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A WAVELET METHOD FOR THE FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS WITH CONVOLUTION KERNEL[*1)]

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Abstract

We study the Fredholm integro-differential equation

$$D_x^{2s}\sigma(x) + \int_{-\infty}^{+\infty} k(x-y)\sigma(y)dy = g(x)$$

by the wavelet method. Here $\sigma(x)$ is the unknown function to be found, $k(y)$ is a convolution kernel and $g(x)$ is a given function. Following the idea in [7], the equation is discretized with respect to two different wavelet bases. We then have two different linear systems. One of them is a Toeplitz-Hankel system of the form $(H_n + T_n)x = b$ where T_n is a Toeplitz matrix and H_n is a Hankel matrix. The other one is a system $(B_n + C_n)y = d$ with condition number $\kappa = O(1)$ after a diagonal scaling. By using the preconditioned conjugate gradient (PCG) method with the fast wavelet transform (FWT) and the fast iterative Toeplitz solver, we can solve the systems in $O(n \log n)$ operations.

Key words: Fredholm integro-differential equation, Kernel, Wavelet transform, Toeplitz matrix, Hankel matrix, Sobolev space, PCG method.

1. Introduction

In this paper, we study the Fredholm integro-differential equation

$$A(\sigma(x)) \equiv D_x^{2s}\sigma(x) + \int_{-\infty}^{+\infty} k(x-y)\sigma(y)dy = g(x) \tag{1}$$

by the wavelet method. The applications of the equation in image restoration could be found in [10]. For the history of numerical methods for the Fredholm integro-differential equations, we refer to [4]. Following the idea in [7], the equation is discretized with respect to two different orthonormal wavelet bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of $L^2(R)$. The $\mathcal{B}_1$ comes from the father wavelet $\varphi(x)$ and the $\mathcal{B}_2$ comes from the mother wavelet $\psi(x)$. After discretizing of the equation with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ on a finite interval, we then have

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two different n-by-n linear systems. One of them is a Toeplitz-Hankel system of the form

$$(H_n + T_n)x = b \tag{2}$$

where T_n is a Toeplitz matrix and H_n is a Hankel matrix. The other one is a system

$$(B_n + C_n)y = d \tag{3}$$

with condition number

$$\kappa(D_n^{-1/2}(B_n + C_n)D_n^{-1/2}) = O(1) \tag{4}$$

after a diagonal scaling D_n. The relation between $H_n + T_n$ and $B_n + C_n$ is $B_n + C_n = W_n(H_n + T_n)W_n^{-1}$ where W_n is the wavelet transform matrix between $\mathcal{B}_1$ and $\mathcal{B}_2$.

We then solve (2) by solving its equivalent form (3) with $y = W_n x$ and $d = W_n b$. For solving (3), we use the PCG method with the diagonal preconditioner D_n. The condition number of the preconditioned system is, by (4),

$$\kappa(D_n^{-1}(B_n + C_n)) = \kappa(D_n^{-1/2}(B_n + C_n)D_n^{-1/2}) = O(1).$$

When the PCG method is applied to solve the preconditioned system, the convergence rate will be linear, see [5]. By using the FWT, see [2], and fast iterative Toeplitz solver, see [1] and [9], we can solve the system $(B_n + C_n)y = d$ and also $(H_n + T_n)x = b$ in $O(n \log n)$ operations.

2. Discretization of Fredholm Equation

The Fredholm integro-differential equation is given as follows, $A\sigma = g$, where A is defined by (1), $g \in L^2(R)$ and $k(x-y) \in L^2(R)$ is symmetric and positive, i.e., $k(x-y) = k(y-x) > 0$. For solving the equation, we need to find $\sigma \in C_0^{2s}(R)$ such that (1) is to be satisfied. The equivalent variational form of (1) is: find $\sigma \in H_0^s(R)$ such that

$$B(\sigma, \mu) = F(\mu) \tag{5}$$

for $\forall \mu \in H_0^s(R)$. Here $B(\sigma,\mu) = B_0(\sigma,\mu) + B_1(\sigma,\mu)$ with

$$\begin{aligned} B_0(\sigma,\mu) &= \int_{-\infty}^{+\infty} D_x^s\sigma(x) D_x^s\mu(x)dx, \\ B_1(\sigma,\mu) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k(x-y)\sigma(y)\mu(x)dydx \end{aligned}$$

and

$$F(\mu) = \int_{-\infty}^{+\infty} g(x)\mu(x)dx.$$

We assume that $B(\sigma,\mu)$ is a continuous elliptic bilinear form on $H_0^s(R) \times H_0^s(R)$, i.e., there exist two constants $\beta \geq \alpha > 0$, such that $\alpha\|\sigma\|_{H_0^s}^2 \leq B(\sigma,\sigma)$ and $B(\sigma,\mu) \leq \beta\|\sigma\|_{H_0^s}\|\mu\|_{H_0^s}$. For instance, when $s = 0$ (or $s = 1$) and $+\infty > C \geq k(x-y) \geq c > 0$, then obviously, $B(\sigma,\mu)$ is a continuous elliptic bilinear form on $L^2(R) \times L^2(R)$ (or $H_0^1(R) \times H_0^1(R)$).

2.1 Wavelet Bases

Now, following the idea in [7], we are going to discretize the Fredholm integro-differential equation with respect to two different orthonormal wavelet bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of $L^2(R)$. First of all, we introduce a function $\varphi(x) \in L^2(R)$ called the father wavelet (or scaling function), with a compact support $[0, a]$, $a > 0$, see [3]. The $\varphi(x)$ has the property that

$$\varphi(x-k), \qquad k \in Z \tag{6}$$

form an orthonormal sequence in $L^2(R)$. Let V_0 be the closed linear subspace of $L^2(R)$ generated by (6). A chain of closed subspaces in $L^2(R)$ is given as

$$\cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots.$$

The multiresolution analysis (MRA), depending on $\varphi(x)$, is given as follows:

(i) $f(x) \in V_0$ if and only if $f(2^j x) \in V_j$;

(ii) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$;

(iii) $\overline{\bigcup_{-\infty}^{\infty} V_j} = L^2(R)$ and $\bigcap_{-\infty}^{\infty} V_j = 0$;

(iv) The sequence (6) forms an orthonormal basis of V_0.

Let W_j denote the orthogonal complement of V_j in V_{j+1}, i.e., $V_{j+1} = V_j \oplus W_j$.

From MRA (iii), we also have $\oplus_{-\infty}^{\infty} W_j = L^2(R)$. There exists at least one function $\psi(x) \in W_0$ such that $\psi(x-k)$, $k \in Z$ form an orthonormal basis of W_0, see [2] and [8]. The $\psi(x)$ is called the mother wavelet. We then construct following two wavelet sequences: $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$, $j, k \in Z$, and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in Z$. The $\{\varphi_{j,k}(x)\}$ and $\{\psi_{j,k}(x)\}$ form two wavelet bases of $L^2(R)$ and $\{\psi_{j,k}(x)\}$ also constructs an orthonormal basis of $H_0^s(R)$ for $0 \le s < r$ where r is the regularity of the MRA, see [6] and [8]. The bilinear form B defined by (5) can be projected on the subspace V_J (J is fixed) with respect to the following two bases in V_J:

$$\mathcal{B}_1 = \{\varphi_{J,k}(x)\} \quad \text{and} \quad \mathcal{B}_2 = \bigcup_{-\infty < j \le J-1} \{\psi_{j,k}(x)\}.$$

The following lemma could be found in [6] and [8].

Lemma 1. *Let* $f = \sum_{j,k} \langle f, \psi_{j,k}\rangle \psi_{j,k}$. *Then* $f \in H_0^s(R)$ *if and only if*

$$\|f\|_{H_0^s}^2 \equiv \sum_{j,k} |\langle f, \psi_{j,k}\rangle|^2 (1 + 2^{2j})^s < +\infty, \quad 0 \le s < r \tag{7}$$

where r is the regularity of the MRA.

2.2 Projection of B with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$

Let B_J denote the projection of $B(\sigma, \mu)$ on $V_J \times V_J$. The matrix representation of B_J corresponding to the basis $\mathcal{B}_1$ has the elements given by

$$m_{p,q} = B(\varphi_{J,p}, \varphi_{J,q}) \tag{8}$$

where $\forall p, q \in Z$. For $\forall \sigma, \mu \in H_0^s(R)$, let σ_J, μ_J denote the the projections of σ, μ on V_J respectively. Then the equation (5) becomes

$$B(\sigma_J, \mu_J) = F(\mu_J) \tag{9}$$

Let

$$\sigma_J = \sum_{p \in Z} x_p \varphi_{J,p} \quad \text{and} \quad \mu_J = \varphi_{J,q}, \qquad \forall q \in Z. \tag{10}$$

Substituting (10) into (9), we have the following linear system

$$M_\infty x = b \tag{11}$$

where $(M_\infty)_{p,q} = m_{p,q}$ is given by (8), and

$$(x)_p = x_p, \qquad (b)_q = \int_{-\infty}^{+\infty} g(x)\varphi_{J,q}(x)dx.$$

Let H_∞ and T_∞ be matrices with $(H_\infty)_{p,q} = h_{p,q}$ and $(T_\infty)_{p,q} = t_{p,q}$ where $h_{p,q} = B_0(\varphi_{J,p}, \varphi_{J,q})$ and $t_{p,q} = B_1(\varphi_{J,p}, \varphi_{J,q})$. Then we have $M_\infty = H_\infty + T_\infty$. Because,

$$\begin{aligned} h_{p,q} = h_{q,p} = B_0(\varphi_{J,p}, \varphi_{J,q}) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{D_x^s\varphi_{J,p}}(\xi)\widehat{D_x^s\varphi_{J,q}}(\xi)d\xi \\ &= \frac{1}{2\pi 4^J}\int_{-\infty}^{+\infty}(i\xi)^{2s}e^{-i(p+q)2^{-J}\xi}|\hat{\varphi}(2^{-J}\xi)|d\xi = h_{p+q}. \end{aligned}$$

Hence, H_∞ is a Hankel matrix. For matrix T_∞, since $k(x-y)$ is symmetric and $\varphi(x)$ has the compact support $[0, a]$, we have

$$\begin{aligned} t_{p,q} &= B_1(\varphi_{J,p}, \varphi_{J,q}) \\ &= 2^J \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k(x-y)\varphi(2^Jx-p)\varphi(2^Jy-q)dydx \\ &= 2^J \int_{2^{-J}p}^{2^{-J}(a+p)}\int_{2^{-J}q}^{2^{-J}(a+q)} k(x-y)\varphi(2^Jx-p)\varphi(2^Jy-q)dydx \\ &= 2^{-J}\int_0^a\int_0^a k[2^{-J}(x-y+p-q)]\varphi(x)\varphi(y)dydx = t_{p-q} = t_{q,p}. \end{aligned}$$

Hence, T_∞ is a Toeplitz matrix. Therefore, (11) is a Toeplitz-Hankel system.

The matrix representation of B_J corresponding to the basis $\mathcal{B}_2$ has the elements given by

$$n_{p,l;q,m} = B(\psi_{p,q}, \psi_{l,m}) \tag{12}$$

for $-\infty < p, l < J$ and $-\infty < q, m < \infty$. Let

$$\sigma_J = \sum_{p,q} y_{p,q}\psi_{p,q} \quad \text{and} \quad \mu_J = \psi_{l,m}, \quad -\infty < l < J, \quad \forall m \in Z. \tag{13}$$

Substituting (13) into (9), we have the following linear system

$$N_\infty y = d \tag{14}$$

where $(N_\infty)_{p,l;q,m} = n_{p,l;q,m}$, given by (12), denotes the (p, l)th entry of the (q, m)th block of N_∞, $y = (y_{p,q})^T$ and $d = (d_{p,q})^T$ are vectors with $d_{p,q} = \int_{-\infty}^{+\infty} g(x)\psi_{p,q}(x)dx$. Let B_∞ and C_∞ be matrices with

$$(B_\infty)_{p,l;q,m} = B_0(\psi_{p,q}, \psi_{l,m}) \quad \text{and} \quad (C_\infty)_{p,l;q,m} = B_1(\psi_{p,q}, \psi_{l,m}).$$

Then we have $N_\infty = B_\infty + C_\infty$.

3. Condition Number and Operation Cost

Now we consider the condition number of the system (14) by following the idea in [6]. Let $\phi \in V_J$ with $\phi = \sum_{j,k} w_{j,k}\psi_{j,k}$. We have

$$B(\phi,\phi) = \sum_{j,k}\sum_{p,q} w_{j,k}w_{p,q}n_{j,p;k,q} = w^T N_\infty w \tag{15}$$

where $w = (w_{j,k})^T$ is a vector. By the assumption that $B(\sigma,\mu)$ is a continuous elliptic bilinear form on the space $H_0^s(R) \times H_0^s(R)$, we have

$$C_1\|\phi\|^2_{H_0^s} \le B(\phi,\phi) \le C_2\|\phi\|^2_{H_0^s} \tag{16}$$

where $C_2 \ge C_1 > 0$ are constants. Combining (15) and (16), we have

$$C_1\|\phi\|^2_{H_0^s} \le w^T N_\infty w \le C_2\|\phi\|^2_{H_0^s}.$$

By using (7), one can easily obtain

$$C_3 \sum_{j,k} |2^{js}w_{j,k}|^2 \le w^T N_\infty w \le C_4 \sum_{j,k} |2^{js}w_{j,k}|^2$$

where $C_4 \ge C_3 > 0$ are constants. After a diagonal scaling D, we have

$$C_3\|w\|^2 \le w^T D^{-1/2} N_\infty D^{-1/2} w \le C_4\|w\|^2$$

where $\|\cdot\|$ is the l_2-norm. Thus, the condition number of N_∞ after a diagonal scaling is

$$\kappa(D^{-1/2}N_\infty D^{-1/2}) = O(1). \tag{17}$$

The relation between M_∞ given by (11) and N_∞ given by (14) is $N_\infty = WM_\infty W^{-1}$ where W is the wavelet transform matrix between two orthonormal wavelet bases $\mathcal{B}_1$ and $\mathcal{B}_2$. We then solve the Toeplitz-Hankel system (11) by solving its equivalent form $(WM_\infty W^{-1})Wx = Wb$ i.e., $N_\infty y = d$ where $y = Wx$ and $d = Wb$. We use the PCG method with the diagonal preconditioner D to solve the preconditioned system $D^{-1}N_\infty y = D^{-1}d$. Since by (17), the condition number of the preconditioned system is $O(1)$, the convergence rate will be linear, see [5].

In practice, we usually use a finite interval instead of $(-\infty,+\infty)$. We then have an n-by-n system

$$M_n x = b \tag{18}$$

where M_n is the finite section of M_∞. Let W_n be the finite section of the wavelet transform matrix W. The system (18) can be solved by solving its equivalent form

$$(W_n M_n W_n^{-1})W_n x = W_n b$$

i.e.,

$$N_n y = d \tag{19}$$

where N_n is the finite section of N_∞, $y = W_n x$ and $d = W_n b$. The PCG method is applied to solve the system (19) with the diagonal preconditioner D_n which is the finite section of D. In each iteration of the PCG method, we have to compute the matrix-vector multiplication $N_n v$ for some vector v and solve the system $D_n z = t$, see [5]. For $N_n v$, we know that

$$N_n v = (B_n + C_n)v = B_n v + C_n v.$$

For $C_n v = W_n T_n W_n^{-1} v$, By using the FWT, $u = W_n^{-1} v$ could be computed in $O(n)$ operations. The $T_n u$ could be computed by using the fast iterative Toeplitz solver in $O(n \log n)$, see [1] and [9]. By using FWT again, we note that the operation cost for $C_n v$ will be $O(n \log n)$. Similarly, $B_n v$ could also be computed in $O(n \log n)$ operations. Hence, the operation cost for $N_n v$ will remain $O(n \log n)$. It requires only $O(n)$ operations to solve the system $D_n z = t$. Thus, the total operation cost per iteration is $O(n \log n)$. Since the number of iteration is independent of n, we therefore can solve the system (19) and also the system (18) in $O(n \log n)$ operations.

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