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THE STABILITY AND CONVERGENCE OF COMPUTING LONG-TIME BEHAVIOUR*1)

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Abstract

The object of this paper is to establish the relation between stability and convergence of the numerical methods for the evolution equation $u_t - Au - f(u) = g(t)$ on Banach space V, and to prove the long-time error estimates for the approximation solutions. At first, we give the definition of long-time stability, and then prove the fact that stability and compatibility imply the uniform convergence on the infinite time region. Thus, we establish a general frame in order to prove the long-time convergence. This frame includes finite element methods and finite difference methods of the evolution equations, especially the semilinear parabolic and hyperbolic partial differential equations. As applications of these results we prove the estimates obtained by Larsson [5] and Sanz-serna and Stuart [6].

Key words: Stability, Compatibility, Covergence, Reaction-diffusion equation, Long-time error estimates.

1. Introduction

In 1978, Hoff[3] considered the long-time behavior computation of nonlinear reaction-diffusion equations, which is supposed to have an invariant region S, i.e. any local solution arising from a point in S is constrained to lie in. Hoff constructed a family of finit difference schemes for the equations. Under some assumptions he proved that any trajectory starting in S will converge to an asymptotically stable equilibrium, and S is also an invariant region of the difference equations. So Hoff obtained error estimates uniform in time for the difference equations. In 1989, Larsson[5] studied the long-time error estimates of finite-element approximations of reaction-diffusion equation (below dimension 3). The distinction between [5] and [3] is that Larsson didn't assume the equation has a invariant region but has an asymptotically stable hyperbolic equilibrium, and so the trajectories constrict to some neighbourhood of the equilibrium.

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However, by the standard finite time error estimates one can show that the discretization solution will enter this neighbourhood. In 1992, Sanz-Serna and Stuart[6] obtained an error estimates uniform in time for explicit difference scheme of one-dimensional reaction-diffusion equation by using analogous technique of [5]. Note that all of above results are obtained under the condition that the continuous trajectories converge to an asymptotically stable, hyperbolic equilibrium and they are difficult to be generalized.

In this paper, we'll establish the relation between stability and convergence, and then obtain a sufficient condition of long-time convergence of discrete methods for more general equation. Under such a condition we get an error estimate on the infinite time region. Therefore, we provide an abstract frame to prove convergence. This method dosen't assume the existence of an equilibrium. It can be used to both the finite element methods and the difference methods, the explicit schemes and the implicit schemes.

The paper is outlined as follows: In 2, we give definitions of stability, compatibility and covergence of discrete schemes on the infinite time region. Especially, we obtain a theorem which states that the stability and compatibility imply convergence. In 3 and 4, we apply this theorem to the problem in [5] and obtain the similar results; to the problem in [6] and obtain the same results. In 5 we give an example whose continuous solution is periodical in time. It can't include in the frame of [3], [5] or [6]. But we can prove the long-time convergence of an explicit difference scheme from this theorem.

2. The Relation Between Stability and Convergence

Let V be a Banach space with norm $\|\cdot\|$. We consider the evolutionary equation such as:

$$\begin{aligned} &u_t - Au - f(u) = g(t), \\ &u(0) = u_0, \end{aligned} \tag{2.1}$$

here A and f are operators on a dense subset of V to V. Let $u(t) \in V$ be a solution of (2.1).

Let V_h be a finite dimensional Banach space with norm $\|\cdot\|_h$. It may or may not be the subspace of V. Let p_h be a operator from V to V_h. We denote the discretization of (2.1) as follows:

$$\begin{aligned} &B_{h,\tau}(u_{h,\tau}^{n+1}) = C_{h,\tau}(u_{h,\tau}^n) + \tau g_{h,\tau}^n, \quad n = 0, 1, 2, \cdots \\ &u_{h,\tau}^0 = u_{0,h}, \end{aligned} \tag{2.2}$$

where $g_{h,\tau}^n$, $u_{0,h} \in V_h$, $B_{h,\tau}$ and $C_{h,\tau}$ are operators on V_h, h is space step-size and τ is time step-size, $u_{h,\tau}^n$ is the approximation to $u(t_n)(t_n = n\tau)$.

Let

$$L_{h,\tau}(u_{h,\tau}^n) = (B_{h,\tau}(u_{h,\tau}^{n+1}) - C_{h,\tau}(u_{h,\tau}^n))/\tau, \tag{2.3}$$

$$R_{h,\tau}^n(u) = L_{h,\tau}(p_h u(n\tau)) - g_{h,\tau}^n, \tag{2.4}$$

$$N = \{1, 2, \cdots\}, \quad N_0 = \{0\} \cup N, \tag{2.5}$$

$$S_{h,\tau} = \sup_{n \in N_0} \|R_{h,\tau}^n(u)\|_h, \tag{2.6}$$

$$e_{h,\tau}^n = u_{h,\tau}^n - p_h u(n\tau), \tag{2.7}$$

$$\bar{B}_{h,\tau}^n(v_h) = B_{h,\tau}(p_h u(n\tau) + v_h) - B_{h,\tau}(p_h u(n\tau)), \tag{2.8}$$

$$\bar{C}_{h,\tau}^n(v_h) = C_{h,\tau}(p_h u(n\tau) + v_h) - C_{h,\tau}(p_h u(n\tau)). \tag{2.9}$$

Condition (A)**.** There exist positive numbers θ_τ^n, $\varepsilon_{1,h,\tau}, \varepsilon_{2,h,\tau}, \rho, M, T$, such that, for any $n \in N$,

$(A1)$ $\bar{B}_{h,\tau}^n$ has an inverse on $\{w_h | w_h \in V_h, \|w_h\|_h \le \varepsilon_{2,h,\tau}\}$, and

$$\|(\bar{B}_{h,\tau}^n)^{-1}(w_h)\|_h \le M\|w_h\|_h, \quad \forall w_h \in V_h;$$

$(A2)_1$ For $\|v_h\|_h \le \varepsilon_{1,h,\tau}$ we have $\|\bar{C}_{h,\tau}^n(v_h)\|_h \le \theta_\tau^n \|\bar{B}_{h,\tau}^n(v_h)\|_h$ and

$$\|\bar{C}_{h,\tau}^0(v_h)\|_h \le M\|v_h\|_h;$$

$(A2)_2$ If $n\tau \le T$, then $\theta_\tau^n \le 1 + M\tau$; if $n\tau > T$, then $\theta_\tau^n \le 1 - \rho\tau$;

$(A3)$ $\varepsilon_{1,h,\tau} \ge M(aS_{h,\tau} + b\|e_{h,\tau}^0\|_h)$, $\varepsilon_{2,h,\tau} \ge aS_{h,\tau} + b\|e_{h,\tau}^0\|_h$,

where

$$a = \frac{1}{\rho} + \frac{e^{M(T+\tau)} - 1}{M}, b = Me^{MT}. \tag{2.10}$$

We have

Theorem 2.1. *Assume that scheme* (2.2) *satisfies condition* (A) *for a pair of* h, τ, *and*

$$\|e_{h,\tau}^0\|_h \le \varepsilon_{1,h,\tau}, \tag{2.11}$$

Then the solution of (2.2) *exists and satisfies*

$$\|e_{h,\tau}^n\|_h \le M(aS_{h,\tau} + b\|e_{h,\tau}^0\|_h), \quad (n \in N), \tag{2.12}$$

where a *and* b *are defined by* (2.10).

Proof. From (2.3)–(2.9), we have

$$-\tau R_{h,\tau}^j(u) = \tau(L_{h,\tau}(u_{h,\tau}^j) - L_{h,\tau}(p_h u(j\tau))) = \bar{B}_{h,\tau}^{j+1}(e_{h,\tau}^{j+1}) - \bar{C}_{h,\tau}^j(e_{h,\tau}^j)$$

for any $j \in N_0$ and so

$$\bar{B}_{h,\tau}^{j+1}(e_{h,\tau}^{j+1}) = \bar{C}_{h,\tau}^j(e_{h,\tau}^j) - \tau R_{h,\tau}^j(u). \tag{2.13}$$

Now, we are going to prove the following proposition by mathematical induction: there exists $e^j_{h,\tau}(j \geq 0)$ satisfing (2.13) and

$$\|e^j_{h,\tau}\|_h \leq \varepsilon_{1,h,\tau}, (j \in N_0). \tag{2.14}$$

From (2.11) we know the proposition is correct for $j = 0$.

Assume that the proposition are correct for $j \leq n$, i.e. there exists $e^j_{h,\tau}$ $(0 \leq j \leq n)$ satisfing (2.13) and

$$\|e^j_{h,\tau}\|_h \leq \varepsilon_{1,h,\tau}, (0 \leq j \leq n). \tag{2.15}$$

Now, we are going to prove the proposition for $j = n+1$. At first, we give the estimate on the norm of $\bar{C}^n_{h,\tau}(e^n_{h,\tau}) - \tau R^n_{h,\tau}(u)$.

If $n = 0$,then from $(A2)_1$ and (2.10) we have

$$\|\bar{C}^0_{h,\tau}(e^0_{h,\tau}) - \tau R^0_{h,\tau}(u)\|_h \leq \tau S_{h,\tau} + \|\bar{C}^0_{h,\tau}(e^0_{h,\tau})\|_h, \leq aS_{h,\tau} + b\|e^0_{h,\tau}\|_h. \tag{2.16}$$

For $n \geq 1$, then from $(A2)_1$ and (2.15), we have

$$\|\bar{C}^j_{h,\tau}(e^j_{h,\tau})\|_h \leq \theta^j_\tau \|\bar{B}^j_{h,\tau}(e^j_{h,\tau})\|_h, (1 \leq j \leq n). \tag{2.17}$$

Moreover, from (2.13),

$$\|\bar{B}^{j+1}_{h,\tau}(e^{j+1}_{h,\tau})\|_h \leq \|\bar{C}^j_{h,\tau}(e^j_{h,\tau})\|_h + \tau S_{h,\tau} \quad (0 \leq j \leq n-1). \tag{2.18}$$

Combining (2.17) and (2.18) we get

$$\begin{aligned}\|\bar{C}^n_{h,\tau}(e^n_{h,\tau}) - \tau R^n_{h,\tau}(u)\|_h \leq & \tau S_{h,\tau} + \theta^n_\tau \|\bar{B}^n_{h,\tau}(e^n_{h,\tau})\|_h \\ \leq & \tau S_{h,\tau} + \theta^n_\tau (\|\bar{C}^{n-1}_{h,\tau}(e^{n-1}_{h,\tau})\|_h + \tau S_{h,\tau}) \leq \cdots \\ \leq & (1 + \theta^n_\tau + \cdots + \theta^n_\tau \cdots \theta^1_\tau)\tau S_{h,\tau} + \theta^n_\tau \cdots \theta^1_\tau \|\bar{C}^0_{h,\tau}(e^0_{h,\tau})\|_h.\end{aligned}$$

If $n\tau \leq T$,then from $(A2)_2$ we have

$$\begin{aligned}1 + \theta^n_\tau + \cdots + \theta^n_\tau \cdots \theta^1_\tau \leq & 1 + (1 + M\tau) + \cdots + (1 + M\tau)^n \\ = & \frac{(1 + M\tau)^{n+1} - 1}{M\tau} \leq \frac{e^{M(T+\tau)} - 1}{M\tau}, \\ \theta^n_\tau \cdots \theta^1_\tau \leq & (1 + M\tau)^n < e^{MT}.\end{aligned}$$

If $n\tau > T$, let $m = [T/\tau]$, then from $(A2)_2$,

$$\begin{aligned}1 + \theta^n_\tau + \cdots + \theta^n_\tau \cdots \theta^1_\tau \leq & 1 + (1 - \rho\tau) + \cdots + (1 - \rho\tau)^{n-m}\Big(\frac{e^{M(T+\tau)} - 1}{M\tau}\Big) \\ < & \frac{1}{\tau}\Big(\frac{1}{\rho} + \frac{e^{M(T+\tau)} - 1}{M}\Big),\end{aligned}$$

$$\theta_\tau^n \cdots \theta_\tau^1 \leq (1+M\tau)^m < e^{MT}.$$

Thus, we have the following estimate for both $n=0$ and $n \geq 1$:

$$\|\bar{C}_{h,\tau}^n(e_{h,\tau}^n) - \tau R_{h,\tau}^n(u)\|_h \leq aS_{h,\tau} + b\|e_{h,\tau}^0\|_h. \tag{2.19}$$

Finally, by $(A3)$ we get further estimate

$$\|\bar{C}_{h,\tau}^n(e_{h,\tau}^n) - \tau R_{h,\tau}^n(u)\|_h \leq \varepsilon_{2,h,\tau}.$$

Now we can conclude from $(A1)$ that there exists $e_{h,\tau}^{n+1}$ such that

$$\bar{B}_{h,\tau}^{n+1}(e_{h,\tau}^{n+1}) = \bar{C}_{h,\tau}^n(e_{h,\tau}^n) - \tau R_{h,\tau}^n(u).$$

Hence, from (2.19) we have

$$\|e_{h,\tau}^{n+1}\|_h = \|(\bar{B}_{h,\tau}^{n+1})^{-1}(\bar{C}_{h,\tau}^n(e_{h,\tau}^n) - \tau R_{h,\tau}^n(u))\|_h, \leq M(aS_{h,\tau} + b\|e_{h,\tau}^0\|_h). \tag{2.20}$$

By noting $(A3)$ we get

$$\|e_{h,\tau}^{n+1}\|_h \leq \varepsilon_{1,h,\tau}, \tag{2.21}$$

so the proposition is hold for $j=n+1$. By induction, for $\forall j \in N_0$, there exists $e_{h,\tau}^j$ satisfing (2.13) and also (2.14). Therefore, we have proved the existence of the discrete solution. And from the procedure of the proof we can easily conclude (from (2.20)) that (2.12) holds for $n \in N$. This completes the proof of Theorem 2.1. #

Assume that h and τ satisfy a relation (R) (such as the mesh size satisfies some conditions). Now, let us give the definitions of compatibility, stability and convergence

Compatibility: $\lim\limits_{h,\tau\to 0} S_{h,\tau} = 0$.

Convergence: $\lim\limits_{h,\tau\to 0} \sup\limits_{n\in N_0} \|e_{h,\tau}^n\|_h = 0$.

Stability: There exist $h_0, \tau_0 > 0$, such that, if h, τ satisfy relation (R), and $0 < h \leq h_0$, $0 < \tau \leq \tau_0$, then scheme (2.2) satisfies condition $(A1)$ and $(A2)$, here the constants ρ, M, T don't depend on h, τ.

In [2] Guo Ben-yu discussed the stability of the approximation to the general nonlinear equation, but his discussions were all on finite time interval. However, our definition of stability is more concerned with the case of infinite time interval

Theorem 2.2. *Assume that the scheme* (2.2) *satisfies the conditions of stability, compatibility and* $(A3)$ *for* $0 < h \leq h_0$, $0 < \tau \leq \tau_0$ *(*h, τ *satisfy* (R)*). Moreover, the initial approximation satisfies the property*

$$\|e_{h,\tau}^0\|_h \leq \varepsilon_{1,h,\tau}, \quad \textit{and} \quad \lim_{h,\tau\to 0}\|e_{h,\tau}^0\|_h = 0. \tag{2.22}$$

Then the solution of (2.2) *exists and satisfies the error estimate*

$$\|e^n_{h,\tau}\|_h \le M(aS_{h,\tau} + b\|e^0_{h,\tau}\|_h), (n \in N), \tag{2.23}$$

where a, b *defined by* (2.10).

As a consequence, the approximation $u^n_{h,\tau}$ converges to the solution u of (2.1).

Proof. Obviously the definition of stability and $(A3)$ imply the condition (A). Next, from (2.22) we know the conditions of the theorem 2.1 are satisfied, and hence the estimate (2.23) holds. Finally from compatibility and (2.22) we can deduce the convergence of the approximation solution.

Remark 1. If $\varepsilon_{1,h,\tau}, \varepsilon_{2,h,\tau}$ do not depend on h, τ then from compatibility and (2.22) the condition $(A3)$ is satisfied for sufficiently small h, τ.

Remark 2. Let V_1 be a Banach space with norm $\|\cdot\|_1$, q_h: $V_h \to V_1$ and r:$V \to V_1$ be continuous operators, satisfy

$$\|q_h\| = \sup_{v_h \in V_h} \frac{\|q_h v_h\|_1}{\|v_h\|_h} \le C_1, \quad (h \le h_0), \tag{2.24}$$

$$\lim_{h\to 0} \sup_{t\ge 0} \|q_h p_h u - ru\|_1 = 0, \quad u \in V. \tag{2.25}$$

Then from the definition of the convergence we have

$$\lim_{h,\tau\to 0} \sup_{n\in N_0} \|q_h u^n_{h,\tau} - ru(n\tau)\|_1 = 0. \tag{2.26}$$

Especially, If $V_h \subset V$, $\lim\limits_{h\to 0} \sup\limits_{t\ge 0} \|p_h u - u\|_1 = 0$ ($p_h : V \to V_h$ is a projection), $\|\cdot\|_h = \|\cdot\|$, and let $V_1 = V, q_h = I, r = I$, then

$$\lim_{h,\tau\to 0} \sup_{n\in N_0} \|u^n_{h,\tau} - u(n\tau)\| = 0. \tag{2.27}$$

Remark 3. In the case of variable time-step discretization, we can obtain the analogous results by almost the same argument.

3. Application to the FEM

Larsson[5] considered the long-time error estimates of finite-element approximations of semilinear parabolic problems of the form:

$$\begin{aligned} &u_t = \Delta u + f(u), && \text{in} \quad \Omega \times (0,\infty),\\ &u = 0, && \text{on} \quad \partial\Omega \times (0,\infty),\\ &u(\cdot,0) = u_0, && \text{in} \quad \Omega, \end{aligned} \tag{3.1}$$

here Ω is a bounded domain in R^n $(n \le 3)$ with smooth boundary $\partial\Omega$, u_0 is smooth on $\bar{\Omega}$ with zero boundary values, and f is a smooth function satisfying

$$f'(s) > 0, \quad (s \in R). \tag{3.2}$$

Larsson obtained the long-time convergence and error estimates of implicit completely discrete finite-element method. In this section, we obtain the similar results from theorem 2.2.

The corresponding stationary problem is

$$\begin{aligned} -\Delta u = f(u), &\quad \text{in} \quad \Omega, \\ u = 0, &\quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.3}$$

For any given positive function $w \in C(\bar{\Omega})$, we consider the eigenvalue problem with weight w:

$$\begin{aligned} -\Delta \varphi = \mu w \varphi, &\quad \text{in} \quad \Omega, \\ \varphi = 0, &\quad \text{on} \quad \partial\Omega. \end{aligned}$$

let $\mu_1[w]$ denote its smallest eigenvalue, then $\mu_1[w]$ can be characterized as

$$\mu_1[w] = \inf_{\chi \in H_0^1} \frac{\| \nabla \chi\|^2}{(w\chi, \chi)}, \tag{3.4}$$

and that $\mu_1[w] > 0$. Here $\| \cdot \|$ and $(\cdot, \cdot)$ denote the usual norm and inner product in $L_2(\Omega)$ and $H_0^1(\Omega)$ is the subspace of the standard Sobolev space $H^1(\Omega)$ satisfying the homogeneous Dirichlet boundary condition.

We will assume that (3.3) has a classical solution $\underline{u}$, which is linearized stable in the sense that, for some real number $\delta > 0$,

$$\frac{1}{\mu_1[f'(\underline{u})]} \le \delta < 1. \tag{3.5}$$

Note that $\mu_1[f'(\underline{u})]$ is well defined because of (3.2). We also assume that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$is such that (3.1) has a global classical solution u which satisfies $u(t) \to \underline{u}$ in $L_2(\Omega)$ as $t \to \infty$ and

$$\|u(\cdot, t)\|_{L_\infty} \le B_1, 0 \le t < \infty, \tag{3.6}$$

$$\|u(t)\|_{H^2} + \|u_t(t)\|_{H^2} + \|u_{tt}(t)\|_{H^2} \le B_2, \quad 0 < t < \infty. \tag{3.7}$$

In the discrete problem to be described below we replace the function f by a smooth function $\tilde{f}$, which satisfies

$$\tilde{f}(s) = f(s), |s| \le B_1 + 1, \tag{3.8}$$

$$|\tilde{f}(s)|, \; |\tilde{f}'(s)|, \; |\tilde{f}''(s)| \le K, \quad \forall s \in R, \tag{3.9}$$

where K depends on B_1, of course. Obviously, this replacement does not affect the exact solution of (3.1).

Let $V = H_0^1(\Omega), V_h \subset V$ be the finite-element space, p_h be the Ritz projection from V to V_h satisfying

$$(\bigtriangledown p_h u, \bigtriangledown v_h) = (\bigtriangledown u, \bigtriangledown v_h), \quad \forall v_h \in V_h, \tag{3.10}$$

$$\|p_h u - u\| \le Ch^2 \|u\|_{H^2}, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3.11}$$

Let q_h the $L_2(\Omega)$ projection from V to V_h satisfying

$$(q_h u, v_h) = (u, v_h), \forall v_h \in V_h. \tag{3.12}$$

Now the implicit completely discrete finite-element method for (3.1) with the time step-size τ reads as follows. Find $u_{h,\tau}^n \in V_h$ such that

$$\Big(\frac{u_{h,\tau}^{n+1} - u_{h,\tau}^n}{\tau}, \chi\Big) + (\bigtriangledown u_{h,\tau}^{n+1}, \bigtriangledown \chi) = (\tilde{f}(u_{h,\tau}^{n+1}), \chi), \quad \forall \chi \in V_h, n \in N_0,$$
$$u_{h,\tau}^0 = u_{0,h}. \tag{3.13}$$

Let A_h be the linear operator defined on V_h as follows:

$$(A_h u_h, v_h) = (\bigtriangledown u_h, \bigtriangledown v_h), \quad \forall u_h, v_h \in V_h. \tag{3.14}$$

Set

$$B_{h,\tau}(v_h) = v_h - \tau A_h v_h - \tau q_h \tilde{f}(v_h), \quad \forall v_h \in V_h, \tag{3.15}$$

$$C_{h,\tau}(v_h) = v_h, \forall v_h \in V_h. \tag{3.16}$$

Then we can rewrite (3.13) as

$$B_{h,\tau}(u_{h,\tau}^{n+1}) = C_{h,\tau}(u_{h,\tau}^n), \quad n \in N_0.$$
$$u_{h,\tau}^0 = u_{0,h}. \tag{3.17}$$

Theorem 3.1.[5] *Let f be a fixed smooth function satisfying* (3.2). *For* $\forall B_1, B_2 > 0$, *and* $0 < \delta < 1$, *there are* $C_0 > 0$ *and* $h_0 > 0$, $\tau_0 > 0$ *such that whenever*

($H1$) $\underline{u}$ *is a solution of* (3.3) *satisfying* (3.5);

($H2$) $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *such that* $u(t) \to \underline{u}$ *in* $L_2(\Omega)$ *as* $t \to \infty$;

($H3$) u *satisfies* (3.6), (3.7);

($H4$) *the initial approximation* $u_{0,h}$ *is chosen such as that*

$$\|u_{0,h} - u_0\| \le Ch^2 \|u_0\|_{H^2}, h < h_0. \tag{3.18}$$

Then, for $0 < h \le h_0$, $0 < \tau \le \tau_0$, *the solution* $u^n_{h,\tau}$ *of discrete problem* (3.17) *exists and satisfies*

$$\|u^n_{h,\tau} - u(n\tau)\| \le C_0(\tau + h^2), n \in N_0. \tag{3.19}$$

Lemma 3.2.[5] *There is a constant* C *such that*

$$\Big|\frac{1}{\mu_1[w_1]} - \frac{1}{\mu_1[w_2]}\Big| \le C\|w_1 - w_2\| \tag{3.20}$$

for any pair of weight functions w_1 *and* w_2.

Now we are going to prove the the compatibility and stability of scheme (3.17), and then give the proof of Theorem 3.1.

Lemma 3.3. *Under the assumptions of Theorem* 3.1, *the scheme* (3.17) *is compatable and*

$$S_{h,\tau} \le C(\tau + h^2), \tag{3.21}$$

here C *is a positive constant.*

Proof. Form (3.15)–(3.17)

$$\begin{aligned} R^n_{h,\tau}(u) =& \frac{(B_{h,\tau}(p_h u((n+1)\tau)) - C_{h,\tau}(p_h u(n\tau)))}{\tau} \\ =& \frac{p_h(u((n+1)\tau) - u(n\tau))}{\tau} - A_h p_h u((n+1)\tau) - q_h\tilde{f}(p_h u((n+1)\tau)). \end{aligned}$$

From (3.1), for $\forall v_h \in V_h$

$$(u_t, v_h) + (\bigtriangledown u, \bigtriangledown v_h) - (\tilde{f}(u), v_h) = 0.$$

Hence, by (3.11) and $(H3)$ we have

$$\begin{aligned} (R^n_{h,\tau}(u), v_h) =& \Big(\frac{p_h(u((n+1)\tau) - u(n\tau))}{\tau}, v_h) + (\bigtriangledown u((n+1)\tau), \bigtriangledown v_h\Big) \\ & - (\tilde{f}(p_h u((n+1)\tau)), v_h) \\ =& \Big(\frac{p_h(u((n+1)\tau) - u(n\tau))}{\tau} - u_t((n+1)\tau), v_h\Big) \\ & + (\tilde{f}(u((n+1)\tau)) - \tilde{f}(p_h u((n+1)\tau)), v_h) \\ \le& \Big(\Big\|p_h\Big(\frac{(u((n+1)\tau) - u(n\tau))}{\tau} - u_t((n+1)\tau)\Big)\Big\| \\ & + \|(p_h - I)u_t((n+1)\tau)\| + K\|(p_h - I)u((n+1)\tau)\|\|v_h\| \\ \le& \Big(\Big\|p_h\int_{n\tau}^{(n+1)\tau}\frac{s - n\tau}{\tau}u_{tt}(s)ds\Big\| + C_1h^2(\|u_t((n+1)\tau)\|_{H^2} \\ & + \|u((n+1)\tau)\|_{H^2})\|v_h\| \le C(\tau + h^2)\|v_h\|, \end{aligned}$$

where C is a positive constant depending on $\|u\|_{H^2}, \|u_{tt}\|, \|u_t\|_{H^2}$. Taking $v_h = R^n_{h,\tau}(u)$, we get

$$\|R^n_{h,\tau}(u)\| \le C(\tau + h^2).$$

This proves the lemma.

Lemma 3.4. *Let*

$$w_n = \int_0^1 \tilde{f}'(p_h u(n\tau) + s v_h) ds.$$

Then there exist $\varepsilon_1, T, h_0, 0 < \delta_1 < 1$ *such that, for* $h < h_0$, $n\tau > T$, $\|v_h\| \le \varepsilon_1$,

$$\frac{1}{\mu_1[w_n]} \le \delta_1. \tag{3.22}$$

Proof. By Lemma 3.2,

$$\begin{aligned}\frac{1}{\mu_1[w_n]} \le& \frac{1}{\mu_1[f'(\underline{u})]} + C\|w_n - f'(\underline{u})\| \\ =& \frac{1}{\mu_1[f'(\underline{u})]} + C\|w_n - \tilde{f}'(\underline{u})\| \\ \le& \delta + C(\|w_n - \tilde{f}'(p_h u(n\tau))\| + \|\tilde{f}'(p_h u(n\tau)) - \tilde{f}'(u(n\tau))\| \\ &+ \|\tilde{f}'(u(n\tau)) - \tilde{f}'(\underline{u})\|) \\ \le& \delta + C\Big(\Big\| \int_0^1 (\tilde{f}'(p_h u(n\tau) + s v_h) - \tilde{f}'(p_h u(n\tau)))ds \Big\| \\ &+ \|\tilde{f}'(p_h u(n\tau)) - \tilde{f}'(u(n\tau))\| + \|\tilde{f}'(u(n\tau)) - \tilde{f}'(\underline{u})\| \\ \le& \delta + C\Big(\Big\| \int_0^1 \tilde{f}''(\xi_1) s v_h ds \Big\| + \|\tilde{f}''(\xi_2)(p_h - I)u(n\tau)\| + \|\tilde{f}''(\xi_3)(u(n\tau) - \underline{u})\|\Big).\end{aligned}$$

From (3.9),(3.11)and $(H3)$ we have

$$\frac{1}{\mu_1[w_n]} \le \delta + \bar{C}(\|v_h\| + h^2 + \|u(n\tau) - \underline{u}\|).$$

Note that $\delta < 1$ and $(H2)$, then (3.22) can be deduced from above inquality.

From (3.15) and (3.16) we have, for $\forall v_h \in V_h$

$$\bar{B}^n_{h,\tau}(v_h) = v_h - \tau A_h v_h - \tau q_h(\tilde{f}(p_h u(n\tau) + v_h) - \tilde{f}(p_h u(n\tau))), \tag{3.23}$$

$$\bar{C}^n_{h,\tau}(v_h) = v_h. \tag{3.24}$$

Lemma 3.5. *Scheme* (3.17) *is stable under the assumptions of Theorem* 3.1.

Proof. Let $D_h = I - \tau A_h$, then from (3.14) we can conclude easily that D_h has an inverse and $\|D_h^{-1}\| < 1$, $(\tau > 0)$. Obviously,

$$\bar{B}^n_{h,\tau}(v_h) = D_h v_h - \tau q_h(\tilde{f}(p_h u(n\tau) + v_h) - \tilde{f}(p_h u(n\tau))).$$

From (3.9), $\tilde{f}$ satisfies global Lipschitz condition, so there is τ_0 such that $\bar{B}^n_{h,\tau}$ has an inverse for $0 < \tau \le \tau_0$.

From (3.23),

$$(\bar{B}^n_{h,\tau}(v_h), v_h) = \|v_h\|^2 + \tau|v_h|_1^2 - \tau(\tilde{f}'(\xi)v_h, v_h) \ge \|v_h\|^2 - \tau K\|v_h\|^2,$$

and hence

$$\|\bar{B}^n_{h,\tau}(v_h)\| \geq (1-\tau K)\|v_h\|. \tag{3.25}$$

Let $\tau_0 < \dfrac{1}{2K}$, then for $\tau \leq \tau_0$ we find

$$\|\bar{B}^n_{h,\tau}(v_h)\| \geq \frac{1}{2}\|v_h\|.$$

Hence, with $\forall v_h \in V_h$, $0 < \tau \leq \tau_0$ we have

$$\|(\bar{B}^n_{h,\tau})^{-1}(v_h)\| \leq 2\|v_h\|. \tag{3.26}$$

Taking $\varepsilon_{2,h,\tau} = \infty$, then the condition $(A1)$ is satisfied for $0 < \tau \leq \tau_0$.

If the assumptions of Lemma 3.4 are satisfied, then from (3.4) we have

$$(q_h(w_n v_h), v_h) = (w_n v_h, v_h) \leq \frac{|v_h|_1^2}{\mu_1[w_n]} \leq \delta_1 |v_h|_1^2, \tag{3.27}$$

hence for $0 < h \leq h_0$, $\|v_h\| \leq \varepsilon_1$, $n\tau \geq T$,

$$(\bar{B}^n_{h,\tau}(v_h), v_h) = \|v_h\|^2 + \tau|v_h|_1^2 - \tau(q_h(w_n v_h), v_h) > \|v_h\|^2 + \tau(1-\delta_1)|v_h|_1^2,$$

thus, there exists ρ_1 such that

$$(\bar{B}^n_{h,\tau}(v_h), v_h) \geq (1+\rho_1\tau)\|v_h\|^2.$$

Taking $\rho = \dfrac{\rho_1}{1+\rho_1\tau_0}$, then for $0 < h \leq h_0$, $\|v_h\| \leq \varepsilon_1$, $n\tau \geq T$, we have

$$\begin{aligned}\|v_h\| \leq & \frac{1}{1+\rho_1\tau}\|\bar{B}^n_{h,\tau}(v_h)\| \\ = & \Big(1 - \frac{\rho_1\tau}{1+\rho_1\tau}\Big)\|\bar{B}^n_{h,\tau}(v_h)\| \leq (1-\rho\tau)\|\bar{B}^n_{h,\tau}(v_h)\|.\end{aligned} \tag{3.28}$$

On the other hand, from (3.24)

$$\|\bar{C}^n_{h,\tau}(v_h)\| = \|v_h\|. \tag{3.29}$$

The inequality (3.25) shows that

$$\begin{aligned}\|v_h\| \leq \frac{1}{1-\tau K}\|\bar{B}^n_{h,\tau}(v_h)\| = & \Big(1 + \frac{K\tau}{1-K\tau}\Big)\|\bar{B}^n_{h,\tau}(v_h)\| \\ \leq & (1+2K\tau)\|\bar{B}^n_{h,\tau}(v_h)\|.\end{aligned} \tag{3.30}$$

In a words, if we take $\varepsilon_{1,h,\tau} = \varepsilon_1$, $M = \max(2, 2K)$, and $\theta^n_\tau = 1 + M\tau$ for $n\tau \leq T$; while $\theta^n_\tau = 1 - \rho\tau$ for $n\tau > T$, then from (3.26), (3.28)–(3.30), we see the conditions

$(A2)_1$ and $(A2)_2$ satisfied. Thus scheme (3.17) is stable. This completes the proof of Lamma 3.5.

Proof of Theorem 3.1. From $(H2), (H4)$ and (3.11) we have

$$\|e_{h,\tau}^0\| = \|u_{h,\tau}^0 - p_h u_0\| = \|u_{0,h} - p_h u_0\| \le Ch^2 \|u_0\|_{H^2}. \tag{3.31}$$

Let h_0 be small enough, then for $0 < h \le h_0$, we have

$$\|e_{h,\tau}^0\| \le \varepsilon_{1,h,\tau}, \lim_{h\to 0} \|e_{h,\tau}^0\| = 0.$$

From Lemma 3.3 and 3.5, scheme (3.17) is stable and compatable. Moreover, since $\varepsilon_{1,h,\tau}$, $\varepsilon_{2,h,\tau}$ do not depend on h, τ, so Theorem 2.2 holds by Remark 1. Hence for $0 < h \le h_0$, $0 < \tau \le \tau_0$ and $n \ge 1$, the approximations $u_{h,\tau}^n$ exist and satisfy

$$\|e_{h,\tau}^n\| \le M(aS_{h,\tau} + b\|e_{h,\tau}^0\|). \tag{3.32}$$

From (2.10)

$$a \le \frac{1}{\rho} + \frac{e^{M(T+\tau_0)} - 1}{M}.$$

By using (3.21), (3.31) to evalute the right hand of (3.32), then we can show that there exists constant C such that

$$\|e_{h,\tau}^n\| \le C(\tau + h^2),$$

or

$$\|u_{h,\tau}^n - p_h u(n\tau)\| \le C(\tau + h^2). \tag{3.33}$$

Finally, from (3.11) and $(H3)$ we get

$$\begin{aligned}\|u_{h,\tau}^n - u(n\tau)\| \le& \|u_{h,\tau}^n - p_h u(n\tau)\| + \|(I - p_h)u(n\tau)\| \\ \le& C(\tau + h^2) + Ch^2 \|u(n\tau)\|_{H^2}.\end{aligned}$$

This completes the proof of Theorem 3.1.

4. Application to the FDM

Sanz-Serna and Stuart[6] considered the difference method for the reaction-diffusion problem of the form:

$$\begin{aligned}&u_t = u_{xx} + f(u), \quad 0 < x < 1,\ t > 0, \\ &u(0,t) = u(1,t) = 0, \quad t > 0, \\ &u(x,0) = u_0(x), \quad 0 < x < 1.\end{aligned} \tag{4.1}$$

The associate explicit finite difference scheme is as follows:

$$\begin{aligned}
&\frac{u_j^{n+1}-u_j^n}{\tau}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}+f(u_j^n), \quad j=1,\cdots,J-1, n\in N_0,\\
&u_0^n=u_J^n=0, \quad n\in N,\\
&u_j^0=u_0(jh), \quad j=0,\cdots,J,
\end{aligned} \tag{4.2}$$

where u_j^n denotes the approximation to $u(x_j,t_n)$, $x_j=jh, j=0,\cdots,J, t_n=n\tau$.

Let $V_h=\{v|v=(v_1,\cdots,v_{J-1})^T\}$. Define the norms of v such as

$$\|v\|_h=\Big(\sum_{j=1}^{J-1}hv_j^2\Big)^{1/2} \tag{4.3}$$

and

$$\|v\|_\infty=\max_{1\le j\le J-1}|v_j|. \tag{4.4}$$

Obviously

$$\|v\|_\infty\le\frac{1}{\sqrt{h}}\|v\|_h. \tag{4.5}$$

Let $V=H_0^1([0,1])$, p_h be the operator from V to V_h:

$$p_hu=(u(h),u(2h),\cdots,u((J-1)h))^T. \tag{4.6}$$

We are going to state the assumptions and main theorem given by Sanz-Serna and Stuart.

Assumptions[6]**:**

$(H1)$ $f(\cdot)\in C^2((a,b),R)$ for some interval $(a,b)\subset R$.

$(H2)$ Equation (1.1) have a solution u for which the derivatives u_{xxxx} and u_{tt} exist and are uniformly bounded for $0\le x\le 1$, $0\le t\le\infty$. Furthermore, there exists $\delta>0$ such that

$$a+\delta\le u(x,t)\le b-\delta, \quad \forall(x,t)\in[0,1]\times[0,\infty).$$

$(H3)$ As $t\to\infty$, u approaches an equilibrium $\bar{u}$. More precisely, $\|u(\cdot,t)-\bar{u}(\cdot)\|_\infty\to 0$, where $\bar{u}$ satisfies

$$\begin{aligned}
&\bar{u}_{xx}+f(\bar{u})=0, \quad 0<x<1,\\
&\bar{u}(0)=\bar{u}(1)=0.
\end{aligned} \tag{4.7}$$

$(H4)$ $\bar{u}$ is an asymptotically stable equilibrium in the sence that

$$\lambda_{\max}=\max_{\phi\in H_0^1}\frac{\displaystyle\int_0^1(-(\phi_x)^2+f'(\bar{u}(x))\phi^2)dx}{\displaystyle\int_0^1\phi^2dx}<0. \tag{4.8}$$

Note that $\lambda_{\max}$ is the largest eigenvalue of the problem

$$\begin{aligned} &\lambda\phi = \phi_{xx} + f'(\bar{u})\phi, \quad 0 < x < 1 \\ &\phi(0) = \phi(1) = 0. \end{aligned} \tag{4.9}$$

$(H5)$ The grids are refined in such a way that

$$\tau/h^2 \leq \mu < 1/2 (\text{this is relation}(R)). \tag{4.10}$$

Theorem 4.1.[6] *Under the assumptions above, there exist constants h_0 and C, depending only upon f, μ and u, such that, for $0 < h \leq h_0$, the numerical solution $u_{h,\tau}^n = (u_1^n, \cdots, u_{J-1}^n)^T$ exists for all positive integers n and satisfies the error bound*

$$\|u_{h,\tau}^n - p_h u(n\tau)\|_h \leq C(\tau + h^2), \quad n \in N_0. \tag{4.11}$$

Now we are going to prove Theorem 4.1 by the general frame established in §2. For $v_h \in V_h$, set

$$B_{h,\tau}(v_h) = v_h, \tag{4.12}$$

$$\begin{aligned} &C_{h,\tau}(v_h) = \Big(\cdots, v_j + \frac{\tau}{h^2}(v_{j+1} - 2v_j + v_{j-1}) + \tau f(v_j), \cdots\Big)^T. \\ &(v_0 = v_J = 0) \end{aligned} \tag{4.13}$$

Then the scheme (4.2) can be rewritten as

$$\begin{aligned} &B_{h,\tau}(u_{h,\tau}^{n+1}) = C_{h,\tau}(u_{h,\tau}^n), \quad n \geq 0. \\ &u_{h,\tau}^0 = p_h u_0 \end{aligned} \tag{4.14}$$

Lemma 4.2. *Under the assumptions above, scheme* (4.14) *is compatable.*

Proof. Set $w = R_{h,\tau}^n(u)$. Then from (2.3) and (2.4) we have

$$w = [B_{h,\tau}(p_h u((n+1)\tau)) - C_{h,\tau}(p_h u(n\tau))]/\tau.$$

From (4.6),(4.12) and (4.13),

$$\begin{aligned} w_j =& \frac{u((n+1)\tau, jh) - u(n\tau, jh)}{\tau} - \frac{u(n\tau,(j+1)h) - 2u(n\tau, jh) + u(n\tau,(j-1)h)}{h^2} \\ &- f(u(n\tau, jh)) \\ =& u_t(n\tau, jh) + \frac{u_{tt}(\xi_1, jh)}{2}\tau - u_{xx}(n\tau, jh) \\ &- \frac{h^2}{24}(u_{xxxx}(n\tau, \xi_2) + u_{xxxx}(n\tau, \xi_3)) - f(u(n\tau, jh)). \end{aligned}$$

From (4.1) and $(H2)$, there exists a constant C, such that

$$\begin{aligned} |w_j| =& \Big|\frac{u_{tt}(\xi_1, jh)}{2}\tau - \frac{h^2}{24}(u_{xxxx}(n\tau,\xi_2) + u_{xxxx}(n\tau,\xi_3))\Big| \\ \le & C(\tau + h^2), \quad 1 \le j \le J-1, \end{aligned} \tag{4.15}$$

hence

$$\|R^n_{h,\tau}(u)\|_h = \|w\|_h \le \|w\|_\infty \le C(\tau + h^2). \tag{4.16}$$

Thus, scheme (4.14)is compatable. Furthermore,

$$S_{h,\tau} \le C(\tau + h^2). \tag{4.17}$$

This completes the proof of Lemma 4.2.

From (4.12) and (4.13), for $v_h \in V_h$, it is easy to see

$$\bar{B}^n_{h,\tau}(v_h) = v_h, \tag{4.18}$$

$$\begin{aligned} \bar{C}^n_{h,\tau}(v_h) =& \Big(\cdots, v_j + \frac{\tau}{h^2}(v_{j+1} - 2v_j + v_{j-1}) \\ & + \tau(f(u(n\tau, jh) + v_j) - f(u(n\tau, jh))), \cdots\Big)^T. \end{aligned} \tag{4.19}$$

Lemma 4.3. *Under the assumptions above, scheme* (4.14) *is stable.*

Proof. With $\varepsilon_{2,h,\tau} = +\infty$, $M \ge 1$, then from (4.18) the condition $(A1)$ holds for $\forall h, \tau$.

Let $v_h = (v_1, \cdots, v_{J-1})^T \in V_h$. From (4.19) we conclude that

$$\begin{aligned} [\bar{C}^n_{h,\tau}(v_h)]_j =& v_j + \frac{\tau}{h^2}(v_{j+1} - 2v_j + v_{j-1}) + \tau f'(\bar{u}(jh))v_j \\ & + \tau f''(\eta^n_j)(u(n\tau, jh) - \bar{u}(jh))V_j + \tau f''(\xi^n_j)v_j^2/2, \end{aligned}$$

where

$$\begin{aligned} \eta^n_j &= s^n_j \bar{u}(jh) + (1 - s^n_j)u(n\tau, jh); \\ \xi^n_j &= u(n\tau, jh) + r^n_j v_j \end{aligned}$$

and $0 \le s^n_j \le 1$, $0 \le r^n_j \le 1$. By using (4.5), the norm of $\bar{C}^n_{h,\tau}(v_h)$ satisfies

$$\|\bar{C}^n_{h,\tau}(v_h)\|_h \le \|I + \tau A\|_h \|v_h\|_h + \tau K_1(\|(u(n\tau, jh) - \bar{u}(jh)\|_\infty + \frac{\|v_h\|_h}{2\sqrt{h}}\|v_h\|_h), \tag{4.20}$$

where $K_1 = \sup_{a+\delta \le x \le b-\delta} |f''(x)|$,

$$A = \begin{pmatrix} \theta_1 & h^{-2} & & \\ h^{-2} & \theta_2 & h^{-2} & \\ & \ddots & \ddots & \ddots \\ & & h^{-2} & \theta_{J-1} \end{pmatrix} \tag{4.21}$$

and $\theta_j = -\dfrac{2}{h^2} + f'(\bar{u}(jh))$. From the known results we can conclude that, as $h \to 0$, the largest eigenvalue of A converges to the eigenvalue $\lambda_{\max}$ defined in assumption $(H4)$ (see, for example, [4]). Furthermore, the smallest eigenvalue satisfies

$$\lambda^* \ge -\frac{4}{h^2} - K_2,$$

where $K_2 = \max\{f'(\bar{u}(x)),\ 0 < x < 1\}$. Since A is symmetric, hence, by $(H5)$ we can obtain

$$\|I + \tau A\|_h \le 1 - \alpha\tau, \quad \alpha > 0 \tag{4.22}$$

for h, τ sufficiently small. From $(H3)$, there exists $T > 0$, such that for $n\tau > T$

$$K_1 \|u(n\tau, jh) - \bar{u}(jh)\|_\infty \le \alpha/4. \tag{4.23}$$

Let $\varepsilon_{1,h,\tau} = \dfrac{\alpha\sqrt{h}}{2K_1}$, $\rho = \dfrac{\alpha}{2}$. From (4.20), (4.22) and (4.23), there exist h_0, τ_0, such that for $0 < h \le h_0$, $\dfrac{\tau}{h^2} \le \mu$, $n\tau > T$, $\|v_h\|_h \le \varepsilon_{1,h,\tau}$ we have

$$\|\bar{C}^n_{h,\tau}(v_h)\|_h \le (1 - \rho\tau)\|v_h\|_h. \tag{4.24}$$

From (4.20), (4.22) and the assumption, there exists $M > 0$ such that, for $0 < h \le h_0$, $\dfrac{\tau}{h^2} \le \mu$, $n\tau \le T$, $\|v_h\|_h \le \varepsilon_{1,h,\tau}$, we have

$$\|\bar{C}^n_{h,\tau}(v_h)\|_h \le (1 + M\tau)\|v_h\|_h. \tag{4.25}$$

Next, by formula (4.18), we obtain

$$\|\bar{B}^n_{h,\tau}(v_h)\|_h = \|v_h\|_h. \tag{4.26}$$

Thus, if we take $\theta^n_\tau = 1 + M\tau$ for $n\tau \le T$; $\theta^n_\tau = 1 - \rho\tau$ for $n\tau \ge T$, then condition $(A2)_1$ and $(A2)_2$ hold for $0 < h \le h_0$, $\dfrac{\tau}{h^2} \le \mu$. This completes the proof of Lemma (4.3).

Proof of Theorem 4.1. Since $\|e^0_{h,\tau}\|_h = 0$, $S_{h,\tau} \le C_1 h^2$ (from (4.17)), we know the condition $(A3)$ holds for sufficiently small h_0 and τ_0. Furthermore, from Lemma 4.2 and 4.3, the conditions in Theorem 2.2 are satisfied. This completes the proof of Theorem 4.1 by using (4.17).

5. An Example

Consider the semilinear parabolic problems as follows

$$\begin{aligned} &u_t = u_{xx} - u^2 + g(t,x), \quad 0 < x < 1,\ t > 0, \\ &u(0,t) = u(1,t) = 0,\ t > 0, \\ &u(x,0) = 0,\ 0 < x < 1, \end{aligned} \tag{5.1}$$

where

$$g(t,x) = \sin(2\pi x)(\cos t + 4\pi^2 \sin t + \sin t \sin(2\pi x)). \tag{5.2}$$

The solution of (5.1) is

$$u(t,x) = \sin t \sin(2\pi x). \tag{5.3}$$

We consider the forward Euler method with central differences in space:

$$\begin{aligned} &\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} - \tau(u_j^n)^2 + \tau g(n\tau, jh), \quad j = 1, \cdots, J-1, n \in N_0, \\ &u_0^n = u_J^n = 0, \quad n \in N, \\ &u_j^0 = 0, \quad j = 0, \cdots, J, \end{aligned} \tag{5.4}$$

where u_j^n denotes the approximation to $u(n\tau, jh)$, $Jh = 1$.

Since the solution of (5.1) does not converge as $t \to \infty$, it can't include in the frames of [3], [5] and [6]. But, by using the frame of §2, we can prove: For sufficiently small h and τ satisfying $\frac{\tau}{h^2} \le \frac{1}{2}$, the scheme (5.4) possesses convergence uniformly in time.

Theorem 5.1. *There exist constants h_0 and C, such that, for $0 < h \le h_0$, $\frac{\tau}{h^2} \le \frac{1}{2}$, the numerical solution $u_{h,\tau}^n = (u_1^n, \cdots, u_{J-1}^n)^T$ exists for all positive integers n and satisfies the error bound*

$$\|u_{h,\tau}^n - p_h u(n\tau)\|_h \le C(\tau + h^2), n \in N_0, \tag{5.5}$$

where $\|\cdot\|_h$ and p_h are defined by (4.3) *and* (4.6) *respectively.*

The proof of Theorem 5.1 is analogous as that of Theorem 4.1.

For $v_h = (v_1, \cdots, v_{J-1})^T \in V_h$, we take

$$B_{h,\tau}(v_h) = v_h, \tag{5.6}$$

$$C_{h,\tau}(v_h) = \Big(\cdots, v_j + \frac{\tau}{h^2}(v_{j+1} - 2v_j + v_{j-1}) - \tau v_j^2, \cdots\Big)^T, \tag{5.7}$$

$$g_{h,\tau}^n = (g(n\tau, h), \cdots, g(n\tau, (J-1)h))^T, \tag{5.8}$$

then we can rewrite scheme (5.4) as

$$B_{h,\tau}(u_{h,\tau}^{n+1}) = C_{h,\tau}(u_{h,\tau}^n) + \tau g_{h,\tau}^n, \quad n \in N_0,$$

$$u^0_{h,\tau} = 0. \tag{5.9}$$

Lemma 5.2. *Scheme* (5.9) *is compatable, and have*

$$S_{h,\tau} \leq C(\tau + h^2), \tag{5.10}$$

where C *is a constant.*

Proof. We ommit the proof because it is straightward.

From (5.6) and (5.7) we have

$$\bar{B}^n_{h,\tau}(v_h) = v_h, \tag{5.11}$$

$$\bar{C}^n_{h,\tau}(v_h) = \Big(\cdots, v_j + \frac{\tau}{h^2}(v_{j+1} - 2v_j + v_{j-1}\Big) - 2\tau\sin(n\tau)\sin(2j\pi h)v_j - \tau v_j^2, \cdots\Big)^T. \tag{5.12}$$

Lemma 5.3. *If* $\frac{\tau}{h^2} \leq \frac{1}{2}$, *then the scheme* (5.9) *is stable.*

Proof. Let

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{pmatrix}$$

The eigenvalues of A are $\frac{-4\sin^2(k\pi h/2)}{h^2}$, $(k = 1, \cdots, J-1)$ (see [1]), hence, from (5.12) we have

$$\|\bar{C}^n_{h,\tau}(v_h)\|_h \leq (1 - \tau\lambda_h)\|v_h\|_h + 2\tau\|v_h\|_h + \frac{\tau}{\sqrt{h}}\|v_h\|_h^2, \tag{5.13}$$

where $\lambda_h = \frac{\pi^2\sin^2(\pi h/2)}{(\pi h/2)^2}$. Since $\sin x \geq x - \frac{1}{6}x^3$, $(x \geq 0)$, there exists h_0 such that, $\lambda_h \geq 9$ for $0 < h \leq h_0$. Let $\varepsilon_{1,h,\tau} = \frac{\sqrt{h}}{2}$, then for $0 < h \leq h_0$ and $\|v_h\|_h < \varepsilon_{1,h,\tau}$, we have

$$\|\bar{C}^n_{h,\tau}(v_h)\|_h \leq (1 - 9\tau + 2\tau + \tau/2)\|v_h\|_h \leq (1 - 6\tau)\|v_h\|_h.$$

Note that (5.11), and take $\rho = 6$, $M = 1$, $T = 0$, $\theta^n_\tau = 1 - \rho\tau$, then the conditions $(A2)_1$ and $(A2)_2$ hold for $0 < h \leq h_0$, $\frac{\tau}{h^2} \leq \frac{1}{2}$. This completes the proof of lemma 5.3.

Proof of Theorem 5.1. From (5.10) and $\|e^0_{h,\tau}\|_h = 0$ we can conclude that, the condition $(A3)$ holds for sufficiently small h_0 and $0 < h \leq h_0$, $\frac{\tau}{h^2} \leq \frac{1}{2}$. On the other hand, from lemma 5.2 and 5.3 we see Theorem 2.2 holds. Then from (5.10) we can deduce Theorem 5.1.

The following numerical results verified the long-time convergence of scheme (5.4) in a sense.

time t	error (L_2 norm)	error (maximum norm)
10.00000	5.6708982E-04	8.3100796E-04
20.00000	9.1236364E-04	1.3898611E-03
30.00000	9.6483884E-04	1.4952421E-03
40.00000	7.0018758E-04	1.0771155E-03
50.00000	2.1183636E-04	3.1808019E-04
60.00000	3.3865144E-04	4.8586726E-04
70.00000	7.8310986E-04	1.1746287E-03
80.00000	9.8119921E-04	1.5131235E-03
90.00000	8.5874501E-04	1.3295412E-03
100.0000	4.5624792E-04	6.9254637E-04
5000.000	9.7848591E-04	1.5147328E-03
5010.000	7.5847126E-04	1.1598468E-03
5020.000	2.9577641E-04	4.3708086E-04
5030.000	2.5466483E-04	3.6710501E-04
5040.000	7.2789006E-04	1.0963678E-03
5050.000	9.7230350E-04	1.5004277E-03
5060.000	8.9846307E-04	1.3860464E-03
5070.000	5.3234241E-04	8.0072880E-04
5080.000	2.2076156E-06	3.6954880E-06
5090.000	5.2837259E-04	7.8123808E-04
5100.000	8.9519034E-04	1.3697147E-03
10000.00	1.7128926E-04	2.4917722E-04
10010.00	6.7269686E-04	1.0180473E-03
10020.00	9.6278504E-04	1.4868975E-03
10030.00	9.3699095E-04	1.4405847E-03
10040.00	6.0683367E-04	9.0640783E-04
10050.00	8.7746092E-05	1.2588128E-04
10060.00	4.5683002E-04	6.7991018E-04
10070.00	8.6307735E-04	1.3242960E-03
10080.00	9.9076249E-04	1.5313625E-03
10090.00	7.9297303E-04	1.2033582E-03
10100.00	3.4287336E-04	4.9999356E-04

Here $h = \dfrac{1}{49}$, $\tau = 0.0001$.

Now let us explain the fact that the numerical solution has long-time convergence in another way. In fact, we can prove, in a sense, scheme (5.4) has a solution with period 2π (the same as the period of the continuous solution).

Theorem 5.4. *There exist h_0, $C>0$, such that, for $\frac{\tau}{h^2}\le\frac{1}{2}$ and $\tau=\frac{2\pi}{m}$ $(m\in N)$, scheme (5.9) has a solution $\bar{u}^n_{h,\tau}$ with period 2π, i.e.*

$$\bar{u}^0_{h,\tau}=\bar{u}^m_{h,\tau}, \tag{5.14}$$

and having the following estimate:

$$\|\bar{u}^n_{h,\tau}-p_h u(n\tau)\|_h\le C(\tau+h^2),\quad n\in N_0. \tag{5.15}$$

Remark. Since $g^n_{h,\tau}=g^{n+m}_{h,\tau}$, $(n\ge 0)$, therefore (5.14) implies $\bar{u}^n_{h,\tau}=\bar{u}^{n+m}_{h,\tau}$.

Proof. From §2, $e^k_{h,\tau}=u^k_{h,\tau}-p_h u(k\tau)$ satisfies the following equation

$$e^{k+1}_{h,\tau}=\bar{C}^k_{h,\tau}(e^k_{h,\tau})+\tau R^k_{h,\tau}(u). \tag{5.16}$$

Since $p_h u(n\tau)=(\cdots,\sin(n\tau)\sin(2j\pi h),\cdots)^T$ has period 2π, it is sufficient to prove equation (5.16) has a solution of period 2π satisfying (5.15). Let

$$D^k(v_h)=\bar{C}^k_{h,\tau}(v_h)+\tau R^k_{h,\tau}(u), \tag{5.17}$$

then one can transform (5.16) into

$$e^{k+1}_{h,\tau}=D^k(e^k_{h,\tau}). \tag{5.18}$$

Let $Z=\{v_h|v_h\in V_h,\ \|v_h\|_h\le\varepsilon_{1,h,\tau}\}$. Then for $\forall v_h\in Z$ we have

$$\begin{aligned}\|D^k(v_h)\|_h\le&(1-\rho\tau)\varepsilon_{1,h,\tau}+\tau S_{h,\tau}\\ =&\varepsilon_{1,h,\tau}-\rho\tau(\varepsilon_{1,h,\tau}-\frac{S_{h,\tau}}{\rho})\le\varepsilon_{1,h,\tau},\end{aligned}$$

hence, D^k is a operator from Z to Z.

For $\forall v_{1h},v_{2h}\in Z$, from (5.12) and (5.17) we conclude that

$$\begin{aligned}\|D^k(v_{1h})-D^k(v_{2h})\|_h\le&\|I+\tau A\|\|v_{1h}-v_{2h}\|_h+2\tau\|v_{1h}-v_{2h}\|_h+\tau\|p_h(v_{1h}^2-v_{2h}^2)\|_h\\ \le&(1-\tau\lambda_h+2\tau+\frac{2\tau}{\sqrt{h}}\varepsilon_{1,h,\tau})\|v_{1h}-v_{2h}\|_h.\end{aligned}$$

Let h_0 be sufficiently small such that lemma 5.3 holds. Then from the proof of the lemma we conclude that

$$\begin{aligned}\|D^k(v_{1h})-D^k(v_{2h})\|_h\le&(1-9\tau+2\tau+\tau)\|v_{1h}-v_{2h}\|_h\\ =&(1-\rho\tau)\|v_{1h}-v_{2h}\|_h.\end{aligned} \tag{5.19}$$

Define T^k by

$$T^k = D^k D^{k-1} \cdots D^0, \quad (k \geq 0). \tag{5.20}$$

Obviously T^k is a strict contraction in Z. Moreover, for $\forall v_{1h}, v_{2h} \in Z$ we have

$$\|T^k(v_{1h}) - T^k(v_{2h})\|_h \leq (1-\rho\tau)^{k+1}\|v_{1h} - v_{2h}\|_h. \tag{5.21}$$

From (5.18) and (5.20),

$$e^{k+1}_{h,\tau} = T^k(e^0_{h,\tau}), \quad (k \geq 0). \tag{5.22}$$

Since T^{m-1} is a strict contraction in Z, it has a fixed point. So there exists $\bar{e}^k_{h,\tau} \in Z$ satisfies (5.22) and

$$\bar{e}^0_{h,\tau} = \bar{e}^m_{h,\tau}, \quad (\text{this implies} \quad \bar{u}^0_{h,\tau} = \bar{u}^m_{h,\tau}), \tag{5.23}$$

hence $\bar{e}^k_{h,\tau}$ is a solution of period 2π.

From (5.21) and (5.22), for $\forall e^0_{h,\tau} \in Z$, $e^n_{h,\tau}$ defined by (5.22) satisfies

$$\|\bar{e}^n_{h,\tau} - e^n_{h,\tau}\|_h = \|T^{n-1}(\bar{e}^n_{h,\tau)} - T^{n-1}(e^n_{h,\tau})\|_h \leq 2(1-\rho\tau)^n \varepsilon_{1,h,\tau}. \tag{5.24}$$

If taking $e^0_{h,\tau} = 0$, then from theorem 5.1 we get

$$\|e^n_{h,\tau}\|_h \leq C_1(\tau + h^2), \quad n \geq 0. \tag{5.25}$$

From (5.24), there exists $N > 0$ such that, for $n > N$,

$$\|\bar{e}^n_{h,\tau} - e^n_{h,\tau}\|_h \leq C_1(\tau + h^2),$$

hence, for $n > N$, we have

$$\|\bar{e}^n_{h,\tau}\|_h \leq C(\tau + h^2), \quad (C = 2C_1).$$

By using (5.23) we conclude that

$$\|\bar{e}^n_{h,\tau}\|_h \leq C(\tau + h^2), \quad \forall n \geq 0 \tag{5.26}$$

This is (5.15). Thus we complete the proof of theorem 5.4.

In fact, from the proof of theorem 5.4 we can also conclude that, this periodic solution is asymptotically stable (see (5.24)).

By theorem 5.4, we can understand easily why the numerical solution has long-time convergence.

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