Journal of Computational Mathematics, Vol.17, No.5, 1999, 449–456.

A NEW GENERALIZED ASYNCHRONOUS PARALLEL MULTISPLITTING ITERATION METHOD*1)

Zhong-zhi Bai
(*State Key Laboratory of Scientific and Engineering Computing, ICMSEC, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

For the large sparse systems of linear and nonlinear equations, a new class of generalized asynchronous parallel multisplitting iterative method is presented, and its convergence theory is established under suitable conditions. This method not only unifies the discussions of various existing asynchronous multisplitting iterations, but also affords new algorithmic and theoretical results for the parallel solution of large sparse system of linear equations. Besides its generality, this method is also much more suitable for implementing on the MIMD multiprocessor systems.

Key words: Systems of linear and nonlinear equations, Asynchronous multisplitting iteration, Relaxed method, Convergence theory.

1. Introduction

To solve large sparse systems of linear and nonlinear equations on the multiprocessor systems, many authors presented and studied various parallel iterative methods in the sense of multisplitting in recent years. For details one can refer to [1]-[9] and references therein. Among these methods the chaotic multisplitting iterative methods proposed by Bru, Elsner and Neumann[4] are meaningful on both theory and application since it aims at avoiding the synchronous wait among processors of a multiprocessor system and making use of the efficiency of the MIMD parallel computer. However, because more restrictions are imposed upon these chaotic multisplitting iterative methods (see [7,6]), the maximum efficiency in exploiting the resources of the multiprocessor systems has not yet been attained. To overcome this shortcoming, Evans, Wang and Bai (see [2,7]) further modified and developed Bru, Elsner and Neumann's work from the angles of both algorithmic model and theoretical analysis, and presented a series of asynchronous parallel multisplitting iterative methods. Recently, Su[6] also presented another generalization of Bru, Elsner and Neumann's chaotic multisplitting methods, which is called as generalized multisplitting asynchronous iteration. Since the designs of these asynchronous multisplitting methods take into account not only the good parallelism of the multiple splittings, but also the concrete characteristics of the multiprocessor systems, they can sufficiently exploit the parallel computational efficiency of the multiprocessor systems.

In this paper, by summarizing the advantages of the aforementioned asynchronous multisplitting iteration methods, we propose a new asynchronous parallel iterative method in the sense of multiple splittings, called as a new generalized asynchronous

* Received October 30, 1996.
1) Project 19601036 Supported by the National Natural Science Fundation of China.

multisplitting iterative method (GAMI-method), for solving large sparse systems of linear and nonlinear equations. This new method has the properties of convenient implementation, and flexible and free communication, etc., and can also make full use of the efficiency of the multiprocessor systems. Meanwhile, the above stated existing asynchronous parallel multisplitting iterative methods are its special cases. Under similar conditions to [7] and [6], we establish convergence theory for our new method.

Since a system of equations can be equivalently transformed to several fixed point equations having a common fixed point by the multisplitting technique under certain conditions, without loss of generality, in the sequal we will mainly consider the iteration for getting a common fixed point of an operator class.

2. Description of the GAMI-Method

To mathematically describe our new generalized asynchronous multisplitting iterative method for parallely solving system of equations, we first introduce the following notations and concept.

Assume $\alpha (1 \le \alpha \le n)$ be a given positive integer. For all $i \in \{1,2,\cdots,\alpha\}$, let $T_{p,i} : R^n \to R^n (p = 0,1,2,\cdots)$ be mappings having a common fixed point $x^* \in R^n$, and E_i be nonnegative, nonzero, diagonal matrices satisfying $\sum\limits_{i=1}^{\alpha} E_i = E$ nonsingular. Denote $N_0 := \{0,1,2,\cdots\}$ and $\mathcal{O}_T = \{T_{p,i} : R^n \to R^n \mid i \in \{1,2,\cdots,\alpha\}; p \in N_0\}$. For any $p \in N_0$, we let $J(p)$ be a nonempty subset of the number set $\{1,2,\cdots,\alpha\}$, and $s_j^{(i)}(p)$, $t_j^{(i)}(p) (j = 1,2,\cdots,n; i = 1,2,\cdots,\alpha)$ be nonnegative numbers satisfying:

(a) for $\forall i \in \{1,2,\cdots,\alpha\}$, the set $\{p \in N_0 \mid i \in J(p)\}$ is infinite;

(b) for $\forall i \in \{1,2,\cdots,\alpha\}$, $\forall j \in \{1,2,\cdots,n\}$, $\forall p \in N_0$, there hold $s_j^{(i)}(p) \le p$ and $t_j^{(i)}(p) \le p$;

(c) for $\forall i \in \{1,2,\cdots,\alpha\}$, $\forall j \in \{1,2,\cdots,n\}$, there hold $\lim\limits_{p\to\infty} s_j^{(i)}(p) = \infty$ and $\lim\limits_{p\to\infty} t_j^{(i)}(p) = \infty$.

If we additionally define

$$\tau(p) = \min_{\substack{1\le j\le n \\ 1\le i\le \alpha}} \left\{ s_j^{(i)}(p), \quad t_j^{(i)}(p) \right\},$$

then there obviously have $\tau(p) \le p$ and $\lim\limits_{p\to\infty} \tau(p) = \infty$.

With the above preparations, we can now describe the generalized asynchronous multisplitting iterative method (GAMI-method) for parallely solving systems of equations as follows.

GAMI-method. *Given an initial vector $x^0 \in R^n$, and suppose that we have got approximations $x^1, x^2, \cdots, x^p$ of a common fixed point $x^* \in R^n$ of the operator class $\mathcal{O}_T = \{T_{p,i} : R^n \to R^n \mid i \in \{1,2,\cdots,\alpha\}; p \in N_0\}$. Then the next approximation x^{p+1} of x^* can be got by the following formula:*

$$x^{p+1} = \sum_{i\in J(p)} E_i T_{p,i}\left(x^{s^{(i)}(p)}\right) + \left(I - \sum_{i\in J(p)} E_i\right) x^p + \sum_{i\in J(p)} (I - E^{-1}) E_i \left(x^p - x^{t^{(i)}(p)}\right), \tag{2.1}$$

where

$$\begin{cases} x^{s^{(i)}(p)} = \left(x_1^{s_1^{(i)}(p)}, x_2^{s_2^{(i)}(p)}, \cdots, x_n^{s_n^{(i)}(p)}\right)^T, \\ x^{t^{(i)}(p)} = \left(x_1^{t_1^{(i)}(p)}, x_2^{t_2^{(i)}(p)}, \cdots, x_n^{t_n^{(i)}(p)}\right)^T, \end{cases}$$

and I is the $n \times n$ identity matrix.

Clearly, when we let $E = I$, the GAMI-method naturally reduces to a generalized version of the asynchronous parallel multisplitting iterative methods studied by Bai, Wang and Evans in [2], and when $t_j^{(i)}(p) = p(j = 1, 2, \cdots, n; i = 1, 2, \cdots, \alpha; p \in N_0)$, it automatically turns to the one discussed by Su in [6]. Moreover, the case $t_j^{(i)}(p) = s_j^{(i)}(p)(j = 1, 2, \cdots, n; i = 1, 2, \cdots, \alpha; p \in N_0)$ really produces a new asynchronous parallel multisplitting method for solving systems of equations.

3. Preliminary Knowledge

For a vector $x \in R^n$, $x > 0(\geq 0)$ will denote that all its components are positive (nonnegative). Similarly, for $x, y \in R^n$, $x > y(x \geq y)$ will mean that $x - y > 0(x - y \geq 0)$. For $x \in R^n$, $|x|$ will denote the vector whose components are the absolute value of the corresponding components of x. We shall employ similar notations for matrices.

Let $v = (v_1, v_2, \cdots, v_n)^T > 0$. Then the monotone norm $\| \bullet \|_v$ of a vector $x = (x_1, x_2, \cdots, x_n)^T$ is defined as

$$\|x\|_v = \max_{1 \leq j \leq n} \left| \frac{x_j}{v_j} \right|.$$

This vector norm is monotone in the sense that $|x| \leq |y|$ implies $\|x\|_v \leq \|y\|_v$. If we denote by $\|B\|_v$ the matrix norm of $B \in L(R^n)$ induced by the monotone vector norm $\| \bullet \|_v$, then there obviously holds $\||B|v\|_v = \|B\|_v$. Moreover, it easily follows that $\|x\|_v \leq \beta$ if and only if $|x| \leq \beta v$, and $\|B\|_v \leq \beta$ if $|B|v \leq \beta v$, where $\beta \in R^1$ is some nonnegative constant. The monotone norm and its properties will play an important role in the establishment of the convergence theorem of the GAMI-method.

Write

$$\begin{cases} \mathcal{P}_i = E_i E_i^+, \qquad i = 1, 2, \cdots, \alpha, \\ I_p = \sum\limits_{i \in J(p)} E_i, \quad \mathcal{I}_p = I_p I_p^+, \qquad p \in N_0, \end{cases}$$

where E_i^+ and I_p^+ denote the Moore-Penrose inverses of the matrices E_i and I_p, respectively. Then we easily know that these matrices have the following useful properties:

(1) $\mathcal{P}_i E_i = E_i \mathcal{P}_i = E_i$, $\mathcal{P}_i E_i^+ = E_i^+ \mathcal{P}_i = E_i^+$, $i = 1, 2, \cdots, \alpha$;

(2) $\mathcal{I}_p I_p = I_p \mathcal{I}_p = I_p$, $\mathcal{I}_p I_p^+ = I_p^+ \mathcal{I}_p = I_p^+$, $p = 0, 1, 2, \cdots$;

(3) $\mathcal{I}_p E_i = E_i$, $(I - \mathcal{I}_p) E_i = 0$, $i \in J(p)$, $p \in N_0$.

In addition, define an infinite integer sequence $\{m_l\}_{l \in N_0}$ according to the following rule: m_0 is the least positive integer such that $\cup_{0 \leq \tau(p) \leq p < m_0} J(p) = \{1, 2, \cdots, \alpha\}$, and in general, m_{l+1} is the least positive integer such that $\cup_{m_l \leq \tau(p) \leq p < m_{l+1}} J(p) = \{1, 2, \cdots, \alpha\}$. By the definitions of the set $J(p)$ and the nonnegative integer sequences $\left\{s_j^{(i)}(p)\right\}$, $\left\{t_j^{(i)}(p)\right\}$ $(j = 1, 2, \cdots, n; i = 1, 2, \cdots, \alpha)$ and $\{\tau(p)\}$, this nonnegative integer sequence

$\{m_l\}$ is well-defined and possesses the following properties (For their proofs, one can refer to [6] for detail):

(1) $\sum_{p=m_l}^{m_{l+1}-1} \mathcal{I}_p (l = 0, 1, 2, \cdots)$ are nonsingular diagonal matrices;

(2) $\prod_{p=m_l}^{m_{l+1}-1} (I - \mathcal{I}_p) = 0, \qquad l = 0, 1, 2, \cdots.$

The following fact cited from [2] is elementary for our subsequent discussion.

Lemma 3.1 (see [2]). *Given $\bar{x}^* \in R^n$ and $\{\bar{x}^q\}_{q=0}^p \subset R^n (\forall p \in N_0)$. Assume that for all $t \in \{0, 1, 2, \cdots, p\}$, there exist positive number δ and positive vector $v \in R^n$ such that $\|\bar{x}^t - \bar{x}^*\|_v \le \delta$. Then there identically hold*

$$\left\|\bar{x}^{s^{(i)}(p)} - \bar{x}^*\right\|_v \le \delta, \quad \left\|\bar{x}^{t^{(i)}(p)} - \bar{x}^*\right\|_v \le \delta, \qquad i = 1, 2, \cdots, \alpha,$$

provided $s_j^{(i)}(p) \le p$ and $t_j^{(i)}(p) \le p (j = 1, 2, \cdots, n; i = 1, 2, \cdots, \alpha)$, where

$$\begin{cases} \bar{x}^{s^{(i)}(p)} = \left(\bar{x}_1^{s_1^{(i)}(p)}, \bar{x}_2^{s_2^{(i)}(p)}, \cdots, \bar{x}_n^{s_n^{(i)}(p)}\right)^T, \\ \bar{x}^{t^{(i)}(p)} = \left(\bar{x}_1^{t_1^{(i)}(p)}, \bar{x}_2^{t_2^{(i)}(p)}, \cdots, \bar{x}_n^{t_n^{(i)}(p)}\right)^T, \end{cases} \qquad i = 1, 2, \cdots, \alpha.$$

Lemma 3.2. *Let $x^* \in R^n$ be a common fixed point of the operator class $\mathcal{O}_T$, and the sequence $\{x^p\}_{p \in N_0}$ be generated by the GAMI-method. Assume that there hold*

$$\begin{cases} \mathcal{I}_p |x^{p+1} - x^*| \le \Delta_p \mathcal{I}_p v, \\ (I - \mathcal{I}_p)|x^{p+1} - x^*| \le (I - \mathcal{I}_p)|x^p - x^*|, \end{cases} \qquad p = 0, 1, 2, \cdots, \tag{3.1}$$

where $v \in R^n$ is a positive vector, and $\{\Delta_p\}$ is a nonnegative number sequence satisfying

$$\Delta_{p+1} \le \Delta_p, \qquad p = 0, 1, 2, \cdots. \tag{3.2}$$

Then, for any positive integer $q \ge m_l (l \in \{-1\} \cup N_0, m_{-1} = 0)$, there hold

$$\left(I - \prod_{p=m_l}^{q} (I - \mathcal{I}_p)\right) \left|x^{q+1} - x^*\right| \le \left(I - \prod_{p=m_l}^{q} (I - \mathcal{I}_p)\right) \Delta_{m_l} v, \quad m = -1, 0, 1, 2, \cdots. \tag{3.3}$$

Proof. Analogously to the proof of Theorem 1 in [6], we can inductively demonstrate this lemma.

Lemma 3.3. *Let $x^* \in R^n$ be a common fixed point of the operator class $\mathcal{O}_T$, and the sequence $\{x^p\}_{p \in N_0}$ be generated by the GAMI-method. If we denote*

$$\gamma_p := \max\left\{\|x^p - x^*\|_v, \quad \max_{i \in J(p)} \left\|x^{s^{(i)}(p)} - x^*\right\|_v, \quad \max_{i \in J(p)} \left\|x^{t^{(i)}(p)} - x^*\right\|_v\right\}, \qquad \forall p \in N_0,$$

then, for $p = 0, 1, 2, \cdots$, there hold

(i) $\mathcal{I}_p |x^{p+1} - x^*| \le \sum_{i \in J(p)} E_i \left|T_{p,i}\left(x^{s^{(i)}(p)}\right) - x^*\right| + \left[\mathcal{I}_p - \left(2 \min\{I, E^{-1}\} - I\right) I_p\right] \gamma_p v;$

(ii) $(I - \mathcal{I}_p)|x^{p+1} - x^*| = (I - \mathcal{I}_p)|x^p - x^*|,$

where we use the notation

$$\min\{I, E^{-1}\} = diag\left(\min\{1, e_{11}^{-1}\}, \min\{1, e_{22}^{-1}\}, \cdots, \min\{1, e_{nn}^{-1}\}\right),$$

with $E = diag(e_{11}, e_{22}, \cdots, e_{nn})$.

Proof. Equivalently, we can express (2.1) as

$$x^{p+1} = \sum_{i\in J(p)} E_i T_{p,i}\left(x^{s^{(i)}(p)}\right) + \sum_{i\in J(p)} (I-E)E^{-1}E_i x^{t^{(i)}(p)} + \left(I - \sum_{i\in J(p)} E^{-1}E_i\right) x^p. \tag{3.4}$$

Note that $x^* \in R^n$ is a common fixed point of the operator class $\mathcal{O}_T$, according to (3.4) we have

$$x^* = \sum_{i\in J(p)} E_i T_{p,i}(x^*) + \sum_{i\in J(p)} (I-E)E^{-1}E_i x^* + \left(I - \sum_{i\in J(p)} E^{-1}E_i\right) x^*. \tag{3.5}$$

Now, subtracting (3.5) from (3.4) we immediately obtain

$$\begin{aligned} x^{p+1} - x^* = &\sum_{i\in J(p)} E_i\left(T_{p,i}\left(x^{s^{(i)}(p)}\right) - x^*\right) + \sum_{i\in J(p)} (I-E)E^{-1}E_i\left(x^{t^{(i)}(p)} - x^*\right) \\ &+ \left(I - \sum_{i\in J(p)} E^{-1}E_i\right)(x^p - x^*), \qquad p = 0, 1, 2, \cdots. \end{aligned} \tag{3.6}$$

By making use of the properties of the operators $\mathcal{I}_p (p \in N_0)$ and $\mathcal{P}_i (i = 1, 2, \cdots, \alpha)$, and through direct manipulations we have

$$\begin{aligned} \mathcal{I}_p|x^{p+1} - x^*| \le &\sum_{i\in J(p)} E_i\left|T_{p,i}\left(x^{s^{(i)}(p)}\right) - x^*\right| + \sum_{i\in J(p)} |I-E|E^{-1}E_i\left|x^{t^{(i)}(p)} - x^*\right| \\ &+ \left(\mathcal{I}_p - \sum_{i\in J(p)} E^{-1}E_i\right)|x^p - x^*| \\ \le &\sum_{i\in J(p)} E_i\left|T_{p,i}\left(x^{s^{(i)}(p)}\right) - x^*\right| + \left[|I-E|E^{-1}I_p + (\mathcal{I}_p - E^{-1}I_p)\right]\gamma_p v. \end{aligned}$$

Up to now, to prove (i) we only need to test that there holds

$$|I-E|E^{-1}I_p + (\mathcal{I}_p - E^{-1}I_p) = \mathcal{I}_p - \left(2\min\{I, E^{-1}\} - I\right) I_p. \tag{3.7}$$

In fact, for any $j \in \{1, 2, \cdots, n\}$, if $e_{jj}^{-1} \ge 1$, we easily know that

$$\begin{aligned} \left[|I-E|E^{-1}I_p + (\mathcal{I}_p - E^{-1}I_p)\right]_{jj} &= |1 - e_{jj}|e_{jj}^{-1}[I_p]_{jj} + \left([\mathcal{I}_p]_{jj} - e_{jj}^{-1}[I_p]_{jj}\right) \\ &= (e_{jj}^{-1} - 1)[I_p]_{jj} + \left([\mathcal{I}_p]_{jj} - e_{jj}^{-1}[I_p]_{jj}\right) \\ &= [\mathcal{I}_p]_{jj} - [I_p]_{jj} \\ &= \left[\mathcal{I}_p - \left(2\min\{I, E^{-1}\} - I\right) I_p\right]_{jj}, \end{aligned}$$

where we use $[\bullet]_{jj}$ to denote the j-th diagonal component of the corresponding diagonal matrix; and if $e_{jj}^{-1} < 1$, we easily see that

$$\begin{aligned}\left[|I-E|E^{-1}I_p+(\mathcal{I}_p-E^{-1}I_p)\right]_{jj} &= |1-e_{jj}|e_{jj}^{-1}[I_p]_{jj}+\left([\mathcal{I}_p]_{jj}-e_{jj}^{-1}[I_p]_{jj}\right)\\ &= (1-e_{jj}^{-1})[I_p]_{jj}+\left([\mathcal{I}_p]_{jj}-e_{jj}^{-1}[I_p]_{jj}\right)\\ &= [\mathcal{I}_p]_{jj}-(2e_{jj}^{-1}-1)[I_p]_{jj}\\ &= \left[\mathcal{I}_p-\left(2\min\{I,E^{-1}\}-I\right)I_p\right]_{jj}.\end{aligned}$$

Therefore, the identity (3.7) holds, and we have fulfilled the proof of (i).

From (3.6) we can directly get (ii) by applying the properties of the operators $\mathcal{I}_p (p \in N_0)$ and $\mathcal{P}_i (i = 1, 2, \cdots, \alpha)$, too.

4. Convergence Theory of the GAMI-Method

Theorem 4.1. *Let $x^* \in R^n$, and assume that for $\forall i \in J(p)$, there exist constant $\beta \in (0,1)$ and $\delta \in (0,1)$ independent of i and p such that for any $z \in R^n$, when $\|z - x^*\|_v \le \delta$, there holds either*

$$\|T_{p,i}(z) - x^*\|_v \le \beta\|z - x^*\|_v \tag{4.1}$$

or

$$\|\mathcal{P}_i(T_{p,i}(z) - x^*)\|_v \le \beta\|\mathcal{P}_i(z - x^*)\|_v. \tag{4.2}$$

Then for any $x^0 \in R^n$ satisfying $\|x^0 - x^*\|_v \le \delta$, the sequence $\{x^p\}$ generated by the GAMI-method converges to x^* provided $E < 2I/(1+\beta)$.

Proof. Evidently, each of the inequalities (4.1) and (4.2) implies that x^* is a common fixed point of the operator class $\mathcal{O}_T$. In accordance with Lemma 3.2 and Lemma 3.3, to fulfill this proof we only need to demonstrate that there exists a nonnegative number sequence $\{\Delta_p\}$ such that

(a) $\sum\limits_{i\in J(p)} E_i \left|T_{p,i}\left(x^{s_i(p)}\right) - x^*\right| + \left[\mathcal{I}_p - (2\min\{I, E^{-1}\} - I)I_p\right]\gamma_p v \le \Delta_p \mathcal{I}_p v,\ p = 0, 1, 2, \cdots;$

(b) $\Delta_{p+1} \le \Delta_p$, $p = 0, 1, 2, \cdots$; and

(c) $\lim\limits_{p\to\infty} \Delta_p = 0$.

As a matter of fact, in light of the properties of the operators $\mathcal{P}_i (i = 1, 2, \cdots, \alpha)$ as well as the assumptions (4.1) and (4.2) we get for all $i \in \{1, 2, \cdots, \alpha\}$ and all $z \in R^n$ satisfying $\|z - x^*\|_v \le \delta$ that there hold

$$E_i|T_{p,i}(z) - x^*| = E_i\mathcal{P}_i|T_{p,i}(z) - x^*| \le E_i\beta\|z - x^*\|_v v. \tag{4.3}$$

Now, define

$$\begin{cases} e_{min} = \min\left\{[E_i]_{jj} \mid [E_i]_{jj} > 0; \quad j = 1, 2, \cdots, n; i = 1, 2, \cdots, \alpha\right\}, \\ \Omega = 2\min\{I, E^{-1}\} - (1+\beta)I, \qquad \Gamma = I - e_{min}\Omega, \end{cases} \tag{4.4}$$

and

$$\Delta_p = \Gamma\gamma_p, \qquad p = 0, 1, 2, \cdots. \tag{4.5}$$

Then we easily see that it holds $[\Omega]_{jj} > 0$ and $[\Gamma]_{jj} \in [0, 1)$, $j = 1, 2, \cdots, n$, when $E < 2I/(1+\beta)$. Moreover, through direct computations we obtain by (4.3) that

$$\begin{aligned}
&\sum_{i\in J(p)} E_i \left| T_{p,i}\left(x^{s_i(p)}\right) - x^*\right| + \left[\mathcal{I}_p - \left(2\min\{I, E^{-1}\} - I\right) I_p\right] \gamma_p v \\
&\quad \leq \left[\mathcal{I}_p - \left(2\min\{I, E^{-1}\} - (1+\beta)I\right) I_p\right] \gamma_p v \\
&\quad = [\mathcal{I}_p - \Omega I_p]\,\gamma_p v
\end{aligned}$$

hold for both cases (4.1) and (4.2). Note that for any $p \in N_0$ and any $j \in \{1, 2, \cdots, n\}$, $[\mathcal{I}_p]_{jj} = 0$ if and only if $[I_p]_{jj} = 0$, we see that

$$\begin{aligned}
[\mathcal{I}_p - \Omega I_p]\,\gamma_p v &= [\mathcal{I}_p - \Omega I_p]\,\gamma_p \mathcal{I}_p v \\
&\leq \Gamma\gamma_p\mathcal{I}_p v = \Delta_p \mathcal{I}_p v.
\end{aligned}$$

Therefore, (a) holds for all $p = 0, 1, 2, \cdots$. In accordance with Lemma 3.3 we know that there have

$$\mathcal{I}_p|x^{p+1} - x^*| \leq \Gamma\gamma_p\mathcal{I}_p v, \qquad p = 0, 1, 2, \cdots.$$

To test (b) we only need to demonstrate the validity of the inequalities

$$\gamma_{p+1} \leq \gamma_p, \qquad p = 0, 1, 2, \cdots. \tag{4.6}$$

Because of

$$\begin{aligned}
|x^{p+1} - x^*| &= |\mathcal{I}_p(x^{p+1} - x^*) + (I - \mathcal{I}_p)(x^{p+1} - x^*)| \\
&\leq \mathcal{I}_p|x^{p+1} - x^*| + (I - \mathcal{I}_p)|x^{p+1} - x^*| \\
&\leq \Gamma\gamma_p\mathcal{I}_p v + (I - \mathcal{I}_p)|x^p - x^*| \\
&\leq \Gamma\gamma_p\mathcal{I}_p v + (I - \mathcal{I}_p)\gamma_p v \\
&= [\Gamma\mathcal{I}_p + (I - \mathcal{I}_p)]\gamma_p v \\
&\leq \gamma_p v,
\end{aligned} \tag{4.7}$$

by applying Lemma 3.1, we easily see that when $p = 0$ there holds $\gamma_1 \leq \gamma_0$. Now, based upon (4.7) and Lemma 3.1 again, and by making use of induction, we can immediately deduce the validity of (4.6).

Now, we turn to (c). Evidently, we only need to verify that it holds

$$\gamma_p \leq \Gamma^l\gamma_0, \qquad \forall p \geq m_l, \quad l = 0, 1, 2, \cdots. \tag{4.8}$$

In fact, when $l = 0$, (4.8) is trivial. Suppose that (4.8) holds for all $p \geq m_l$. Then, when $p \geq m_{l+1}$, from Lemma 3.2 we see that

$$|x^{p+1} - x^*| \leq \Delta_{m_l} v = \Gamma\gamma_{m_l} v,$$

and from the definition of $\{\gamma_p\}$ as well as the induction assumption we get that

$$\gamma_{p+1} \leq \Gamma\gamma_{m_l} \leq \Gamma \times \Gamma^l\gamma_0 = \Gamma^{l+1}\gamma_0.$$

That is, (4.8) also holds for all $p \geq m_{l+1}$. The above discussion shows the validity of (4.8).

(4.7) and (4.8) immediately give $\lim_{p\to\infty} x^p = x^*$, and the proof of this theorem is hence completed.

We use the following remarks to end this paper.

Remark 4.1. The iteration formula (2.1) can be equivalently expressed as

$$x^{p+1} = \sum_{i\in J(p)} \left(E^{-1}E_i\right)\left[ET_{p,i}\left(x^{s^{(i)}(p)}\right) + (I-E)x^{t^{(i)}(p)}\right] + \sum_{i\notin J(p)} \left(E^{-1}E_i\right)x^p, \quad p = 0,1,2,\cdots.$$

Note that $\sum_{i=1}^{\alpha} E^{-1}E_i = I$, we see that the requirement that the sum of the weighting matrices $E_i(i = 1,2,\cdots,\alpha)$ does not equal to the identity matrix is just equivalent to relaxing the original iteration with the diagonal matrix E, the sum of the weighting matrices. Hence, whether $E = I$ or not is not relevant for multisplitting iterative methods from the theoretical point of view.

Remark 4.2. Different constructions of the operator class $\mathcal{O}_T$ can result in various asynchronous parallel multisplitting iterative methods for solving systems of linear and nonlinear equations. Some representatives of the choices of the operator class $\mathcal{O}_T$ have been shown in [1, 3, 9].

Remark 4.3. The existing results in the papers [1-9] are special cases of that in this paper.

References

[1] Z.Z. Bai, D.R. Wang, Generalized matrix multisplitting relaxation methods and their convergence, *Numer. Math. J. Chinese Univ. (English Ser.),* **2** (1993), 87-100.

[2] Z.Z. Bai, D.R. Wang, D.J. Evans, Models of asynchronous parallel matrix multisplitting relaxed iterations, *Parallel Comput.,* **21** (1995), 565-582.

[3] Z.Z. Bai, D.R. Wang, D.J. Evans, Models of asynchronous parallel nonlinear multisplitting relaxed iterations, *J. Comp. Math.,* **13** (1995), 369-386.

[4] R. Bru, L. Elsner, M. Neumann, Models of parallel chaotic iteration methods, *Linear Algebra Appl.,* **103** (1988), 175-192.

[5] D.P. O'Leary, R.E. White, Multisplittings of matrices and parallel solution of linear systems, *SIAM J. Alg. Disc. Meth.,* **6** (1985), 630-640.

[6] Y.F. Su, Generalized multisplitting asynchronous iteration, *Linear Algebra Appl.,* **235** (1996), 77-92.

[7] D.R. Wang, Z.Z. Bai, D.J. Evans, A class of asynchronous parallel matrix multisplitting relaxation methods, *Parallel Algorithms Appl.,* **2** (1994), 173-192.

[8] D.R. Wang, Z.Z. Bai, D.J. Evans, A class of asynchronous parallel nonlinear multisplitting relaxation methods, *Parallel Algorithms Appl.,* **2** (1994), 209-228.

[9] D.R. Wang, Z.Z. Bai, D.J. Evans, Asynchronous multisplitting relaxed iterations for weakly nonlinear systems, *Intern. J. Computer Math.,* **54** (1994), 57-76.