}(\phi - \phi^I)\mathbf{n} \cdot \mathbf{grad}\Phi \mathrm{ds}\right\} \\ =& \sum_{T \subset G_h} \int_{\partial T} \phi^I \mathbf{n} \cdot \mathbf{grad}\Phi \mathrm{ds},\end{aligned}$$

where we use (3.8). Then, by (3.2) and (3.9) we obtain

$$|(\phi-\phi^I,K)_{G_h}|\leq Ch^\alpha(\|\mathbf{grad}\Phi\|_{0,G_h}+\|\Delta\Phi\|_{0,G_h})\|\phi^I\|^h_{1,G_h}\leq Ch^\alpha\|K\|_{0,G_h}\|\phi^I\|^h_{1,G_h},$$

which results in (3.7).

Lemma 3.3. *Let G be a disk, $G_h, G_h \Subset G$, be a union of elements, and $\omega \in C_0^\infty(G_h)$. Further assume that $\phi\in\bar{V}_h(G_h)$ and $\psi \in V_h$. Then we have*

$$|R(\omega,\phi,\psi)|\leq C(\|\psi\|_{0,G_h}+h^\alpha\|\psi\|^h_{1,G)h})\|\phi\|^h_{1,G_h}. \tag{3.10}$$

Proof. By definition (Ref. (2.1))

$$R(\omega,\phi,\psi)=(\phi(\mathbf{grad}\omega)^t,\mathbf{grad}_h,\psi)-(\mathbf{grad}_h\phi,\psi(\mathbf{grad}\omega)^t),$$

so it is enough to prove that the first tem above is bounded by the right hand side of (3.10). From the proof of Lemma 3.2, there exists a function $\phi^0\in\overset{\circ}{H}{}^1(G_h)$ with the properties

$$\|\phi^0\|_{1,G_h}\leq C\|\phi\|^h_{1,G_h},\quad \|\phi-\phi^0\|_{0,G)h}\leq Ch^\alpha\|\phi\|^h_{1,G_h}. \tag{3.11}$$

Obviously

$$(\phi(\mathbf{grad}\omega)^t,\mathbf{grad}_h\psi)=((\phi-\phi^0)(\mathbf{grad}\omega)^t,\mathbf{grad}_h\psi)+(\phi^0(\mathbf{grad}\omega)^t,\mathbf{grad}_h\psi) \tag{3.12}$$

and

$$|((\phi-\phi^0)(\mathbf{grad}\omega)^t,\mathbf{grad}_h\psi)|\leq Ch^\alpha\|\phi\|^h_{1,G_h}\|\psi\|^h_{1,G_h}. \tag{3.13}$$

In addition,

$$\begin{aligned}(\phi^0(\mathbf{grad}\omega)^t,\mathbf{grad}_h\psi)=&-\sum_{T\subset G_h}\Big\{(\mathrm{div}(\phi^0(\mathbf{grad}\omega)^t),\psi)_T-\int_T n^t\phi^0(\mathbf{grad}\omega)^t\psi\mathrm{ds}\Big\}\\ \leq& C\|\phi^0\|_{1,G_h}(\|\psi\|_{0,G_h}+h^\alpha\|\psi\|^h_{1,G_h}),\end{aligned} \tag{3.14}$$

where we used (3.2) in the last step. Combining (3.11)–(3.14) yields the desired result.

Before we get to the next section, we introduce some notations. Let $\boldsymbol{L}$ and Q be liner functionals on the space $\overset{\circ}{\boldsymbol{V}}_h(G)$ and $W_h(G)$, respectively, we define

$$\|\boldsymbol{L}\|_{-1,G}=\sup_{\substack{\psi\in\overset{\circ}{\mathbf{V}}_h(G)\\ \psi\neq0}}\frac{|\boldsymbol{L}(\psi)|}{\|\psi\|^h_{1,G}}$$

and

$$\|Q\|_{0,G}=\sup_{\substack{q\in\overset{\circ}{W}_h(G)\\ q\neq0}}\frac{|Q(q)|}{\|q\|_{0,G}}.$$

4. Interior Duality Estimates

Let $(\phi,p)\in\boldsymbol{H}^1\times L^2$ be some solution to the generalized Stokes equations

$$-\Delta\phi+\mathbf{grad}p=\boldsymbol{F},$$

$$\text{div } \phi = K.$$

Regardless of the boundary conditions used to specify the particular solution, (ϕ, p) satisfies

$$(\mathbf{grad}\phi, \mathbf{grad}\psi) - (\text{div}\psi, p) = (\boldsymbol{F}, \psi), \quad \text{for all } \psi \in \overset{\circ}{\boldsymbol{H}}{}^1,$$
$$(\text{div}\phi, q) = (K, q), \quad \text{for all } q \in L^2.$$

Similarly, regardless of the particular boundary conditions, the finite element solution $(\phi_h, p_h) \in \boldsymbol{V}_h \times W_h$ satisfies

$$(\mathbf{grad}_p\phi_h, \mathbf{grad}_h\psi) - (\text{div}_h\psi, p_h) = (\boldsymbol{F}, \psi), \quad \text{for all } \psi \in \overset{\circ}{\boldsymbol{V}}_h,$$
$$(\text{div}_h\phi_h, q) = (K, q), \quad \text{for all } q \in W_h.$$

Applying integration by parts yields

$$(\mathbf{grad}_h(\phi - \phi_h), \mathbf{grad}_h\psi) - (\text{div}_h, \psi, p - p_h)$$
$$= \sum_{T\subset\Omega} \int_{\partial T} \mathbf{n}^t(\mathbf{grad}\phi - pI)\psi \text{ds}, \quad \text{for all } \psi \in \overset{\circ}{\boldsymbol{V}}_h \tag{4.1}$$
$$(\text{div}_h(\phi - \phi_h), q) = 0, \quad \text{for all } q \in W_h, \tag{4.2}$$

where I is the two by two identity matrix. The interior error analysis only depends on the above interior discretization equations.

Our goal here is to estimate function satisfying (4.1) and (4.2). We will use the same duality techniques as in [1] and [14]. To simplify notations, we consider a pair of functions $(\phi, p) \in (\boldsymbol{H}^1 + \boldsymbol{V}_h) \times L^2$ that satisfies

$$(\mathbf{grad}_h\phi, \mathbf{grad}_h\psi) - (\text{div}_h, \psi, p) = \boldsymbol{L}(\psi), \quad \text{for all } \psi \in \overset{\circ}{\boldsymbol{V}}_h \tag{4.3}$$
$$(\text{div}_h\phi, q) = Q(q), \quad \text{for all } q \in W_h, \tag{4.4}$$

for some given linear functionals $\boldsymbol{L}$ and Q, which may represent consistency errors out of using nonconforming elements or numerical integrations.

Theorem 4.1. *Let $G_0 \Subset G$ be concentric open disks with closures contained in Ω and s an arbitrary nonnegative integer. Then there exists a constant C such that if $(\phi, p) \in (\boldsymbol{H}^1 + \boldsymbol{V}_h) \times L^2$ satisfies* (4.3) *and* (4.4)*, we have*

$$\|\phi\|_{0,G_0} + \|p\|_{-1,G_0} \le C(h^\alpha\|\phi\|_{1,G}^h + h^\alpha\|p\|_{0,G} + \|\phi\|_{-s,G}$$
$$+ \|p\|_{-1-s,G} + \|\boldsymbol{L}\|_{-1,G} + \|Q\|_{0,G}), \tag{4.5}$$

where α is defined in A5. Moreover, if $\boldsymbol{L}(\psi) = 0$ for all $\psi \in \overset{\circ}{Z}_h(G)$ and $Q(q) = 0$ for any $q \in Y_h$, then

$$\|\phi\|_{0,G_0} + \|p\|_{-1,G_0} \le C(h^\alpha\|\phi\|_{1,G}^h + h^\alpha\|p\|_{0,G} + \|\phi\|_{-s,G} + \|p\|_{-1-s,G}). \tag{4.6}$$

In order to prove the theorem we first establish a lemma.

Lemma 4.2. *Under the hypotheses of Theorem* 4.1, *there exists a constant* C *for which*

$$\begin{aligned}\|\phi\|_{-s,G_0}+\|p\|_{-s-1,G_0}\leq & C(h^\alpha\|\phi\|^h_{1,G}+h^\alpha\|p\|_{0,G}+\|\phi\|_{-s-1,G}\\ &+\|p\|_{-s-2,G}+\|\boldsymbol{L}\|_{-1,G}+\|Q\|_{0,G}).\end{aligned}$$

Moreover, the last two terms can be taken away if $\boldsymbol{L}(\psi)=0$ *for any* $\psi\in Z_h$ *and* $Q(q)=0$ *for any* $q\in Y_h$.

Proof. Let G' be a disk such that $G_0\Subset G'\subset G$. Choose a function $\omega\in C_0^\infty(G')$ which is identically 1 on G_0. Also choose a function $\delta\in C_0^\infty(G_0)$ with integral 1. Then

$$\|p\|_{-s-1,G_0}\leq\|\omega p\|_{-s-1,G}=\sup_{\substack{g\in \overset{\circ}{H}{}^{s+1}(G)\\ g\neq 0}}\frac{(\omega p,g)}{\|g\|_{s+1,G}}. \tag{4.7}$$

Now

$$(\omega p,g)=\Big(\omega p,g-\delta\int_G g\mathrm{dx}\Big)+(\omega p,\delta)\int_G g\mathrm{dx}$$

and clearly

$$\Big|(\omega p,\delta)\int_G g\mathrm{dx}\Big|\leq C\|p\|_{-s-2,G}\|g\|_{0,G}.$$

Since $g-\delta\int_G d\mathrm{dx}\in H^{s+1}(G)\cap\hat{L}^2(G)$ it follows from Lemma 2.1 that exist $\Phi\in\boldsymbol{H}^{s+2}(G)\cap\overset{\circ}{\boldsymbol{H}}{}^1(G)$ and $P\in H^{s+1}(G)\cap\hat{L}^2(G)$ such that

$$-\Delta\Phi+\mathbf{grad}P=0, \tag{4.8}$$

$$\mathrm{div}\Phi=g-\delta\int_G g\mathrm{dx}. \tag{4.9}$$

Furthermore,

$$\|\Phi\|_{s+2,G}+\|P\|_{s+1,G}\leq C\|g\|_{s+1,G}. \tag{4.10}$$

Then, by (4.9), we obtain

$$\begin{aligned}\Big(g-\delta\int_G g\mathrm{dx},\omega p\Big)=&(\mathrm{div}\Phi,\omega p)=(\mathrm{div}(\omega\Phi),p)-(\mathbf{grad}\omega,p\Phi)\\ =&(\mathrm{div}(\omega\Phi)^I,p)+\{(\mathrm{div}[w\Phi-(\omega\Phi)^I],p)-(\mathbf{grad}w,p\Phi)\}\\ =&:A_1+B_1. \end{aligned}\tag{4.11}$$

Here the superscript I is the approximation operator specified in property A1 of Section 3. Choosing $\psi=(\omega\Phi)^I$ in (4.3), we get

$$\begin{aligned}A_1:=&(\mathrm{div}(\omega\Phi)^I,p)=(\mathbf{grad}_h\phi,\mathbf{grad}(\omega\Phi)^I)-\boldsymbol{L}((\omega\phi)^I)\\ =&(\mathbf{grad}_h\phi,\mathbf{grad}(\omega\Phi))+(\mathbf{grad}_h\phi,\mathbf{grad}[(\omega\Phi)^I-\omega\Phi])-\boldsymbol{L}((\omega\phi)^I)\\ =&(\mathbf{grad}_h(\omega\phi),\mathbf{grad}\Phi)-\boldsymbol{L}((\omega\phi)^I)+\{R(\omega,\Phi,\phi)\\ &+(\mathbf{grad}_h,\phi,\mathbf{grad}[(\omega\Phi)^I-\omega\Phi])\}\end{aligned}$$

$$= : A_2 - \boldsymbol{L}((\omega\phi)^I) + B_2, \tag{4.12}$$

where R is defined in (2.1). Next, by using (4.8) and integration by parts we obtain

$$\begin{aligned} A_2 :=&(\mathbf{grad}_h(\omega\phi), \mathbf{grad}\Phi) = (\mathrm{div}_h(\omega\phi), P) + \Big\{ \sum_{T\subset G_h} \int_{\partial T} \omega \mathbf{n}^t(\mathbf{grad}\Phi - PI)\phi \mathrm{ds}\Big\} \\ =& : (\mathrm{div}_h\phi, \omega P) + (\mathbf{grad}\omega, P\phi) + A_3 \\ =&\{(\mathrm{div}_h\phi, \omega P - (\omega P)^I) + (\mathbf{grad}\omega, P\phi)\} + Q((\omega P)^I) + A_3 \\ =& : B_3 + Q((\omega P)^I) + A_3 \end{aligned}$$

where we applied (4.4) in the last step.

Applying the approximation property $A1$, the consistency property $A5$, (2.2), and the Cauchy-Schwartz inequality, we get

$$\begin{aligned} &|B_1| \le C(h\|\Phi\|_{2,G}\|p\|_{0,G} + \|\phi\|_{s+2,G}\|p\|_{-s-2,G}), \\ &|B_2| \le C(\|\phi\|_{-s-1,G}\|\Phi\|_{s+2,G} + h\|\phi\|^h_{1,G}\|\Phi\|_{2,G}), \\ &|B_3| \le C(h\|\phi\|^h_{1,G}\|P\|_{1,G} + \|\phi\|_{-s-1}\|P\|_{s+1,G}), \\ &|A_3| \le Ch^\alpha(\|\Phi\|_{2,G} + \|P\|_{1,G})\|\phi\|^h_{1,G}, \\ &|\boldsymbol{L}((\omega\Phi)^I)| \le C\|\boldsymbol{L}\|_{-1,G}(\|\Phi\|_{1,G} + h\|\Phi\|_{2,G}), \\ &|Q((\omega P)^I)| \le C\|Q\|_{0,G}(\|P\|_{0,G} + h\|P\|_{1,G}). \end{aligned} \tag{4.13}$$

Substituting (4.10) into (4.13) and combining the result with (4.7), (4.11), and (4.12), we arrive at (4.5) (note that usually $\alpha \le 1$). Using a similar duality argument, one can prove that $\|\phi\|_{-s,G_0}$ is also bounded above by the right hand side of (4.5). We omit it here. For the treatment when $\boldsymbol{V}_h$ is conforming, please refer to [1].

Inequality (4.6) is obvious from the above proof.

Proof of Theorem 4.1. This can be achieved by iteriation ([1], [14]).

5. Interior Error Estimates

In this section we state and prove the mian result of this paper, Theorem 5.3. First we obtain in Lemma 5.1 a bound on solutions of the homogeneous discrete system. In Lemma 5.2 this bound is iterated to get a better bound, which is then used to establish the desired local estimate on disks. Finally Theorem 5.3 is extends this estimate to arbitrary interior domains.

Lemma 5.1. *Let $\boldsymbol{L}$ be a linear functional on $\boldsymbol{V}_h$ and Q a linear functional on W_h. Assume that $(\phi, p) \in \boldsymbol{V}_h \times W_h$ satisfies*

$$(\mathbf{grad}_h\phi, \mathbf{grad}_h\psi) - (\mathrm{div}_h, \psi, p) = \boldsymbol{L}(\psi), \quad \textit{for all } \psi \in \mathring{\boldsymbol{V}}_h, \tag{5.1}$$

$$(\mathrm{div}_h\phi, q) = Q(q), \quad \textit{for all } q \in \mathring{W}_h \tag{5.2}$$

Then for any concentric disks $G_0 \Subset G \Subset \Omega$, and any nonnegative integer t, we have

$$\|\phi\|^h_{1,G_0} + \|p\|_{0,G_0} \le C(h^\alpha\|\phi\|^h_{1,G} + h^\alpha\|p\|_{0,G}$$

$$+\|\phi\|_{-t,G}+\|p\|_{-t-1,G}+\|\boldsymbol{L}\|_{-1,G}+\|Q\|_{0,G}), \tag{5.3}$$

where $C=C(t,G_0,G)$.

Proof. Let G_h, $G_0 \Subset G_h \Subset G$, be as in Assumption A4. Let G' and G_1 be disk concentric with G_0 and G, such that $G_0 \Subset G_1 \Subset G_h \Subset G' \Subset G$, and construct $\omega \in C_0^\infty(G_1)$ with $\omega \equiv 1$ on G_0. Set $\tilde{\phi}=\omega\phi \in \boldsymbol{H}_h^1(G_h)$, $\tilde{p}=\omega p \in L^2(G_h)$. By Lemma 3.1, we cna define functions $\pi\tilde{\phi} \in \bar{\boldsymbol{V}}_h(G_h)$ and $\pi\tilde{p} \in \hat{W}_h(G_h)$ by the equations

$$(\mathbf{grad}_h(\tilde{\phi}-\pi\tilde{\phi}),\mathbf{grad}_h\psi)-(\mathrm{div}_h\tilde{p}-\pi\tilde{p})=0, \quad \text{for all } \psi \in \bar{\boldsymbol{V}}_h(G_h), \tag{5.4}$$

$$(\mathrm{div}_h(\tilde{\phi}-\pi\tilde{\phi}),q)=0, \quad \text{for all } q \in \hat{W}_h(G_h). \tag{5.5}$$

Furthermore, there exists a constant C such that

$$\begin{aligned}
&\|\tilde{\phi}-\pi\tilde{\phi}\|_{1,G_h}^h+\left\|\tilde{p}-\pi\tilde{p}-\int_{G_h}\tilde{p}\mathrm{dx}\right\|_{0,G_h}\\
\le& C\Big(\inf_{\psi\in\bar{\boldsymbol{V}}_h(G_h)}\|\tilde{-}\psi\|_{1,G_h}^h+\inf_{q\in W_h(G_h)}\|\tilde{p}-q\|_{0,G_h}\Big)\\
\le& Ch(\|\phi\|_{1,G'}^h+\|p\|_{0,G'}),
\end{aligned} \tag{5.6}$$

where we used the superapproximation property in the last step.

To prove (5.3), note that

$$\begin{aligned}
\|\phi\|_{1,G_0}^h+\|p\|_{0,G_0}\le&\|\tilde{\phi}\|_{1,G_h}^h+\|\tilde{p}\|_{0,G_h}\le\|\tilde{\phi}-\pi\tilde{\phi}\|_{1,G_h}^h\\
&+\left\|\tilde{p}-\pi\tilde{p}-\int_{G_h}\tilde{p}\mathrm{dx}\right\|_{0,G_h}+\|\pi\tilde{\phi}\|_{1,G_h}^h+\|\pi\tilde{p}\|_{0,G_h}+\left\|\int_{G_h}\omega p\mathrm{dx}\right\|_{G_h}\\
\le& Ch(\|\phi\|_{1,G'}^h+\|p\|_{0,G'})+\|p\|_{-1,G'}+\|\pi\tilde{\phi}\|_{1,G_h}^h+\|\pi\tilde{p}\|_{0,G_h}.
\end{aligned} \tag{5.7}$$

Next, we bound $\|\pi\tilde{\phi}\|_{1,G_h}^h$. In (5.4) we take $\psi=\pi\tilde{\phi}$ to obtain, for a positive constant C_1,

$$\begin{aligned}
C_1(\|\pi\tilde{\phi}\|_{1,G_h}^h)^2\le&(\mathbf{grad}_h\pi\tilde{\phi},\mathbf{grad}_h\pi\tilde{\phi})\\
=&(\mathbf{grad}_h\tilde{\phi},\mathbf{grad}_h\pi\tilde{\phi})-(\mathrm{div}_h\pi\tilde{\phi},\tilde{p}-\pi\tilde{p}).
\end{aligned} \tag{5.8}$$

For the first term on the right hand side of (5.8), we have

$$\begin{aligned}
(\mathbf{grad}_h\tilde{\phi},\mathbf{grad}_h\pi\tilde{\phi})=&(\mathbf{grad}_h(\omega\phi),\mathbf{grad}_h\pi\tilde{\phi})\\
=&(\mathbf{grad}_h\phi,\mathbf{grad}_h(\omega\pi\tilde{\phi}))-R(\omega,\pi\tilde{\phi},\phi)=(\mathbf{grad}_h\phi,\mathbf{grad}_h(\omega\pi\tilde{\phi})^I)\\
&+\{(\mathbf{grad}_h\phi,\mathbf{grad}[\omega\pi\tilde{\phi}-(\omega\pi\tilde{\phi})^I])-R(\omega,\pi\tilde{,}\phi)\}\\
=:&\,G_1+H_1.
\end{aligned} \tag{5.9}$$

To bound G_1, we take $\psi=(\omega\pi\tilde{\phi})^I$ in (5.1) to get

$$\begin{aligned}
G_1=&(\mathrm{div}_h(\omega\pi\tilde{\phi})^I,p)+\boldsymbol{L}((\omega\pi\tilde{\phi})^I)\\
=&(\mathrm{div}_h(\omega\pi\tilde{\phi}),p)+(\mathrm{div}_h[(\omega\pi\tilde{\phi})^I-\omega\pi\tilde{\phi}],p)+\boldsymbol{L}((\omega\pi\tilde{\phi})^I)
\end{aligned}$$

$$=(\mathrm{div}_h \pi\tilde{\phi}, \omega p) + \{(\mathbf{grad}\omega, p\pi\tilde{\phi}) + (div[(\omega\pi\tilde{\phi})^I - \omega\pi\tilde{\phi}], p)\} + \boldsymbol{L}((\omega\pi\tilde{\phi})^I)$$
$$=: (\mathrm{div}_h \pi\tilde{\phi}, \tilde{p}) + H_2 + \boldsymbol{L}((\omega\pi\tilde{\phi})^I). \tag{5.10}$$

Combining (5.8), (5.9), and (5.10), we obtian

$$\begin{aligned} C_1\|\pi\tilde{\phi}\|^2_{1,G_h} \leq &(\mathrm{div}_h \pi\tilde{\phi}, \tilde{p}) + H_1 + H_2 - (\mathrm{div}_h \pi\tilde{\phi}, \tilde{p} - \pi\tilde{p}) + \boldsymbol{L}((\omega\pi\tilde{\phi})^I) \\ =&(\mathrm{div}_h \pi\tilde{\phi}, \pi\tilde{p}) + \boldsymbol{L}((\omega\pi\tilde{\phi})^I) + H_1 + H_2. \end{aligned} \tag{5.11}$$

Taking $q = \pi\tilde{p}$ in (5.5) yields

$$\begin{aligned} (\mathrm{div}_h \pi\tilde{\phi}, \pi\tilde{p}) =&(\mathrm{div}_h \tilde{\phi}, \pi\tilde{p}) = (\mathrm{div}_h(\omega\phi), \pi\tilde{p}) = (\mathrm{div}_h \phi, w\pi\tilde{p}) + (\mathbf{grad}\omega, \pi\tilde{p}\phi) \\ =&\{(\mathrm{div}_h \phi, w\pi\tilde{p} - (\omega\pi\tilde{p})^I) + (\mathbf{grad} w, \pi\tilde{p}\phi)\} + Q((\omega\pi\tilde{p})^I) \\ =:& H_3 + Q((\omega\pi\tilde{p})^I) \end{aligned} \tag{5.12}$$

where we used (5.2) at the last step. Applying he Schwarz inequality, Lemma 3.3, inequality (5.6), and the superapproximation property A2, we get

$$\begin{aligned} &|H_1| \leq C(h^\alpha\|\phi\|^h_{1,G'} + \|\phi\|_{0,G'})\|\pi\tilde{\phi}\|^h_{1,G_h}, \\ &|H_2| \leq C(\|p\|_{-1,G_h} + h^\alpha\|p\|_{0,G_h})\|\pi\tilde{\phi}\|^h_{1,G_h}, \\ &|H_3| \leq C(h\|\phi\|^h_{1,G'} + \|\phi\|_{0,G'})\|\pi\tilde{p}\|_{0,G_h}, \\ &|\boldsymbol{L}((\omega\pi\tilde{\phi})^I)| \leq C\|\boldsymbol{L}\|_{-1,G}\|\pi\tilde{\phi}\|^h_{1,G_h}, \\ &|Q((\omega\pi\tilde{\phi})^I)| \leq C\|Q\|_{0,G}\|\pi\tilde{\phi}\|_{0,G_h}, \end{aligned}$$

Combining the above three inequalities with (5.11) and (5.12), and using the arithmetic-geometry mean inequality, we arrive at

$$\begin{aligned} (\|\pi\tilde{\phi}\|^h_{1,G_H})^2 \leq &C_1(h^{2\alpha}(\|\phi\|^h_{1,G'})^2 + \|\phi\|^2_{0,G'} + h^{2\alpha}\|p\|^2_{0,G_h} + \|p\|^2_{-1,G_h}) \\ &+ C_2(\|\phi\|_{0,G'} + h\|\phi\|^h_{1,G'} + \|Q\|_{0,G})\|\pi\tilde{p}\|_{0,G_h} \\ &+ C\|\boldsymbol{L}\|_{-1,G}\|\pi\tilde{\phi}\|^h_{1,G_h}. \end{aligned} \tag{5.13}$$

Next we estimate $\|\pi\tilde{|}\|_{0,G_h}$. By using the inf-sup condition,

$$\|\pi\tilde{p}\|_{0,G_h} \leq C \sup_{\substack{\psi\in\tilde{\boldsymbol{V}}_{h(G_h)} \\ \psi\neq 0}} \frac{(\mathrm{div}_h\psi, \pi\tilde{p})_{G_h}}{\|\psi\|^h_{1,G_h}}. \tag{5.14}$$

To deal with the numberator on the right hand side of (5.14), we apply (5.4),

$$\begin{aligned} (\mathrm{div}_h\psi, \pi\tilde{p}) =&(\mathrm{div}_h\psi, \tilde{p}) - (\mathbf{grad}_h(\tilde{\phi} - \pi\tilde{\phi}), \mathbf{grad}_h\psi) \\ =&(\mathrm{div}_h\psi, \omega p) - (\mathbf{grad}_h(\tilde{\phi} - \pi\tilde{\phi}), \mathbf{grad}_h\psi) \\ =&(\mathrm{div}_h(\omega\psi), p) - (\mathbf{grad}_h(\tilde{\phi} - \pi\tilde{\phi}), \mathbf{grad}_h\psi) - (\mathbf{grad}\omega, p\psi) \\ =&(\mathrm{div}_h(\omega\psi)^I, p) - (\mathbf{grad}_h(\tilde{\phi} - \pi\tilde{\phi}), \mathbf{grad}_h\psi) \\ &+ (\mathrm{div}_h(\omega\psi - (\omega\psi)^I), p) - (\mathbf{grad}\omega, p\psi). \end{aligned} \tag{5.15}$$

We use (5.1) to attack $(\mathrm{div}_h(\omega\psi)^I, p)$ and get

$$\begin{aligned}(\mathrm{div}_h(\omega\psi)^I, p) =&(\mathbf{grad}_h\phi, \mathbf{grad}(\omega\psi)^I) - \boldsymbol{L}((\omega\psi)^I)\\ =&(\mathbf{grad}_h\phi, \mathbf{grad}_h(\omega\psi)) + (\mathbf{grad}_h\phi, \mathbf{grad}_h[(\omega\psi)^I - \omega\psi]) - \boldsymbol{L}((\omega\psi)^I)\\ =&(\mathbf{grad}_h(\omega\phi), \mathbf{grad}_h\psi) + \{R(w,\psi,\phi) + (\mathbf{grad}_h\phi, \mathbf{grad}[(\omega\psi)^I - \omega\psi])\}\\ &- \boldsymbol{L}((\omega\psi)^I) =: (\mathbf{grad}_h\tilde{\phi}, \mathbf{grad}_h\psi) + M_1 - \boldsymbol{L}((\omega\psi)^I).\end{aligned} \tag{5.16}$$

Combining (5.13) and (5.16), we get

$$\begin{aligned}(\mathrm{div}_h\psi, \pi\tilde{p}) =&(\mathbf{grad}_h\pi\tilde{\phi}, \mathbf{grad}_h\psi) + \{(\mathrm{div}_h(\omega\psi - (\omega\psi)^I), p)\\ &- (\mathbf{grad}w, p\psi)\} + M_1 - \boldsymbol{L}((\omega\psi)^I)\\ =:& (\mathbf{grad}_h\pi\tilde{\phi}, \mathbf{grad}_h\psi) + M_2 + M_1 - \boldsymbol{L}((\omega\psi)^I).\end{aligned} \tag{5.17}$$

Then applying the superapproximation property, the Schwarz inequality, and Lemma 3.3, we arrive at

$$\begin{aligned}&|M_1| \le C(\|\phi\|_{0,G'} + h^\alpha\|\phi\|^h_{1,G'})\|\psi\|^h_{1,G_h},\\ &|M_2| \le C(h^\alpha\|p\|_{0,G_h} + \|p\|_{1,G_h})\|\psi\|^h_{1,G_h},\\ &|(\mathbf{grad}_h\pi\tilde{\phi}, \mathbf{grad}_h\psi)| \le C\|\pi\tilde{\phi}\|^h_{1,G_h}\|\psi\|^h_{1,G_h},\\ &|\boldsymbol{L}((\omega\psi)^I)| \le C\|\boldsymbol{L}\|_{-1,G_h}\|\psi\|^h_{1,G_h}.\end{aligned}$$

Combining (5.14) and (5.17) with the above three inequalities, we obtain

$$\begin{aligned}\|\pi\tilde{p}\|_{0,G_h} \le& C(h^\alpha\|\phi\|^h_{1,G'} + \|\phi\|_{0,G'} + h^\alpha\|p\|_{0,G_h}\\ &+ \|p\|_{-1,G_h} + \|\boldsymbol{L}\|_{-1,G'} + \|\pi\tilde{\phi}\|^h_{1,G_h}).\end{aligned} \tag{5.18}$$

Substituting (5.18) into (5.13), we obtain

$$\begin{aligned}\|\pi\tilde{\phi}\|^h_{1,G_h} \le& C(h^\alpha\|\phi\|^h_{1,G'} + \|\phi\|_{0,G'} + h^\alpha\|p\|_{0,G_h}\\ &+ \|p\|_{-1,G)h} + \|\boldsymbol{L}\|_{-1,G'} + \|Q\|_{0,G}).\end{aligned} \tag{5.19}$$

Thus, substituting (5.19) back into (5.18), we find that $\|\pi\tilde{p}\|_{0,G_h}$ is also bounded above by the right hand side of (5.19). Therefore, from (5.7) we obtain

$$\begin{aligned}\|\phi\|^h_{1,G_0} + \|p\|_{0,G_0} \le& C(h^\alpha\|\phi\|^h_{1,G'} + \|\phi\|_{0,G'} + h^\alpha\|p\|_{0,G_h}\\ &+ \|p\|_{-1,G_h} + \|\boldsymbol{L}\|_{-1,G'} + \|Q\|_{0,G}).\end{aligned}$$

Applying Theorem 4.1 for the case with G' inplace of G_0, we finally arrive at

$$\begin{aligned}\|\phi\|^h_{1,G_0} + \|p\|_{0,G_0} \le& C(h^\alpha\|\phi\|^h_{1,G} + \|\phi\|_{-t,G} + h^\alpha\|p\|_{0,G}\\ &+ \|p\|_{-t-1,G} + \|\boldsymbol{L}\|_{-1,G} + \|Q\|_{0,G}).\end{aligned}$$

By a standard iteration argument, one can prove the following[1,14].

Lemma 5.2. *Suppose the conditions of Lemma* 5.1 *are satisfied. Then*

$$\|\phi\|^h_{1,G_0} + \|p\|_{0,G_0} \le C(\|\phi\|_{-t,G} + \|p\|_{-t-1,G} + \|\boldsymbol{L}\|_{-1,G} + \|Q\|_{0,G}). \tag{5.20}$$

We now state the main result of the paper.

Theorem 5.3. *Let $\Omega_0 \Subset \Omega_1 \Subset \Omega$. Suppose that $(\phi, p) \in \boldsymbol{H}^1 \times L^2$ (the exact solution) satisfies $\phi|_{\Omega_1} \in \boldsymbol{H}^m(\Omega_1)$ and $p|_{\Omega_1} \in \boldsymbol{H}^{m-1}(\Omega_1)$ for some integer $m > 0$. Assume that $(\phi_h, p_h) \in \boldsymbol{V}_h \times W_h$ (the finite element solution) is given so that* (4.1) *and* (4.2) *hold. Let t be a nonnegative integer. Then there exists a constant C depending only on Ω_1, Ω_0 and t, such that*

$$\begin{aligned}\|\phi - \phi_h\|^h_{1,\Omega_0} + \|p - p_h\|_{0,\Omega_0} \le &(\|\phi\|_{1,\Omega_1} + \|p\|_{0,\Omega_1} + \chi(\phi, p, \Omega_1)\\ &+ \|\phi - \phi_h\|_{-t,\Omega_1} + \|p - p_h\|_{-t-1,\Omega_1}),\end{aligned} \tag{5.21}$$

$$\begin{aligned}\|\phi - \phi_h\|^h_{0,\Omega_0} + \|p - p_h\|_{-1,\Omega_0} \le &(h^\alpha\|\phi\|_{1,\Omega_1} + h^\alpha\|p\|_{0,\Omega_1} + h^\alpha\chi(\phi, p, \Omega_1)\\ &+ \|\phi - \phi_h\|_{-t,\Omega_1} + \|p - p_h\|_{-t-1,\Omega_1}),\end{aligned} \tag{5.22}$$

with $\chi(\phi, p, \Omega_1)$, defined in (5.32) *below, represents the consistency error of the finite element space $\boldsymbol{V}_h$ (the order of this term depends on both $\boldsymbol{V}_h$ and the smoothness of solution (ϕ, p) on Ω_1), and α is given in* (3.2).

This theorem will follow easily from a slightly more localized version.

Lemma 5.4. *Suppose the hypotheses of Theorem* 5.3 *are fulfilled and, in addition, that $\Omega_0 = G_0$ and $\Omega_1 = G_1$ are concentric disks. Then the conclusion of the theorem holds.*

Proof. Let $G'_0 \Subset G'$ be further concentric disks strictly contained between G_0 and G and let G_h be a domain strictly contained between G' and G for which properties A3 and A4 hold. Thus

$$G_0 \Subset G'_0 \Subset G' \Subset G \Subset \Omega.$$

Take $\omega \in C_0^\infty(G')$ identically 1 on G'_0 and set $\tilde\phi = w\phi$, $\tilde p = \omega p$. Let $\pi\tilde\phi \in \bar{\boldsymbol{V}}_h(G)$, $\pi\tilde p \in W_h(G)$ be defined by

$$(\mathrm{grad}_h(\tilde\phi - \pi\tilde\phi), \mathbf{grad}_h\psi) - (\mathrm{div}_h\psi, \tilde p - \pi\tilde p) = 0, \quad \text{for all } \psi \in \overline{\boldsymbol{V}}_h(G_h), \tag{5.23}$$

$$(\mathrm{div}_h(\tilde\phi - \pi\tilde\phi), q) = 0, \quad \text{for all } q \in W_h(G_h). \tag{5.24}$$

together with $\int_{G_h} \pi\tilde p \mathrm{dx} = \int_{G_h} \tilde p \mathrm{dx}$. Then using Lemma 3.1 and A1 we have

$$\begin{aligned}\|\tilde\phi - \pi\tilde\phi_h\|^h_{1,G_h} + \|\tilde p - \pi\tilde p_h\|_{0,G_h} \le &C\Big(\inf_{\psi \in bar\boldsymbol{V}_h(G_h)} \|\tilde\phi - \psi\|^h_{1,G_h} + \inf_{q \in W^h(G_h)} \|\tilde p - q\|_{0,G_h}\Big)\\ \le &C(\|\phi\|_{1,G_h} + \|p\|_{0,G_h}).\end{aligned} \tag{5.25}$$

Let's now estimate $\|\phi-\phi_h\|_{1,G_0}$ and $\|p-p_h\|_{0,G_0}$. First, the triangle inequality gives us

$$\begin{aligned}\|\phi-\phi_h\|^h_{1,G_0}+\|p-p_h\|_{0,G_0} \le& \|\phi_h-\pi\tilde{\phi}\|^h_{1,G_0}+\|p_h-\pi\tilde{p}\|_{0,G_0}\\ &+\|\pi\tilde{\phi}-\phi_h\|^h_{1,G_0}+\|\pi\tilde{p}-p_h\|_{0,G_0}\\ \le&\tilde{\phi}-\pi\tilde{\phi}\|^h_{1,G_h}+\|\tilde{p}-\pi\tilde{p}\|_{0,G_h}+\|\pi\tilde{\phi}-\phi_h\|^h_{1,G_0}+\|\pi\tilde{p}-p_h\|_{0,G_0}\\ \le&C(\|\phi\|^h_{1,G_h}+\|p\|_{0,G_h}+\|\pi\tilde{\phi}-\phi_h\|^h_{1,G_0}+\|\pi\tilde{p}-p_h\|_{0,G_0}).\end{aligned} \tag{5.26}$$

From (5.23), (5.24) and (4.3), (4.4) we find

$$\begin{aligned}&(\mathbf{grad}_h(\phi_h-\pi\tilde{\phi}),\mathbf{grad}_h\psi)-(\mathrm{div}_h\psi,p_h-\pi\tilde{p})\\ &\quad=\sum_{T\subset G_h}\int_{\partial T}\boldsymbol{n}^t(\mathbf{grad}\phi-pI)\psi\mathrm{ds},\quad \text{for all } \psi\in\mathring{\boldsymbol{V}}(G_0'),\\ &(\mathrm{div}_h(\phi_h-\pi\tilde{\phi}),q)=0,\quad \text{for all } q\in W_h(G_0').\end{aligned}$$

Define

$$\chi(\phi,p,G_h)=\sup_{\substack{\psi\in\mathring{\boldsymbol{V}}(G_h)\\ \psi\neq0}}\frac{\displaystyle\sum_{T\subset G_h}\int_{\partial T}\boldsymbol{n}^t(\mathbf{grad}\phi-pI)\psi\mathrm{ds}}{\|\psi\|^h_{1,G_h}} \tag{5.27}$$

We next apply Lemma 5.2 to $\phi_h-\pi\tilde{\phi}$ and $p_h-\pi\tilde{p}$ with G replaced by G_0'. Then it follows from (5.20) that

$$\begin{aligned}\|\phi_h-\pi\tilde{\phi}\|_{1,G_0}+\|p_h-\pi\tilde{p}\|_{0,G_0}\le& C(\|\phi_h-\pi\tilde{\phi}\|_{-t,G_0'}+\|p_h-\pi\tilde{p}\|_{-t-1,G_0'})+\chi(\phi,p,G_0')\\ \le&C(\|\phi-\phi_h\|_{-t,G_0'}+\|p-p_h\|_{-t-1,G_0'}+\|\phi_h-\pi\tilde{\phi}\|_{-t,G_0'}\\ &+\|p_h-\pi\tilde{p}\|_{-t-1,G_0'}+\chi(\phi,p,G_h))\\ \le&C(\|\phi-\phi_h\|_{-t,G}+\|p-p_h\|_{-t-1,G}+\|\tilde{\phi}-\pi\tilde{\phi}\|^h_{1,G_h}\\ &+\|\tilde{p}-\pi\tilde{p}\|_{0,G_h}+\chi(\phi,p,G_h)).\end{aligned}$$

In the light of (5.26), (5.25), and the above inequality, we have

$$\begin{aligned}\|\phi-\phi_h\|^h_{1,G_0}+\|p-p_h\|_{0,G_0}\le&\|\phi\|_{1,G}+\|p\|_{0,G}+\chi(\phi,p,G)\\ &+\|\phi-\phi_h\|_{-t,G}+\|p-p_h\|_{-t-1,G}).\end{aligned} \tag{5.28}$$

To prove a local version of (5.22), we note that $\boldsymbol{L}(\psi)=0$ for any continuous function ψ. Then apply Theorem 4.1 to the disks G_0 and G' and get

$$\begin{aligned}\|\phi-\phi_h\|_{0,G_0}+\|p-p_h\|_{-1,G_0}\le& C(h^\alpha\|\phi-\phi_h\|^h_{1,G'}+h^\alpha\|p-p_h\|_{0,G'}\\ &+\|\phi-\phi_h\|_{-t,G'}+\|p-p_h\|_{-t-1,G'}).\end{aligned}$$

Then, applying (5.28) with G_0 replaced by G', we obtain the desired result

$$\|\phi-\phi_h\|_{0,G_0}+\|p-p_h\|_{-1,G_0}\le C(h^\alpha\|\phi\|_{1,G}+h^\alpha\|p\|_{0,G}+h^\alpha\chi(\phi,p,G)$$

$$+ \|\phi - \phi_h\|_{-t,G} + \|p - p_h\|_{-t-1,G}).$$

Proof of Theorem 5.3. The argument here is same as in Theorem 5.1 of [14] and [1]. We skip it here.

Let $\overset{\circ}{\boldsymbol{U}}_h$ denote the largest subspace fo continuous functions in $\overset{\circ}{\boldsymbol{V}}_h$. Then it is easily seen that we can derive the following result from Theorem 5.3.

Corollary 5.5. *Under the conditions of Theorem* 5.3, *we have*

$$\begin{aligned}\|\phi - \phi_h\|_{1,\Omega_0}^h + \|p - p_h\|_{0,\Omega_0} \leq & C\Big(\inf_{\psi\in\overset{\circ}{\boldsymbol{U}}_h} \|\phi - \psi\|_{1,\Omega_1} + \inf_{q\in W_h} \|p - q\|_{0,\Omega_1} \\ & + \chi(\phi, p, \Omega_1) + \|\phi - \phi_h\|_{-t,\Omega_1} + \|p - p_h\|_{-t-1,\Omega_1}\Big), \\ \|\phi - \phi_h\|_{0,\Omega_0} + \|p - p_h\|_{-1,\Omega_0} \leq & C\Big(h^\alpha \inf_{\psi\in\overset{\circ}{\boldsymbol{U}}_h} \|\phi - \psi\|_{1,\Omega_1} + h^\alpha \inf_{q\in W_h} \|p - q\|_{0,\Omega_1} \\ & + h^\alpha chi(\phi, p, \Omega_1) + \|\phi - \phi_h\|_{-t,\Omega_1} + \|p - p_h\|_{-t-1,\Omega_1}\Big).\end{aligned}$$

Now we apply the above result to the case when the velocity is approximated by the k-h order Crouzeix-Raviart elements and the pressure by the discontinuous piecewise functions of order $k-1$. It is easy to check that the properties A1–A5 are satisfied. Moreover, the following holds[6]:

$$\chi(\phi, p, \Omega_1) \leq Ch^l(\|\phi\|_{l+1} + \|p\|_{l,\Omega_1}), \quad \text{with } 1 \geq l \geq k,$$

for the k-th order element, if $\phi \in \boldsymbol{H}^{l+1}(\Omega_1)$ and $p \in H^l(\Omega_1)$. We then have (one can also check that $\alpha = 1$):

Theorem 5.6. *Let* $\boldsymbol{V}_h$ *be the space of the* k*-th order Crouzeix-raviart nonconforming triangular elements and* W_h *the space of discontinuous* $(k-1)$*-th order polynomials. For some domians* $\Omega_0 \Subset \Omega_1 \Subset \Omega$, *assume that* $\phi \in \boldsymbol{H}^l(\Omega_1)$ *and* $p \in H^l - 1(\Omega_1)$ *with* $1 \leq l \leq k+1$. *Let* t *be an arbitrary integer. Then there is a constant* C *depending only on* Ω_0, Ω_1, *and* t, *and a positive number* h_1 *such that for* $h \in (0, h_1]$

$$\begin{aligned}\|\phi - \phi_h\|_{1,\Omega_0}^h + \|p - p_h\|_{0,\Omega_0} \leq & C(h^{l-1}\|\phi\|_{l,\Omega_1} + h^{l-1}\|p\|_{l-1,\Omega_1} \\ & + \|\phi - \phi_h\|_{t,\Omega_1} + \|p - p_h\|_{-t-1,\Omega_1}), \\ \|\phi - \phi_h\|_{0,\Omega_0} + \|p - p_h\|_{-1,\Omega_0} \leq & C(h^l\|\phi\|_{l,\Omega_1} + h^l\|p\|_{l-1,\Omega_1} \\ & + \|\phi - \phi_h\|_{-t,\Omega_1} + \|p - p_h\|_{-t-1,\Omega_1}).\end{aligned}$$

6. Convergence of Difference Quotients

This section is based onthe fundamental investigation in Nitsche and Schatz [14, Section 6]. Our goal is to obtain the convergence of difference quotients of the finite element solution to the derivative fo exact solution in some interior domain for which the finite element space is translation invariant. Here we will only consider the Crouzeix-Raviart family of elements. As an application, we will prove the interior superconvergence of some class of difference quotients of the finite element solution to the derivatives of the exact solution.

The notations used here follow those in [16, Section 6].

Let $\nu = (\nu_1, \nu_2)$ and $\alpha = (\alpha_1, \alpha_2)$. For any function $f(x)$, let

$$T_h^\nu f(x) = f(x + \nu h).$$

A general difference operator can be written as

$$D_h^\alpha f = \sum_{|\nu| \le M} C_{\nu\alpha}(h) T_h^\nu f, \tag{6.1}$$

for some integer M. Here the notation convention is that $D_h^\alpha f$ approximates $D^\alpha f$ with D^α the differential operator. As usual, the vector version of D_h^α is expressed by a bold face symbol, i.e., $\mathbf{D}_h^\alpha$.

We will say that the mesh is uniform and translation invariant on a neighborhood of Ω_0, Ω_1, if there is an h_1 (in general depending on ν, Ω_0, and Ω_1) such that for all $h \in (0, h_1]$,

$$T_h^\nu \phi \in \mathring{\boldsymbol{V}}_h(\Omega_1) \text{ for all } \phi \in \mathring{\boldsymbol{V}}_h(\Omega_0), \text{ and } T_h^\nu q \in \mathring{W}_h(\Omega_1) \text{ for all } q \in \mathring{W}_h(\Omega_0), \tag{6.2}$$

with $|\nu| \le M$ for some fixed integer M (as in (6.1)).

We have the following result.

Theorem 6.1. *Let V_h be the space of the k-th order Crouzeix-Raviart element and W_h the space of the $(k-1)$-th order discontinuous element. Moreover, suppose that the mesh is uniform and translation invariant on a neighborhood of Ω_0, i.e., (6.2) is satisfied. Let $\mathbf{D}_h^\alpha(D_h^\alpha)$ be a finite difference operator in the form of (6.1), and t any nonnegative integer, fixed and arbitrary. If $1 \le l \le k+1$, $\phi \in H^{l+|\alpha|}(\Omega_1)$, and $p \in H^{l-1+|\alpha|}(\Omega_1)$, then there exist an h_0 such that for all $h \in (0, h_0]$*

$$\begin{aligned} \|\mathbf{D}_h^\alpha(\phi - \phi_h)\|_{0,\Omega_0} + \|D_h^\alpha(p - p_h)\|_{-1,\Omega_0)} \le C(&h^l \|\phi\|_{l+|\alpha|,\Omega_1} + h^l \|p\|_{l-1+|\alpha|,\Omega_1} \\ &+ \|\phi - \phi_h\|_{-t,\Omega_1} + \|p - p_h\|_{-t-1,\Omega_1} + \|p - p_h\|_{-t-1,\Omega_1}), \end{aligned} \tag{6.3}$$

with $C = C(t, \Omega_0, \Omega_1, |\alpha|)$.

Proof. Choose two intermediate domains Ω_0' and Ω_1' such that $\Omega_0 \Subset \Omega_0' \Subset \Omega_1' \Subset \Omega_1$. Then, by using the fact that the differentiation and the finite difference operator commute, it is easily seen that for h sufficiently small and $\psi \in \mathring{\boldsymbol{V}}_h(\Omega_0)$ and $q \in \mathring{W}_h(\Omega_0)$

$$\begin{aligned} (\mathbf{grad}_h \mathbf{D}_h^\alpha(\phi - \phi_h), \mathbf{grad}_h \psi) - \mathrm{div}_h \psi, D_h^\alpha(p - p_h)) &= (\mathbf{grad}_h(\phi - \phi_h), \mathbf{D}_h^{\alpha*}\psi) \\ &- (\mathrm{div}_h D_h^{\alpha*}\psi, p - p_h) =: \boldsymbol{L}(\mathbf{D}_h^{\alpha*}\psi) \end{aligned}$$

and

$$(\mathrm{div}_h \mathbf{D}_h^\alpha(\phi - \phi_h), q) = (\mathrm{div}_h(\phi - \phi_h), \mathbf{D}_h^{\alpha*} q) = 0,$$

where $\mathbf{D}_h^{\alpha*}(D_h^{\alpha*})$ denotes the difference operator adjoint to $\mathbf{D}_h^\alpha(D_h^\alpha)$ with respect to the L^2 inner product. On the other hand, from using integration by parts, we get for any $\psi \in \mathring{\boldsymbol{V}}_h(\Omega_1)$,

$$\boldsymbol{L}(\mathbf{D}_h^{\alpha*}\psi) = \sum_{T \subset \Omega_1} \int_{\partial T} \boldsymbol{n}^t(\mathbf{grad}\phi - pI)\mathbf{D}_h^{\alpha*}\psi \mathrm{ds}$$

$$= \sum_{T\subset\Omega_1} \int_{\partial T} \boldsymbol{n}^t(\mathbf{grad}\mathbf{D}_h^\alpha \phi - D_h^\alpha pI)\psi \mathrm{ds}.$$

Therefore

$$\|\boldsymbol{L}\|_{-1,\Omega_0'} \leq Ch^l(\|\mathbf{D}_h^\alpha \phi\|_{l,\Omega_1'} + \|D_h^\alpha p\|_{l-1,\Omega_1'}) \leq Ch^l(\|\phi\|_{l+|\alpha|,\Omega_1} + \|p\|_{l-1+|\alpha|,\Omega_1}).$$

Then applying Theorem 5.6 for $t' = t + |\alpha|$ yields

$$\begin{aligned}\|\mathbf{D}_h^\alpha(\phi-\phi_h)\|_{0,\Omega_0} + \|D_h^\alpha(p-p_h)\|_{-1,\Omega_0} \leq & C(h^l(\|\mathbf{D}_h^\alpha\phi\|_{l,\Omega_1} + h^l\|D_h^\alpha p\|_{l-1,\Omega_1} \\ & + \|\boldsymbol{D}_h^\alpha(\phi-\phi_h)\|_{-t',\Omega_0'} + \|D_h^\alpha(p-p_h)\|_{-t'-1,\Omega_0'}) \\ \leq & C(h^l\|\phi\|_{l+|\alpha|,\Omega_1} + h^l\|p\|_{l-1+|\alpha|,\Omega_1} + \|\phi-\phi_h\|_{-t,\Omega_1} + \|p-p_h\|_{-t-1,\Omega_1}),\end{aligned}$$

where we use the fact that

$$\|\boldsymbol{D}_h^\alpha(\phi-\phi_h)\|_{-t',\Omega_0'} \leq C\|\phi-\phi_h\|_{-t,\Omega_1}$$

and

$$\|D_h^\alpha(p-p_h)\|_{-t'-1,\Omega_0'} \leq C\|p-p_h\|_{-t-1,\Omega_1},$$

with C independent of h, ϕ and p, and ϕ_h and p_h. So (6.3) is proved.

The difference operator D_h^α is said to approximate a derivative D^α with order of accuracy r in L^2 if for any pair of domains $\Omega_0 \Subset \Omega_1$

$$\|D^\alpha f - D_h^\alpha f\|_{0,\Omega_0} \leq C(\Omega_0,\Omega_1)h^r\|f\|_{r+|\alpha|,\Omega_1},$$

for all h sufficiently small and $u \in H^{r+|\alpha|}(\Omega_1)$. We have the following results:

Theorem 6.2. *Suppose that the conditions of Theorem* 6.1 *are satisfied and let $D_h^\alpha(\boldsymbol{D}_h^\alpha)$ approximate $D^\alpha(\boldsymbol{D}^\alpha)$ with order of accuracy r in L^2. Furthermore, let t be a nonnegative integer, fixed but arbitrary. Then there exists an h_1 such that for all $h \in (0, h_1]$*

$$\begin{aligned}&\|\boldsymbol{D}^\alpha\phi - \boldsymbol{D}_h^\alpha\phi_h\|_{0,\Omega_0} + \|D^\alpha p - D_h^\alpha p_h\|_{-1,\Omega_0} \\ &\quad\leq C(h^r\|\phi\|_{r+|\alpha|,\Omega_1} + h^r\|p\|_{r-1+|\alpha|,\Omega_1} + \|\phi-\phi_h\|_{-t,\Omega_1} + \|p-p_h\|_{-t-1,\Omega_1}),\end{aligned}$$

with $C = C(t,\Omega_0,\Omega_1)$.

As a concrete example of the above Theorem, let us assume that $\boldsymbol{V}_h$ is the space of nonconforming linear elements and W_h the space of piecewise constants. Let Ω be a convex polygon. Assume further that $\boldsymbol{F}$ is smooth. Then for any $\Omega_0 \Subset \Omega$ and $D_h^\alpha(\boldsymbol{D}_h^\alpha)$ a difference operator of second order accuracy ($r = 2$ and $|\alpha| = 1$),

$$\begin{aligned}&\|\boldsymbol{D}^\alpha\phi - \boldsymbol{D}_h^\alpha\phi_h\|_{0,\Omega_0} + \|D^\alpha p - D_h^\alpha p_h\|_{-1,\Omega_0} \\ &\quad\leq Ch^2(\|\phi\|_{3,\Omega_1} + \|p\|_{2,\Omega_1} + \|\phi\|_{2,\Omega} + \|p\|_{1,\Omega}),\end{aligned}$$

with $\Omega_0 \Subset \Omega_1 \Subset \Omega$. This is an interior superconvergence result in energy norm. To obtain similar result in maximum norm, we use the technique of Bramble, Nitsche, and Wahlbin [bramble nitsche wahlbin] where interior estimates in maximum norm were obtained by using only the inverse inequalities and the interior error estimates in energy norms, we obtain

$$\|\boldsymbol{D}^{\alpha}\phi - \boldsymbol{D}_h^{\alpha}\phi\|_{\infty,\Omega_0} \leq Ch^2.$$

This also implies that the average of the gradients of the finite element solutin at the midpoint of two adjacent triangles approximates the gradient of the exact solution in the order of h^2.

References

[1] D.N. Arnold, X. Liu, Local error estimates for finite element discretizations of the Stokes equations, *RAIRO mathematical Modeling and numerical Analysis,* **29** (1995), 366–389.

[2] I. Babuška, R. R'iguez, The Problem of the Selection of an A-Posteriori Error Indicator Based on Smoothing Techniques, Institute for Physical Science and Technology, BNM-1126 (August 1991).

[3] R.E. Bank, A. Weisser, Some a posteriori error estimators for elliptic partial differential equations, *Math. Comp.,* **44** (1995), 283–301.

[4] J.H. Bramble, J. Nitsche, A. Schatz, Maximum-norm interior estimates for Rits-Galerkin methods, *Math. Comp.,* **29** (1975), 677–688.

[5] F. Brezzi, A. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York-Heidelberg-Berlin, 1991.

[6] M. Crouzeix, P-A. Raviart, Conforming and non-conforming finite element methods for solving the stationary Stokes equations, *RAIRO Anal. Numér.,* **7** R-3 (1973), 33–76.

[7] J. Douglas, Jr., C.P. Gupta, G. Li, Interior and super-convergence estimates for a primal hybrid finite element method for second order elliptic problems, *Calcolo,* **22** (1985), 397–428.

[8] J. Douglas, Jr., F.A. Milner, Interior and superconvergence estimates for mixed methods for second order elliptic problems, *RAIRO Modél. math. Anal. Numér.,* **19** (1985), 397–428.

[9] R. Durán, M.A-M. Muschietti, R. Rodríguez, On the asymptotic exactness of error estimators for linear triangular finite elements, *Numer. Math.,* **59** (1991), 107–127.

[10] R. Durán, R. Rodríguez, On the asymptotic exactness of Bank-Weisser's estimator, *Numer. Math.,* **62** (1992), 292–303.

[11] E. Eriksson, C. Johnson, An adaptive finite element method for linear elliptic problems, *Math. Comp.,* **50** (1988), 361–383.

[12] L. Gastaldi, Uniform interior error estimates for Reissner-Mindlin plate model, *Math. Comp.,* **61** (1993), 539–568.

[13] X. Liu, Interior error estimates for nonconforming finite element methods, To appear in *Numerische Mathematik.*

[14] J.A. Nische, A.H. Schatz, Interior estimate for Ritz-Galerkin methods, *math. Comp.,* **28** (1974), 937–958.

[15] A.H. Schatz, L.B. Wahlbin, Interior maximun norm estimates for finite telement methods, *Math. Comp.,* **31** (1977), 414–442.

[16] R. Temam, Navier-Stokes Equations, Amsterdam, Amsterdam, 1984.

[17] L.B. Wahlbin, Local Behavior in Finite Element Methods, in Hanbook of Numerical Analysis Volume II, P.G. Ciarlet and J.L. Lions, eds., Elsevier, Amsterdam-New York, 1991.

[18] M.F. Wheeler, J.R. Whiteman, Superconvergence recovery of gradients on subdomains from piecewise linear finite-element approximations, *Numerical methods for partial differential equations,* **3** (1987), 357–374.

[19] J.Z. Zhu, O.C. Zienkiewicz, Adaptive techniques in the finite element method, *Comm. in Appl. Num. Methods,* **4** (1988), 197–204.

[20] O.C. Zienkiewicz, J.Z. Zhu, A simple error estimator and adaptive procedure for practical engineering analysis, *Internat. J. Numer. Methods Engrg.,* **24** (1987), 337–357.