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AN ITERATION METHOD FOR INCOMPRESSIBLE VISCOUS/INVISCID COUPLED PROBLEM VIA A SPECTRAL APPROXIMATION*1)

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Abstract

An efficient iteration-by-subdomain method (known as the Schwarz alternating algorithm) for incompressible viscous/inviscid coupled model is presented. Appropriate spectral collocation approximations are proposed. The convergence analysis show that the iterative algorithms converge with a rate independent of the polynomial degree used.

Key words: Coupled equations, Navier-Stokes equations, Euler equations, Cllocation approximation, Schwarz alternating algorithm.

1. Introduction

Domain decomposition methods are useful approximation techniques to face computational fluid dynamics problems, especially in complex physical domains and using parallel computational environments. They have been first employed in finite difference and finite element methods. In the context of spectral methods, they date from the late 1970s (see for instance [3] and the references therein). Earlier applications of the domain decomposition methods are related to split the whole domain into subdomains of simpler shape, and then to reduce the given problem to a sequence of subproblems which include generally same equations. Recently an intensive attention focuses on the study of possibility of using different type of equations within subdomains where different flow characters are observable. There has been some work, done mainly by Quarteroni and his collaborators [4, 8], on the coupling of compressible viscous and inviscid equations. The coupled problem of incompressible viscous and inviscid equations has been first considered by Xu and Maday in [11]. One of main goals of these investigations was to find correct conditions on the interface separating the viscous and inviscid subdomains. However efficient solvers are also of great importance when solving numerically the full time-dependent coupled equations. We propose in this paper an iteration-by-subdomain procedure to solve the coupled problem. The iteration algorithm, which involves the successive resolution of the two subproblems, is a variant of classical Schwarz alternating methods[9, 4, 8]. But the present algorithm uses two news techniques: first the norms of interface's function are defined via some interface

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"lifting" operators, different from the usual L^2-norm; secondly, the interface iteration functions are constructed on weak form, due to the discontinuous velocity/continuous pressure formulation in the inviscid subdomain (in fact we have not been able to prove the convergence of the iterative procedure based on strong form). We give exact convergence analysis and prove that the iterative algorithms using a spectral collocation approximation converge with a rate independent of the polynomial degree used.

We end this introduction by introducing some notations. Hereafter we use letters of boldface type to denote vectors and vector functions. $c, c_1, c_2, \cdots$ are generic positive constants independent of the discretization parameters. Let Ω to be a bounded, connected, open subset of R^2, with Lipschitz continuous boundary $\partial\Omega$ (see fig.1); Ω_- and Ω_+ are two open subsets of Ω, with $\Omega_- \cap \Omega_+ = \emptyset$, $\bar{\Omega}_- \cup \bar{\Omega}_+ = \bar{\Omega}$. Let $\Gamma_k = \partial\Omega \cap \partial\Omega_k, k = -,+; \Gamma = \partial\Omega_- \cap \partial\Omega_+ \neq \emptyset$. $\mathbf{n}_-, \mathbf{n}_+$ are the unit normals to Ω_-, Ω_+ respectively (so $\mathbf{n}_- = -\mathbf{n}_+$ on Γ). We notice by $C^0(\bar{\Omega})$ the space of continuous functions on $\bar{\Omega}$. For any integer m, we notice by $H^m(\Omega)$ the classical Hilbert Sobolev spaces, provided with the usual norm $\|\cdot\|_{m,\Omega}$, and also, with the semi-norm $|\cdot|_{m,\Omega}$. It is well known that the value on the boundary $\partial\Omega$ of all elements of $H^m(\Omega)$ can be given a meaning through a trace operator which maps linearly and continuously $H^m(\Omega)$ onto a subset of $L^2(\partial\Omega)$, denoted by $H^{m-1/2}(\partial\Omega)$, which is a Hilbert space for the quotient norm $\|\cdot\|_{m-1/2,\partial\Omega}$. We use also the space $L_0^2(\Omega)$ defined by

$$L_0^2(\Omega) = \{v \in L^2(\Omega); \int_\Omega v d\mathbf{x} = 0\}.$$

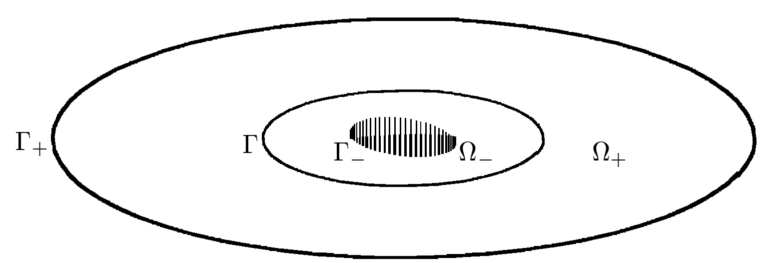

Fig.1 Computational domain

Throughout this paper, with any function $\mathbf{v}$ defined in Ω, we associate the pair $(\mathbf{v}_-, \mathbf{v}_+)$, where $\mathbf{v}_-$ (resp. $\mathbf{v}_+$) denotes the restriction of $\mathbf{v}$ to Ω_- (reps. Ω_+). We define $(\cdot,\cdot)_k, k = -,+$ and $(\cdot,\cdot)_\Gamma$ by

$$(\mathbf{u}_k, \mathbf{v}_k)_k = \int_{\Omega_k} \mathbf{u}_k \mathbf{v}_k \, d\mathbf{x}, \qquad (\Phi, \Psi)_\Gamma = \int_\Gamma \Phi\Psi \, d\sigma.$$

The scalar product on $L^2(\Omega_-)^2 \times L^2(\Omega_+)^2$,

$$(\mathbf{u}, \mathbf{v}) = (\mathbf{u}_-, \mathbf{v}_-)_- + (\mathbf{u}_+, \mathbf{v}_+)_+,$$

coincides with the usual one on $L^2(\Omega)^2$.

2. Viscous/inviscid coupled problem

Consider the following coupled problem: for $\mathbf{f}$ given in $L^2(\Omega)^2$ and α, ν positive constants, find two pairs $(\mathbf{u}_-,\ \mathbf{u}_+)$, $(p_-,\ p_+)$ defined in $(\Omega_-,\ \Omega_+)$ respectively, such that:

$$\left\{\begin{array}{rclll} \alpha\mathbf{u}_- - \nu\triangle\mathbf{u}_- + \nabla p_- &=& \mathbf{f}_-, & \nabla\cdot\mathbf{u}_- = 0, & \text{in } \Omega_-, \\ \alpha\mathbf{u}_+ + \nabla p_+ &=& \mathbf{f}_+, & \nabla\cdot\mathbf{u}_+ = 0, & \text{in } \Omega_+, \\ \mathbf{u}_- &=& 0, & & \text{on } \Gamma_-, \\ \mathbf{u}_+\cdot\mathbf{n}_+ &=& 0, & & \text{on } \Gamma_+. \end{array}\right. \tag{2.1}$$

This problem, that will be hereafter referred to as viscous/inviscid coupled problem, stems from the use of a finite-difference schema in time to the nonlinear Navier-Stokes/Euler coupled equations for incompressible flow[10]. In this respect, ν is the kinematic viscosity, α is the inverse of the time-step, and $\mathbf{f}$ is the source terms.

Obviously, suitable conditions on the interface Γ are required. That can be seen in a trial way that one condition is needed on Γ in order to solve the viscous problem in Ω_-, and that a further condition is required on Γ in order to solve the inviscid problem in Ω_+. In order to find it, we apply the well-known *vanishing viscosity* technique that consists of generating the interface conditions by a limit procedure on *globally viscous* problems when *viscosity* vanishes in Ω_+. It has been proven that the appropriate interface conditions are[11]:

$$\left\{\begin{array}{rcll} \nu\dfrac{\partial\mathbf{u}_-}{\partial\mathbf{n}_-} - p_-\mathbf{n}_- &=& p_+\mathbf{n}_+ & \text{on } \Gamma, \\ \mathbf{u}_-\cdot\mathbf{n}_- &=& -\mathbf{u}_+\cdot\mathbf{n}_+ & \text{on } \Gamma. \end{array}\right. \tag{2.2}$$

The equations (2.1)-(2.2) are well posed in the sense that their weak problem have one unique solution. That can be done by considering the following variational formulation: find $(\mathbf{u},\ p)\in X\times M$, such that for all $\mathbf{v}\in X$, $q\in M$,

$$\begin{array}{rl} \alpha(\mathbf{u},\mathbf{v}) + \nu(\nabla\mathbf{u}_-,\nabla\mathbf{v}_-)_- - (p_-,\nabla\cdot\mathbf{v}_-)_- + (\nabla p_+,\mathbf{v}_+)_+ - (p_+\mathbf{n}_+,\mathbf{v}_-)_\Gamma = & (\mathbf{f},\mathbf{v}), \\ (\nabla\cdot\mathbf{u}_-,q_-)_- - (\mathbf{u}_+,\nabla q_+)_+ - (\mathbf{u}_-\cdot\mathbf{n}_-,q_+)_\Gamma = & 0, \end{array} \tag{2.3}$$

where X, M are two real Hilbert spaces, defined by

$$X = \{\mathbf{v};\mathbf{v}|_{\Omega_-}\in H^1(\Omega_-)^2, \mathbf{v}|_{\Omega_+}\in L^2(\Omega_+)^2, \mathbf{v}|_{\Gamma_-} = 0\}, \tag{2.4}$$

$$M = \{q; q|_{\Omega_-}\in L^2(\Omega_-), q|_{\Omega_+}\in H^1(\Omega_+), \int_\Omega q d\mathbf{x} = 0\}, \tag{2.5}$$

with respective norms

$$\|\mathbf{v}\|_X = \|\mathbf{v}_-\|_{1,\Omega_-} + \|\mathbf{v}_+\|_{0,\Omega_+}, \quad \|q\|_M = \|q_-\|_{0,\Omega_-} + |q_+|_{1,\Omega_+}.$$

Theorem 2.1.[11] *For all α and ν positive, problem* (2.3) *admits one unique solution.*

3. Solution via an iteration-by-subdomain procedure

Our goal in this section is to prove that the solution of the coupled problem (2.1)-(2.2) can be exhibited as a limit of solutions of two subproblems within Ω_- and Ω_+ respectively.

We first remark that the pressure (p_-, p_+) in the coupled problem (2.1)-(2.2) is defined up to an additive constant. In order to fix this constant, we have chosen the pressure space M of functions with zero average in full domain Ω (see (2.5) for the definition of M). In fact, this choice of M is only a matter of convenience, and we can just as well take

$$M = \{q; q|_{\Omega_-} \in L^2(\Omega_-), q|_{\Omega_+} \in H^1(\Omega_+), \int_{\Omega_+} q d\mathbf{x} = 0\} \ . \tag{3.1}$$

The former has been proven suitable for the global Uwaza algorithm[11]. The latter is however preferable to the iteration-by-subdomain method, which will be discussed hereafter.

3.1 The iteration-by-subdomain procedure

Let $\mathbf{u}_-^0, \mathbf{u}_+^0$ to be two functions given in Γ. We define two sequences of function pair $(\mathbf{u}_-^m, p_-^m)_{m\geq 1}$ and $(\mathbf{u}_+^m, p_+^m)_{m\geq 1}$ by solving for each m the following inviscid problem in Ω_+

$$\left\{ \begin{array}{l} \alpha \mathbf{u}_+^m + \nabla p_+^m = \mathbf{f}_+ \ , \quad \nabla \cdot \mathbf{u}_+^m = 0 \ , \quad \text{in } \Omega_+, \\ \mathbf{u}_+^m \cdot \mathbf{n}_+ = 0 \ , \quad \text{on } \Gamma_+ \ , \quad \mathbf{u}_+^m \cdot \mathbf{n}_+ = \varphi^m \ , \quad \text{on } \Gamma, \end{array} \right. \tag{3.2}$$

and then the following viscous problem in Ω_-

$$\left\{ \begin{array}{l} \alpha \mathbf{u}_-^m - \nu \triangle \mathbf{u}_-^m + \nabla p_-^m = \mathbf{f}_- \ , \quad \nabla \cdot \mathbf{u}_-^m = 0 \ , \quad \text{in } \Omega_-, \\ \mathbf{u}_-^m = 0 \ , \quad \text{on } \Gamma_- \ , \quad \nu \dfrac{\partial \mathbf{u}_-^m}{\partial \mathbf{n}_-} - p_-^m \mathbf{n}_- = p_+^m \mathbf{n}_+ \ , \quad \text{on } \Gamma, \end{array} \right. \tag{3.3}$$

where $\varphi^m = \theta \mathbf{u}_-^{m-1} \cdot \mathbf{n}_+|_\Gamma + (1-\theta)\mathbf{u}_+^{m-1} \cdot \mathbf{n}_+|_\Gamma$, $\theta \in [0,1]$ is a relaxation parameter.

Remark 3.1. *In order for* (3.2) *to be well-posed, the interface data of the first step,* φ^1, *has to be chosen to satisfy the compatibility condition:*

$$\int_\Gamma \varphi^1 d\sigma = 0 \ .$$

The iterative procedure will be discussed both in the continuous case, and in its spectral discrete case (see section 4). In both cases, we will prove the solvability of the subproblems and the convergence of the iterative procedure. We first consider the solvability and a priori estimates of the problems (3.2) and (3.3).

The variational formulation of (3.2) writes: find $(\mathbf{u}_+^m, p_+^m) \in X_+ \times M_+$, such that

$$\begin{aligned} A_+[(\mathbf{u}_+^m, p_+^m), (\mathbf{v}_+, q_+)] = (\mathbf{f}_+, \mathbf{v}_+)_+ - (\varphi^m, q_+)_\Gamma, \\ \forall (\mathbf{v}_+, q_+) \in X_+ \times M_+, \end{aligned} \tag{3.4}$$

where

$$X_+ = L^2(\Omega_+)^2 \ , M_+ = H^1(\Omega_+) \cap L^2_0(\Omega_+),$$

and A_+ is defined by

$$A_+[(\mathbf{u}_+^m, p_+^m), (\mathbf{v}_+, q_+)] = \alpha(\mathbf{u}_+^m, \mathbf{v}_+)_+ + (\mathbf{v}_+, \nabla p_+^m)_+ - (\mathbf{u}_+^m, \nabla q_+)_+.$$

Theorem 3.1. *For all $\mathbf{f}_+ \in L^2(\Omega_+)^2, \varphi^m \in L^2(\Gamma)$, the problem (3.4) admits one unique solution; furthermore, its solution $(\mathbf{u}_+^m, p_+^m)$ satisfies*

$$\|\mathbf{u}_+^m\|_{0,\Omega_+} + |p_+^m|_{1,\Omega_+} \leq (\frac{1}{\alpha} + 2)\|\mathbf{f}_+\|_{0,\Omega_+} + 2(1+\alpha)\|\varphi^m\|_{0,\Gamma} \ , \tag{3.5}$$

particularly if $\mathbf{f}_+ = 0$, then

$$\|\mathbf{u}_+^m\|_{0,\Omega_+} \leq 2\|\varphi^m\|_{0,\Gamma} \ , \tag{3.6}$$

$$|p_+^m|_{1,\Omega_+} \leq \alpha\|\mathbf{u}_+^m\|_{0,\Omega_+} \leq 2\alpha\|\varphi^m\|_{0,\Gamma} \ . \tag{3.7}$$

Proof. We note that $A_+[(\mathbf{u}_+^m, p_+^m), (\mathbf{v}_+, q_+)]$ is not coercive in space $X_+ \times M_+$. Consequently, the well-posedness of problem (3.4) can not be derived by the standard Lax-Milgram theorem. However we see that problem (3.4) is equivalent to the saddle-point problem:

$$\begin{cases} \alpha(\mathbf{u}_+^m, \mathbf{v}_+)_+ + (\mathbf{v}_+, \nabla p_+^m)_+ = (\mathbf{f}_+, \mathbf{v}_+)_+, & \forall \mathbf{v}_+ \in X_+, \\ (\mathbf{u}_+^m, \nabla q_+)_+ = (\varphi^m, q_+)_\Gamma, & \forall q_+ \in M_+, \end{cases}$$

whose well-posedness can be proven by applying the saddle-point theory[5]. The estimations (3.5)-(3.7) can also be obtained by using standard estimation techniques.

□

We now consider problem (3.3). Its variational formulation is: find $(\mathbf{u}_-^m, p_-^m) \in X_- \times M_-$, such that

$$A_-[(\mathbf{u}_-^m, p_-^m), (\mathbf{v}_-, q_-)] = (\mathbf{f}_-, \mathbf{v}_-)_- + (p_+^m \mathbf{n}_+, \mathbf{v}_-)_\Gamma, \quad \forall (\mathbf{v}_-, q_-) \in X_- \times M_-, \tag{3.8}$$

where

$$X_- = \{\mathbf{v}_- \in H^1(\Omega_-)^2; \mathbf{v}_-|_{\Gamma_-} = 0\} \ , M_- = L^2(\Omega_-),$$

and A_- is defined by

$$A_-[(\mathbf{u}_-^m, p_-^m), (\mathbf{v}_-, q_-)] = \alpha(\mathbf{u}_-^m, \mathbf{v}_-)_- + \nu(\nabla \mathbf{u}_-^m, \nabla \mathbf{v}_-)_- - (\nabla \mathbf{v}_-, p_-^m)_- + (\nabla \mathbf{u}_-^m, q_-)_-.$$

The following theorem comes from classical results on the Stokes equations (see e.g. [2]).

Theorem 3.2. *For all $\mathbf{f}_- \in L^2(\Omega_-)^2$ and $p_+^m \in L^2(\Gamma)$, the problem (3.8) admits one unique solution; furthermore, its solution $(\mathbf{u}_-^m, p_-^m)$ satisfies*

$$\|\mathbf{u}_-^m\|_{1,\Omega_-} + \|p_-^m\|_{0,\Omega_-} \leq c_0(\|\mathbf{f}_-\|_{0,\Omega_-} + \|p_+^m\|_{0,\Gamma}) \ , \tag{3.9}$$

particularly if $\mathbf{f}_- = 0$, then

$$\|\mathbf{u}_-^m\|_{1,\Omega_-} + \|p_-^m\|_{0,\Omega_-} \leq c_0\|p_+^m\|_{0,\Gamma} \ , \tag{3.10}$$

where c_0 depends on α and ν.

3.2 Convergence of the iteration-by-subdomain procedure

We deal now with the convergence of the iteration-by-subdomain procedure (3.2)-(3.3). We begin by defining the application $L : L^2(\Gamma) \longrightarrow L^2(\Gamma)$,

$$L\lambda = \mathbf{u}_-^{(\lambda)} \cdot \mathbf{n}_+|_\Gamma \ , \quad \forall \lambda \in L^2(\Gamma),$$

and then the application $L_\theta : L^2(\Gamma) \longrightarrow L^2(\Gamma)$,

$$L_\theta \lambda = \theta L\lambda + (1-\theta)\lambda \ , \quad \forall \lambda \in L^2(\Gamma),$$

where $\mathbf{u}_-^{(\lambda)}$ solves the problem: $(\mathbf{u}_-^{(\lambda)}, p_-^{(\lambda)}) \in X_- \times M_-$, such that

$$A_-[(\mathbf{u}_-^{(\lambda)}, p_-^{(\lambda)}), (\mathbf{v}_-, q_-)] = (p_+^{(\lambda)}\mathbf{n}_+, \mathbf{v}_-)_\Gamma, \quad \forall (\mathbf{v}_-, q_-) \in X_- \times M_-,$$

where $p_+^{(\lambda)}$ is the solution of the following problem: $(\mathbf{u}_+^{(\lambda)}, p_+^{(\lambda)}) \in X_+ \times M_+$, such that

$$A_+[(\mathbf{u}_+^{(\lambda)}, p_+^{(\lambda)}), (\mathbf{v}_+, q_+)] = -(\lambda, q_+)_\Gamma, \quad \forall (\mathbf{v}_+, q_+) \in X_+ \times M_+.$$

We define also the "lifting" operator $F : \forall \lambda \in L^2(\Gamma),\ F\lambda \in X_+ \times M_+$ and $F\lambda$ solves:

$$A_+[F\lambda, (\mathbf{v}_+, q_+)] = -(\lambda, q_+)_\Gamma \quad \forall (\mathbf{v}_+, q_+) \in X_+ \times M_+ \tag{3.11}$$

Moreover set for $\lambda, \mu \in L^2(\Gamma)$:

$$((\lambda, \mu)) = A_+[F\lambda, F\mu] \ , \qquad \|\lambda\|_*^2 = ((\lambda, \lambda)) \ . \tag{3.12}$$

Lemma 3.1. *The bilinear form $((\cdot,\cdot))$ defined by (3.12) is symmetric, therefore it defines a scalar product in $L^2(\Gamma)$.*

Proof. Using the notation of (3.11), we have

$$((\lambda, \mu)) = A_+[F\lambda, F\mu] = -(\lambda, p_+^{(\mu)})_\Gamma \ ,$$

but (3.11) implicates

$$-(\mathbf{u}_+^{(\lambda)}, \nabla p_+^{(\mu)})_+ = -(\lambda, p_+^{(\mu)})_\Gamma \ ,$$

furthermore

$$\alpha(\mathbf{u}_+^{(\mu)}, \mathbf{u}_+^{(\lambda)})_+ + (\mathbf{u}_+^{(\lambda)}, \nabla p_+^{(\mu)})_+ = 0 \ ,$$

then

$$((\lambda, \mu)) = \alpha(\mathbf{u}_+^{(\mu)}, \mathbf{u}_+^{(\lambda)})_+ \ ,$$

which gives

$$((\lambda, \mu)) = ((\mu, \lambda)) \ .$$

It is then immediate that $((\cdot,\cdot))$ defines a scalar product in $L^2(\Gamma)$. □

Theorem 3.3. *There exists $\theta_0 \in (0,1]$, such that for all $\theta \in (0,\theta_0)$, it exists $k(\theta) < 1$ such that*

$$\|L_\theta \lambda\|_* \le k(\theta)\|\lambda\|_* \ , \quad \forall \lambda \in L^2(\Gamma) \ . \tag{3.13}$$

Proof. From the symmetry of $((\cdot,\cdot))$ we have

$$\|L_\theta \lambda\|_*^2 = \theta^2\|L\lambda\|_*^2 + 2\theta(1-\theta)((L\lambda,\lambda)) + (1-\theta)^2\|\lambda\|_*^2 \ . \tag{3.14}$$

According to the definitions of F and L, it is verified that

$$\begin{aligned} ((L\lambda,\lambda)) &= A_+[F(L\lambda), F\lambda] = A_+[F(\mathbf{u}_-^{(\lambda)}\cdot \mathbf{n}_+|_\Gamma), F\lambda] \\ &= -(\mathbf{u}_-^{(\lambda)}\cdot\mathbf{n}_+, p_+^{(\lambda)})_\Gamma = -(p_+^{(\lambda)}\mathbf{n}_+, \mathbf{u}_-^{(\lambda)})_\Gamma = -A_-[(\mathbf{u}_-^{(\lambda)}, p_-^{(\lambda)}), (\mathbf{u}_-^{(\lambda)}, p_-^{(\lambda)})] \\ &= -\alpha(\mathbf{u}_-^{(\lambda)}, \mathbf{u}_-^{(\lambda)})_- - \nu(\nabla\mathbf{u}_-^{(\lambda)}, \nabla\mathbf{u}_-^{(\lambda)})_- \le -\min(\alpha,\nu)\|\mathbf{u}_-^{(\lambda)}\|_{1,\Omega_-}^2 \ . \end{aligned}$$

Using (3.6), we get

$$\|L\lambda\|_*^2 = A_+[F(L\lambda), F(L\lambda)] \le 4\|\mathbf{u}_-^{(\lambda)}\cdot\mathbf{n}_-\|_{0,\Gamma}^2 \le c_1\|\mathbf{u}_-^{(\lambda)}\|_{1,\Omega_-}^2 \ , \tag{3.15}$$

where c_1 depends on the trace mapping constant. Combining (3.14)-(3.15), we obtain

$$\|L_\theta\lambda\|_*^2 \le [c_1\theta^2 - 2\min(\alpha,\nu)\theta(1-\theta)]\|\mathbf{u}_-^{(\lambda)}\|_{1,\Omega_-}^2 + (1-\theta)^2\|\lambda\|_*^2 \ . \tag{3.16}$$

Using (3.10), (3.7) and the standard trace's inequalities, we have

$$\begin{aligned} \|\mathbf{u}_-^{(\lambda)}\|_{1,\Omega_-}^2 &\le c_0^2\|p_+^{(\lambda)}\|_{0,\Gamma}^2 \le c_2|p_+^{(\lambda)}|_{1,\Omega_+}^2 \\ &\le c_2\alpha^2\|\mathbf{u}_+^{(\lambda)}\|_{0,\Omega_+}^2 = c_2\alpha A_+[F\lambda, F\lambda] = c_2\alpha\|\lambda\|_*^2 \ , \end{aligned} \tag{3.17}$$

where c_2 depends on c_0^2 and the trace mapping constant. Finally, a combination of (3.16) and (3.17) gives

$$\|L_\theta\lambda\|_*^2 \le [\ c_1c_2\alpha\theta^2 - 2c_2\alpha\min(\alpha,\nu)\theta(1-\theta) + (1-\theta)^2\ ]\ \|\lambda\|_*^2 \ .$$

Let

$$k(\theta) = \sqrt{c_1c_2\alpha\theta^2 - 2c_2\alpha\min(\alpha,\nu)\theta(1-\theta) + (1-\theta)^2} \ ,$$

we obtain (3.13). Furthermore a simple calculation shows

$$k(\theta) < 1 \qquad \text{if and only if} \qquad 0 < \theta < \theta_0 = \min\Big(1, \frac{2(1 + c_2\alpha\min(\alpha,\nu))}{1 + 2c_2\alpha\min(\alpha,\nu) + c_1c_2\alpha}\Big).$$

□

One of the immediate consequences of the above theroem is the following corollary.

Corollary 3.1. *Let $(\mathbf{u}_+, p_+), (\mathbf{u}_-, p_-)$ to be the solution of the coupled equations (2.1) and (2.2); Let $(\mathbf{u}_+^m, p_+^m), (\mathbf{u}_-^m, p_-^m)$ to be the solution of the iteration problems (3.2) and (3.3). Then for all $\theta \in (0,\theta_0)$, $(\mathbf{u}_+^m, p_+^m)$ converges to $(\mathbf{u}_+, p_+)$ in $X_+ \times M_+$ and $(\mathbf{u}_-^m, p_-^m)$ converges to $(\mathbf{u}_-, p_-)$ in $X_- \times M_-$ as $m \to \infty$.*

Proof. We first prove

$$\varphi^m - \mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma = L_\theta(\varphi^{m-1} - \mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma). \tag{3.18}$$

In fact, by the definitions of L_θ and φ^m, we have

$$\begin{aligned}
& L_\theta(\varphi^{m-1} - \mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma) \\
= \;& \theta L(\varphi^{m-1} - \mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma) + (1-\theta)(\varphi^{m-1} - \mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma) \\
= \;& \theta(\mathbf{u}_-^{m-1} \cdot \mathbf{n}_+|_\Gamma) - \theta(\mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma) + (1-\theta)\varphi^{m-1} - \mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma + \theta(\mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma) \\
= \;& \theta(\mathbf{u}_-^{m-1} \cdot \mathbf{n}_+|_\Gamma) + (1-\theta)(\mathbf{u}_+^{m-1} \cdot \mathbf{n}_+|_\Gamma) - \mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma \\
= \;& \varphi^m - \mathbf{u}_- \cdot \mathbf{n}_+|_\Gamma .
\end{aligned}$$

The contraction of L_θ and the equality (3.18) imply

$$\varphi^m \to \mathbf{u}_- \cdot \mathbf{n}_+ \;, \quad \text{as} \quad m \to \infty \;.$$

But (2.3) and (3.4) give

$$A_+[(\mathbf{u}_+^m - \mathbf{u}_+, p_+^m - p_+), (\mathbf{v}_+, q_+)] = (\mathbf{u}_- \cdot \mathbf{n}_+ - \varphi^m, q_+)_\Gamma, \quad \forall(\mathbf{v}_+, q_+) \in X_+ \times M_+,$$

by the estimation (3.6) and (3.7), we get

$$\|\mathbf{u}_+^m - \mathbf{u}_+\|_{0,\Omega_+} + |p_+^m - p_+|_{1,\Omega_+} \le c\|\varphi^m - \mathbf{u}_- \cdot \mathbf{n}_+\|_{0,\Gamma},$$

thus

$$\mathbf{u}_+^m \to \mathbf{u}_+ \;\text{ in }\; X_+ \;, \qquad p_+^m \to p_+ \;\text{ in }\; M_+ \;, \quad \text{as} \;\; m \to \infty \;.$$

and hence

$$p_+^m \to p_+ \;\text{ in }\; L^2(\Gamma) \;, \quad \text{as} \;\; m \to \infty.$$

But (2.3) and (3.8) give

$$A_-[(\mathbf{u}_-^m - \mathbf{u}_-, p_-^m - p_-), (\mathbf{v}_-, q_-)] = (p_-^m \mathbf{n}_+ - p_- \mathbf{n}_+, \mathbf{v}_-)_\Gamma, \; \forall(\mathbf{v}_-, q_-) \in X_- \times M_-.$$

It follows from the estimation (3.10) that

$$\|\mathbf{u}_-^m - \mathbf{u}_-\|_{1,\Omega_-} + \|p_-^m - p_-\|_{0,\Omega_-} \le c\|p_+^m - p_+\|_{0,\Gamma} \to 0 \;,$$

which gives

$$\mathbf{u}_-^m \to \mathbf{u}_- \;\text{ in }\; X_- \;, \qquad p_-^m \to p_- \;\text{ in }\; M_- \;, \quad \text{as} \;\; m \to \infty \;.$$

□

4. An iteration-by-subdomain procedure via a spectral approximation

We approximate the iteration-by-subdomain problems (3.2) and (3.3) by a spectral collocation method. For the sake of simplicify, we consider the domain $\Omega = (-2,2) \times (-1,1)$, which is broken into $\Omega_- = (-2,0) \times (-1,1)$ and $\Omega_+ = (0,2) \times (-1,1)$. We assume also $\mathbf{f} \in C^0(\Omega)^2$. Let us first introduce some notations. We denote by $I\!P_N$ the space of all polynomials of degree $\leq N$ with respect to each variable x_1, x_2. We then denote respectively by Ξ_k^N and Λ_k^N the sets of $(N+1)^2$ Legendre-Gauss-Lobatto points ξ_k^{ij} and $(N-1)^2$ Legendre-Gauss points ζ_k^{ij} within $\bar{\Omega}_k$ (see, e.g. [1] for the exact definitions),

$$
\begin{aligned}
\Xi_k^N &= \{\xi_k^{ij}; \xi_k^{ij} = (\xi_{1,k}^i, \xi_{2,k}^j), 0 \leq i,j \leq N\} \ , \quad k-,+, \\
\Lambda_k^N &= \{\zeta_k^{ij}; \zeta_k^{ij} = (\zeta_{1,k}^i, \zeta_{2,k}^j), 1 \leq i,j \leq N-1\} \ , \quad k-,+.
\end{aligned}
$$

Let $h_{1,k}^i, h_{2,k}^i; 0 \leq i \leq N$, denote respectively the Lagrange polynomials associated to the components $\xi_{1,k}^i$ and $\xi_{2,k}^i$. For any points ξ_k^{ij} and ζ_k^{ij}, we denote respectively by ω_k^{ij} and ρ_k^{ij} the corresponding weights in the Gauss-Lobatto and Gauss integration formula for rectangular regions. For any point ξ_k^{ij} in $\partial\Omega_k \cap \Xi_k^N$ we denote by τ_k^{ij} the corresponding weight in the one-dimensional Gauss-Lobatto integration formula referred to $\partial\Omega_k$. We define the discrete integration rules for all $\Phi, \Psi \in C^0(\bar{\Omega})$,

$$
(\Phi, \Psi)_{k,GL} = \sum_{i=0}^{N} \sum_{j=0}^{N} \Phi(\xi_k^{ij}) \Psi(\xi_k^{ij}) \omega_k^{ij} \ , \quad k = -,+ \ , \tag{4.1}
$$

$$
(\Phi, \Psi)_{GL} = (\Phi, \Psi)_{-,GL} + (\Phi, \Psi)_{+,GL} \ , \tag{4.2}
$$

$$
(\Phi, \Psi)_{-,G} = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \Phi(\zeta_-^{ij}) \Psi(\zeta_-^{ij}) \rho_-^{ij} \ , \tag{4.3}
$$

$$
(\Phi, \Psi)_{\Gamma,GL} = \sum_{\xi_-^{ij} \in \Gamma \cap \Xi_-^N} \Phi(\xi_-^{ij}) \Psi(\xi_-^{ij}) \tau_-^{ij} \ (= \sum_{\xi_+^{ij} \in \Gamma \cap \Xi_+^N} \Phi(\xi_+^{ij}) \Psi(\xi_+^{ij}) \tau_+^{ij}) \ . \tag{4.4}
$$

We introduce the norms associated to (4.1)-(4.4):

$$
\|\Phi\|_{k,GL} = (\Phi, \Phi)_{k,GL}^{\frac{1}{2}} \ , \quad k = -,+ \ , \quad \|\Phi\|_{GL} = \|\Phi\|_{-,GL} + \|\Phi\|_{+,GL} \ ,
$$

$$
\|\Phi\|_{-,G} = (\Phi, \Phi)_{-,G}^{\frac{1}{2}} \ , \quad \|\Phi\|_{\Gamma,GL} = (\Phi, \Phi)_{\Gamma,GL}^{\frac{1}{2}} \ .
$$

The following inequalities are well known (see, e.g. [1] p.70-76):

$$
\|\Phi\|_{k,GL}^2 \leq \|\Phi\|_{0,\Omega_k}^2 \leq 9\|\Phi\|_{k,GL}^2 \ , \quad \forall \Phi \in I\!P_N(\Omega_k) \ , \ k = -,+,
$$

$$
\|\Phi\|_{\Gamma,GL}^2 \leq \|\Phi\|_{0,\Gamma}^2 \leq 3\|\Phi\|_{\Gamma,GL}^2 \ , \quad \forall \Phi \in I\!P_N(\Gamma) \ , \tag{4.5}
$$

$$\|\Phi\|^2_{-,G} = \|\Phi\|^2_{0,\Omega_-} \ , \quad \forall \Phi \in I\!P_{N-2}(\Omega_-) \ .$$

Let $X_{-,N}, M_{-,N}, X_{+,N}$ and $M_{+,N}$ to be four spaces:

$$\begin{array}{ll} X_{-,N} = \{\mathbf{v}_- \in I\!P_N(\Omega_-)^2; \mathbf{v}_-|_{\Gamma_-} = 0\} \ , & M_{-,N} = I\!P_{N-2}(\Omega_-), \\ X_{+,N} = I\!P_N(\Omega_+)^2, & M_{+,N} = I\!P_N(\Omega_+) \cap L^2_0(\Omega_+). \end{array}$$

We now state the spectral collocation approximation to the coupled problem (2.1)-(2.2) as follow: find $(\mathbf{u}_{-,N}, p_{-,N}) \in X_{-,N} \times M_{-,N}$ and $(\mathbf{u}_{+,N}, p_{+,N}) \in X_{+,N} \times M_{+,N}$, such that

$$\left\{ \begin{array}{rcll} \alpha \mathbf{u}_{-,N} - \nu \triangle \mathbf{u}_{-,N} + I_N \nabla p_{-,N} & = & \mathbf{f}_- & \text{at } \xi^{ij}_- \in \Xi^N_- \cap \Omega_-, \\ \nabla \cdot \mathbf{u}_{-,N} & = & 0 & \text{at } \zeta^{ij}_- \in \Lambda^N_-, \\ \mathbf{u}_{-,N} & = & 0 & \text{at } \xi^{ij}_- \in \Xi^N_- \cap \Gamma_-, \\ \alpha \mathbf{u}_{+,N} + \nabla p_{+,N} & = & \mathbf{f}_+ & \text{at } \xi^{ij}_+ \in \Xi^N_+ \cap \Omega_+, \\ \nabla \cdot \mathbf{u}_{+,N} & = & 0 & \text{at } \xi^{ij}_+ \in \Xi^N_+ \cap \Omega_+, \\ \mathbf{u}_{+,N} \cdot \mathbf{n}_+ & = & \frac{\omega^{ij}_+}{\tau^{ij}_+} \nabla \cdot \mathbf{u}_{+,N} & \text{at } \xi^{ij}_+ \in \Xi^N_+ \cap \Gamma_+, \\ \mathbf{u}_{+,N} \cdot \mathbf{n}_+ - \mathbf{u}_{-,N} \cdot \mathbf{n}_+ & = & \frac{\omega^{ij}_+}{\tau^{ij}_+} \nabla \cdot \mathbf{u}_{+,N} & \text{at } \xi^{ij}_+ \in \Xi^N_+ \cap \Gamma, \\ \nu \frac{\partial \mathbf{u}_{-,N}}{\partial \mathbf{n}_-} - p_{-,N} \mathbf{n}_- - p_{+,N} \mathbf{n}_+ & = & \frac{\omega^{ij}_+}{\tau^{ij}_+} R & \text{at } \xi^{ij}_- \in \Xi^N_- \cap \Gamma, \end{array} \right. \tag{4.6}$$

where I_N notices the interpolation operator from the $(N-1)^2$ Gauss points ζ^{ij}_- to $(N+1)^2$ Gauss-Lobatto points ξ^{ij}_-, i.e. $(I_N \phi)(\xi^{ij}_-) = \sum_{l,m=1}^{N-1} h^i_{1,-}(\zeta^l_{1,-}) h^j_{2,-}(\zeta^m_{2,-}) \phi(\zeta^{lm}_-)$. R is the residue duo to discrete integration by part, defined by

$$R = \alpha \mathbf{u}_{-,N} - \nu \triangle \mathbf{u}_{-,N} + I_N \nabla p_{-,N} - \mathbf{f}_-.$$

It is verified that the collocation equations (4.6) is equivalent to the following variational formulation:

$$\begin{array}{rl} \alpha(\mathbf{u}_N, \mathbf{v}_N)_{GL} + \nu(\nabla \mathbf{u}_{-,N}, \nabla \mathbf{v}_{-,N})_{-,GL} - (p_{-,N}, \nabla \cdot \mathbf{v}_{-,N})_{-,G} & \\ +(\nabla p_{+,N}, \mathbf{v}_{+,N})_{+,GL} - (p_{+,N} \cdot \mathbf{n}_+, \mathbf{v}_{-,N})_{\Gamma,GL} & = (\mathbf{f}, \mathbf{v}_N)_{GL}, \\ -(\nabla \cdot \mathbf{u}_{-,N}, q_{-,N})_{-,G} + (\mathbf{u}_{+,N}, \nabla q_{+,N})_{+,GL} - (\mathbf{u}_{-,N} \cdot \mathbf{n}_+, q_{+,N})_{\Gamma,GL} & = 0, \\ \forall \mathbf{v}_N \in X_{-,N} \times X_{+,N} \ , \quad \forall q_N \in M_{-,N} \times M_{+,N} \ , & \end{array} \tag{4.7}$$

therefore the well-posedness of the problem (4.6) can be proved, as in the differential case, by applying the standard saddle-point theory. We refer to [11] for the detailed proof and error estimations to the discrete solutions $(\mathbf{u}_N, p_N)$.

Remark 4.1. We chose the weak form of the interface conditions in (4.6) because this suits better the numerical analysis. The strong form is obtained just by replacing the right-hand-sides of these formulas with zero. The two forms are equivalent from the point of view of accuracy. We recall that the quotient $\frac{\omega^{ij}_+}{\tau^{ij}_+}$ is proportional to $\frac{1}{N^2}$ (see for instance [1]), hence the weak form enforces the interface conditions up to the residue of the equations times a constant tending to zero as $N \to \infty$.

4.3 The discrete iteration-by-subdomain procedure

We propose an iterative procedure to solve the coupled problem (4.6). We define two sequences $(\mathbf{u}^m_{+,N}, p^m_{+,N})_{m\geq 1}$ and $(\mathbf{u}^m_{-,N}, p^m_{-,N})_{m\geq 1}$ such that $(\mathbf{u}^m_{+,N}, p^m_{+,N}) \in X_{+,N} \times M_{+,N}$, $(\mathbf{u}^m_{-,N}, p^m_{-,N}) \in X_{-,N} \times M_{-,N}$ satisfying the discrete inviscid problem:

$$\begin{cases} \alpha(\mathbf{u}^m_{+,N}, \mathbf{v}^m_{+,N})_{+,GL} + (\nabla p^m_{+,N}, \mathbf{v}^m_{+,N})_{+,GL} & = (\mathbf{f}_+, \mathbf{v}^m_{+,N})_{+,GL}, \\ (\nabla q^m_{+,N}, \mathbf{u}^m_{+,N})_{+,GL} & = (\varphi^m_N, q^m_{+,N})_{\Gamma,GL}, \\ \qquad \forall (\mathbf{v}^m_{+,N}, q^m_{+,N}) \in X_{+,N} \times M_{+,N}. \end{cases} \tag{4.8}$$

and the discrete viscous problem

$$\begin{cases} \alpha(\mathbf{u}^m_{-,N}, \mathbf{v}^m_{-,N})_{-,GL} + \nu(\nabla \mathbf{u}^m_{-,N}, \nabla \mathbf{v}^m_{-,N})_{-,GL} - (p^m_{-,N}, \nabla\cdot \mathbf{v}^m_{-,N})_{-,G} \\ \qquad = (p^m_{+,N}\mathbf{n}_+, \mathbf{v}^m_{-,N})_{\Gamma,GL} + (\mathbf{f}_-, \mathbf{v}^m_{-,N})_{-,GL}, \\ (q^m_{-,N}, \nabla\cdot \mathbf{u}^m_{-,N})_{-,G} = 0, \\ \qquad \forall (\mathbf{v}^m_{-,N}, q^m_{-,N}) \in X_{-,N} \times M_{-,N}. \end{cases} \tag{4.9}$$

where $\varphi^m_N, m \geq 1$ is defined by

$$\varphi^m_N = \theta \mathbf{u}^{m-1}_{-,N} \cdot \mathbf{n}_+|_\Gamma + (1-\theta)(\mathbf{u}^{m-1}_{+,N} \cdot \mathbf{n}_+ - \frac{\omega^{ij}_+}{\tau^{ij}_+} \nabla \cdot \mathbf{u}^{m-1}_{+,N})|_\Gamma \ , \ \theta \in [0,1] \ . \tag{4.10}$$

From the seventh equation of (4.6), it is immediate that

$$\varphi^m_N = \theta \mathbf{u}^{m-1}_{-,N} \cdot \mathbf{n}_+|_\Gamma + (1-\theta)\varphi^{m-1}_N. \tag{4.11}$$

Remark 4.2. Here again, in order for (4.8) to be well posed, φ^1_N is required to satisfy the discrete compatibility condition:

$$\sum_{\xi^{ij}_+ \in \Gamma \cap \Xi^N_+} \varphi^1_N(\xi^{ij}_+)\tau^{ij}_+ = 0 \ .$$

Remark 4.3. The use of different degrees of polynomial between the velocity and the pressure in the viscous part (Stokes problem) is due to the well known Babuška-Brezzi's inf-sup condition. In fact, there exists many possible choices for the discrete velocity-pressure space pairs (see, e.g. [6, 1]). The one we used has been referred generally to as $\mathbb{P}_N \times \mathbb{P}_{N-2}$ method. The spectral approximation of the inviscid part is discussed in [7]. It was shown that the discrete spaces $\mathbb{P}_N$ is suitable both for the velocity function and for the pressure function.

In order to prove the convergence of the discrete iteration-by-subdomain procedure (4.8)-(4.10), we need the following stability results.

Theorem 4.1. *The discrete problem (4.8) admits one unique solution $(\mathbf{u}^m_{+,N}, p^m_{+,N})$; furthermore, $(\mathbf{u}^m_{+,N}, p^m_{+,N})$ satisfies*

$$\|\mathbf{u}^m_{+,N}\|_{+,GL} + \|\nabla p^m_{+,N}\|_{+,GL} \leq (\frac{1}{\alpha} + 2)\|\mathbf{f}_+\|_{+,GL} + 2(1+\alpha)\|\varphi^m_N\|_{\Gamma,GL} \ , \tag{4.12}$$

especially if $\mathbf{f}_+ = 0$, *then*

$$\|\mathbf{u}^m_{+,N}\|_{+,GL} \le 2\|\varphi^m_N\|_{\Gamma,GL} \ , \tag{4.13}$$

$$\|\nabla p^m_{+,N}\|_{+,GL} \le \alpha\|\mathbf{u}^m_{+,N}\|_{+,GL} \le 2\alpha\|\varphi^m_N\|_{\Gamma,GL} \ . \tag{4.14}$$

Proof. The proof of the existence and the uniqueness of problem (4.8) is analogous to the one for the differential problem (theorem 3.1). We ignore the details, but give the proof of the estimations (4.12)-(4.14).
Let $\mathbf{u}^* \in X_{+,N}$ to be the polynomial which satisfies

$$(\nabla q^m_{+,N}, \mathbf{u}^*)_{+,GL} = (\varphi^m_N, q^m_{+,N})_{\Gamma,GL} \ , \ \forall q^m_{+,N} \in M_{+,N} \ ,$$

and

$$\|\mathbf{u}^*\|_{+,GL} \le \|\varphi^m_N\|_{\Gamma,GL}. \tag{4.15}$$

(the existence of such a polynomial is guaranteed by the inf-sup condition[1, 7]). Let $\mathbf{z} = \mathbf{u}^m_{+,N} - \mathbf{u}^*$, then $\mathbf{z}$ satisfies

$$\begin{cases} \alpha(\mathbf{z}, \mathbf{v}^m_{+,N})_{+,GL} + (\nabla p^m_{+,N}, \mathbf{v}^m_{+,N})_{+,GL} & = (\mathbf{f}_+, \mathbf{v}^m_{+,N})_{+,GL} - \alpha(\mathbf{u}^*, \mathbf{v}^m_{+,N})_{+,GL}, \\ (\nabla q^m_{+,N}, \mathbf{z})_{+,GL} & = 0, \\ & \forall(\mathbf{v}^m_{+,N}, q^m_{+,N}) \in X_{+,N} \times M_{+,N}. \end{cases} \tag{4.16}$$

From (4.16), we get

$$\alpha(\mathbf{z}, \mathbf{z})_{+,GL} = (\mathbf{f}_+, \mathbf{z})_{+,GL} - \alpha(\mathbf{u}^*, \mathbf{z})_{+,GL},$$

which gives

$$\alpha\|\mathbf{z}\|_{+,GL} \le \|\mathbf{f}_+\|_{+,GL} + \alpha\|\mathbf{u}^*\|_{+,GL}, \tag{4.17}$$

we derive from (4.15) and (4.17),

$$\|\mathbf{u}^m_{+,N}\|_{+,GL} \le \|\mathbf{z}\|_{+,GL} + \|\mathbf{u}^*\|_{+,GL} \le \frac{1}{\alpha}\|\mathbf{f}_+\|_{+,GL} + 2\|\varphi^m_N\|_{\Gamma,GL}. \tag{4.18}$$

Taking $\mathbf{v}^m_{+,N} = \nabla p^m_{+,N}$ in the first equation of (4.8), we have

$$\alpha(\mathbf{u}^m_{+,N}, \nabla p^m_{+,N})_{+,GL} + \|\nabla p^m_{+,N}\|^2_{+,GL} = (\mathbf{f}_+, p^m_{+,N})_{+,GL}, \tag{4.19}$$

Finally, (4.12)-(4.14) follow from (4.18) and (4.19). □

Theorem 4.2.[1] *The discrete problem* (4.9) *admits one unique solution* $(\mathbf{u}^m_{-,N}, p^m_{-,N})$; *furthermore,* $(\mathbf{u}^m_{-,N}, p^m_{-,N})$ *satisfies*

$$\|\mathbf{u}^m_{-,N}\|_{-,GL} + \|\nabla \mathbf{u}^m_{-,N}\|_{-,GL} \le c_0(\|\mathbf{f}_-\|_{-,GL} + \|p^m_{+,N}\|_{\Gamma,GL}),$$

$$\|p^m_{-,N}\|_{-,G} \le \beta_N(\|\mathbf{f}_-\|_{-,GL} + \|p^m_{+,N}\|_{\Gamma,GL}),$$

especially if $\mathbf{f}_- = 0$, *then*

$$\|\mathbf{u}^m_{-,N}\|_{-,GL} + \|\nabla \mathbf{u}^m_{-,N}\|_{-,GL} \le c_0\|p^m_{+,N}\|_{\Gamma,GL}, \tag{4.20}$$

$$\|p^m_{-,N}\|_{-,G} \le \beta_N\|p^m_{+,N}\|_{\Gamma,GL}.$$

where c_0 *is a constant dependent on* α *and* ν, *but independent on* N. β_N *behaves as* $N^{1/2}$.

4.4 Convergence of the iteration-by-subdomain procedure

We prove now the convergence of the discrete iteration-by-subdomain procedure (4.8)-(4.10). We begin by defining a discrete interface operator $L_N : I\!P_N(\Gamma) \longrightarrow I\!P_N(\Gamma)$,

$$L_N\lambda = \mathbf{u}^{(\lambda)}_{-,N} \cdot \mathbf{n}_+|_\Gamma \ , \quad \forall\lambda \in I\!P_N(\Gamma),$$

and then the operator $L_{N,\theta} : I\!P_N(\Gamma) \longrightarrow I\!P_N(\Gamma)$,

$$L_{N,\theta}\lambda = \theta L\lambda + (1-\theta)\lambda \ , \quad \forall\lambda \in I\!P_N(\Gamma), \tag{4.21}$$

where $\mathbf{u}^{(\lambda)}_{-,N}$ solves the discrete problem: $(\mathbf{u}^{(\lambda)}_{-,N}, p^{(\lambda)}_{-,N}) \in X_{-,N} \times M_{-,N}$, such that

$$\begin{cases} \alpha(\mathbf{u}^{(\lambda)}_{-,N}, \mathbf{v}_-)_{-,GL} + \nu(\nabla\mathbf{u}^{(\lambda)}_{-,N}, \nabla\mathbf{v}_-)_{-,GL} - (p^{(\lambda)}_{-,N}, \nabla\cdot\mathbf{v}_-)_{-,G} = (p^{(\lambda)}_{+,N}\mathbf{n}_+, \mathbf{v}_-)_{\Gamma,GL}, \\ (q_-, \nabla\cdot\mathbf{u}^{(\lambda)}_{-,N})_{-,G} = 0, \qquad\qquad \forall(\mathbf{v}_-, q_-) \in X_{-,N} \times M_{-,N} \ , \end{cases} \tag{4.22}$$

where $p^{(\lambda)}_{+,N}$ is the solution of the following discrete problem: $(\mathbf{u}^{(\lambda)}_{+,N}, p^{(\lambda)}_{+,N}) \in X_{+,N} \times M_{+,N}$, such that

$$\begin{cases} \alpha(\mathbf{u}^{(\lambda)}_{+,N}, \mathbf{v}_+)_{+,GL} + (\nabla p^{(\lambda)}_{+,N}, \mathbf{v}_+)_{+,GL} = 0, \\ (\nabla q_+, \mathbf{u}^{(\lambda)}_{+,N})_{+,GL} = (\lambda, q_+)_{\Gamma,GL}, \qquad\qquad \forall(\mathbf{v}_+, q_+) \in X_{+,N} \times M_{+,N} \ . \end{cases} \tag{4.23}$$

Let $A_{+,N}$ denote the bilinear form: $\forall(\mathbf{u}_+, p_+), \forall(\mathbf{v}_+, q_+) \in X_{+,N} \times M_{+,N}$,

$$A_{+,N}[(\mathbf{u}_+, p_+), (\mathbf{v}_+, q_+)] = \alpha(\mathbf{u}_+, \mathbf{v}_+)_{+,GL} + (\nabla p_+, \mathbf{v}_+)_{+,GL} - (\nabla q_+, \mathbf{u}_+)_{+,GL},$$

then problem (4.23) is equivalent to: find $(\mathbf{u}^{(\lambda)}_{+,N}, p^{(\lambda)}_{+,N}) \in X_{+,N} \times M_{+,N}$, such that

$$A_{+,N}[(\mathbf{u}^m_{+,N}, p^m_{+,N}), (\mathbf{v}_+, q_+)] = -(\lambda, q_+)_{\Gamma,GL}, \qquad \forall(\mathbf{v}_+, q_+) \in X_{+,N} \times M_{+,N} \ . \tag{4.24}$$

We define now the discrete "lifting" operator $F_N : \forall\lambda \in I\!P_N(\Gamma)$,

$$F_N\lambda = (\mathbf{u}^{(\lambda)}_{+,N}, p^{(\lambda)}_{+,N}),$$

solution of the problem (4.24). We define furthermore the scalar product, and the associated norm:

$$((\lambda,\mu))_N = A_{+,N}[F_N\lambda, F_N\mu] \ , \quad \|\lambda\|^2_{*,N} = ((\lambda,\lambda))_N \ . \tag{4.25}$$

Theorem 4.3. *There exists $\theta_0 \in (0,1]$, such that for all $\theta \in (0,\theta_0)$, it exists $k(\theta) < 1$ such that*

$$\|L_{\theta,N}\lambda\|_{*,N} \le k(\theta)\|\lambda\|_{*,N}, \quad \forall\lambda \in I\!P_N(\Gamma) \ . \tag{4.26}$$

Proof. It follows from the definition of L_N and F_N that

$$
\begin{aligned}
((L_N\lambda,\lambda))_N &= A_{+,N}[F_N(L_N\lambda),F_N\lambda] = A_{+,N}[F_N(\mathbf{u}^{(\lambda)}_{-,N}\cdot\mathbf{n}_+|_\Gamma),F_N\lambda] \\
&= -(\mathbf{u}^{(\lambda)}_{-,N}\cdot\mathbf{n}_+,p^{(\lambda)}_{+,N})_{\Gamma,GL} = -(p^{(\lambda)}_{+,N}\mathbf{n}_+,\mathbf{u}^{(\lambda)}_{-,N})_{\Gamma,GL} \\
&= -\alpha(\mathbf{u}^{(\lambda)}_{-,N},\mathbf{u}^{(\lambda)}_{-,N})_{-,GL} - \nu(\nabla\mathbf{u}^{(\lambda)}_{-,N},\nabla\mathbf{u}^{(\lambda)}_{-,N})_{-,GL} \\
&\le -c_1(\|\mathbf{u}^{(\lambda)}_{-,N}\|^2_{-,GL} + \|\nabla\mathbf{u}^{(\lambda)}_{-,N}\|^2_{-,GL})\ ,
\end{aligned}
\tag{4.27}
$$

where c_1 depends on α,ν.
By the definition (4.25) of $\|\cdot\|_{*,N}$, the estimation (4.13), and the trace's inequalities, it can be verified that

$$
\begin{aligned}
\|L_N\lambda\|^2_{*,N} = A_{+,N}[F_N(L_N\lambda),F_N(L_N\lambda)] &\le 2\|\mathbf{u}^{(\lambda)}_{-,N}\cdot\mathbf{n}_-\|^2_{\Gamma,GL} \\
&\le c_2(\|\mathbf{u}^{(\lambda)}_{-,N}\|^2_{-,GL} + \|\nabla\mathbf{u}^{(\lambda)}_{-,N}\|^2_{-,GL})\ .
\end{aligned}
\tag{4.28}
$$

where c_2 depends on the trace's mapping constant.
From the definition (4.21) of $L_{\theta,N}$, and using (4.27) and (4.28), we get

$$
\begin{aligned}
\|L_{\theta,N}\lambda\|^2_{*,N} &= \theta^2\|L_N\lambda\|^2_{*,N} + 2\theta(1-\theta)((L_N\lambda,\lambda)) + (1-\theta)^2\|\lambda\|^2_{*,N} \\
&\le [c_2\theta^2 - 2c_1\theta(1-\theta)](\|\mathbf{u}^{(\lambda)}_{-,N}\|^2_{-,GL} + \|\nabla\mathbf{u}^{(\lambda)}_{-,N}\|^2_{-,GL}) + (1-\theta)^2\|\lambda\|^2_{*,N}.
\end{aligned}
\tag{4.29}
$$

But (4.20) and (4.14) imply

$$
\begin{aligned}
&\|\mathbf{u}^{(\lambda)}_{-,N}\|^2_{-,GL} + \|\nabla\mathbf{u}^{(\lambda)}_{-,N}\|^2_{-,GL} \\
&\le c_0^2\|p^{(\lambda)}_{+,N}\|^2_{\Gamma,GL} \le c\|\nabla p^{(\lambda)}_{+,N}\|^2_{+,GL}\ (\text{ using } \textstyle\int_{\Omega_+} p^{(\lambda)}_{+,N}d\mathbf{x} = 0\ ) \\
&\le c\alpha^2\|\mathbf{u}^{(\lambda)}_{+,N}\|^2_{+,GL} = c\alpha\|\lambda\|^2_{*,N}\ .
\end{aligned}
\tag{4.30}
$$

where c depends on c_0^2 and the trace's mapping constant.
Combining (4.29) and (4.30), we obtain

$$
\|L_{\theta,N}\lambda\|^2_{*,N} \le [\ c_2c\alpha\theta^2 - 2c_1c\alpha\theta(1-\theta) + (1-\theta)^2\ ]\ \|\lambda\|^2_{*,N}, \tag{4.31}
$$

By taking

$$
\theta_0 = \min(1, \frac{2(1+c_1c\alpha)}{1+2c_1c\alpha+c_2c\alpha}),
$$

it can be verified that for all $0<\theta<\theta_0$, holds

$$
k(\theta) \overset{\text{def}}{=} \sqrt{c_2c\alpha\theta^2 - 2c_1c\alpha\theta(1-\theta) + (1-\theta)^2} < 1.
$$

and

$$
\|L_{\theta,N}\lambda\|_{*,N} \le k(\theta)\|\lambda\|_{*,N}, \quad \forall\lambda\in I\!P_N(\Gamma)\ .
$$

□

Remark 4.4. *The optimal value of θ is* $\theta^* = \dfrac{1+c_1c\alpha}{1+2c_1c\alpha+c_2c\alpha}$, *which gives a contraction constant* $k(\theta^*) = \sqrt{\dfrac{c_2c\alpha-(c_1c\alpha)^2}{1+2c_1c\alpha+c_2c\alpha}}$.

We can now state the convergence result for the discrete iteration-by- subdomain procedure (4.8)-(4.10).

Corollary 4.1. *Let* $(\mathbf{u}_{+,N}, p_{+,N}), (\mathbf{u}_{-,N}, p_{-,N})$ *to be the solution of the discrete coupled equations* (4.6)*; Let* $(\mathbf{u}^m_{+,N}, p^m_{+,N}), (\mathbf{u}^m_{-,N}, p^m_{-,N})$ *to be the solution of the discrete iteration problems* (4.8) *and* (4.9)*. Then for all* $\theta \in (0, \theta_0)$, $(\mathbf{u}^m_{+,N}, p^m_{+,N})$ *converges to* $(\mathbf{u}_{+,N}, p_{+,N})$ *in* $X_+ \times M_+$ *and* $(\mathbf{u}^m_{-,N}, p^m_{-,N})$ *converges to* $(\mathbf{u}_{-,N}, p_{-,N})$ *in* $X_- \times M_-$ *as* $m \to \infty$.

Proof. The corollary is an analogy of the corollary 3.1. We begin the proof by verifying

$$\varphi^m_N - \mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma = L_{\theta,N}(\varphi^{m-1}_N - \mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma). \tag{4.32}$$

In fact, by the definitions of $L_{\theta,N}$ and φ^m_N, we have

$$\begin{aligned}
& L_{\theta,N}(\varphi^{m-1}_N - \mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma) \\
= \;& \theta L_N(\varphi^{m-1}_N - \mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma) + (1-\theta)\varphi^{m-1}_N - (1-\theta)(\mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma) \\
= \;& \theta(\mathbf{u}^{m-1}_{-,N} \cdot \mathbf{n}_+|_\Gamma) - \theta(\mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma) + (1-\theta)\varphi^{m-1}_N - \mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma + \theta(\mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma) \\
= \;& \theta(\mathbf{u}^{m-1}_{-,N} \cdot \mathbf{n}_+|_\Gamma) + (1-\theta)\varphi^{m-1}_N - \mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma \\
= \;& \varphi^m_N - \mathbf{u}_{-,N} \cdot \mathbf{n}_+|_\Gamma \quad (\text{ by } \ (4.11)).
\end{aligned}$$

The contraction of $L_{\theta,N}$ implies

$$\varphi^m_N \to \mathbf{u}_{-,N} \cdot \mathbf{n}_+ \ , \text{ as } \ m \to \infty \ . \tag{4.33}$$

Combining (4.7) and (4.8) gives

$$\begin{gathered}
A_{+,N}[(\mathbf{u}^m_{+,N} - \mathbf{u}_{+,N}, p^m_{+,N} - p_{+,N}), (\mathbf{v}_+, q_+)] = (\mathbf{u}_{-,N} \cdot \mathbf{n}_+ - \varphi^m_N, q_+)_\Gamma, \\
\forall (\mathbf{v}_+, q_+) \in X_{+,N} \times M_{+,N}.
\end{gathered} \tag{4.34}$$

by the estimations (4.13) and (4.14), we get

$$\|\mathbf{u}^m_{+,N} - \mathbf{u}_{+,N}\|_{+,GL} + \|\nabla p^m_{+,N} - \nabla p_{+,N}\|_{+,GL} \le 2(1+\alpha)\|\varphi^m_N - \mathbf{u}_{-,N} \cdot \mathbf{n}_+\|_{\Gamma,GL} \tag{4.35}$$

then (4.33) and (4.5) imply

$$\mathbf{u}^m_{+,N} \to \mathbf{u}_{+,N} \ \text{ in } \ X_+ \ , \quad p^m_{+,N} \to p_{+,N} \ \text{ in } \ M_+ \ , \ \text{ as } \ m \to \infty \ . \tag{4.36}$$

and hence

$$p^m_{+,N} \to p_{+,N} \ \text{ in } \ L^2(\Gamma) \ , \ \text{ as } \ m \to \infty. \tag{4.37}$$

Another part, combining (4.7) and (4.9) gives

$$\begin{cases} \alpha(\mathbf{u}^m_{-,N} - \mathbf{u}_{-,N}, \mathbf{v}_-)_{-,GL} + \nu(\nabla(\mathbf{u}^m_{-,N} - \mathbf{u}_{-,N}), \nabla\mathbf{v}_-)_{-,GL} \\ -(p^m_{-,N} - p_{-,N}, \nabla\cdot\mathbf{v}_-)_{-,G} = ((p^m_{+,N} - p_{+,N})\mathbf{n}_+, \mathbf{v}_-)_{\Gamma,GL}, \\ (q_-, \nabla\cdot(\mathbf{u}^m_{-,N} - \mathbf{u}_{-,N}))_{-,G} = 0, \qquad \forall(\mathbf{v}_-, q_-) \in X_{-,N} \times M_{-,N}. \end{cases}$$

Applying the estimations (4.13) and (4.14), we get

$$\begin{aligned} & \|\mathbf{u}^m_{-,N} - \mathbf{u}_{-,N}\|_{-,GL} + \|\nabla\mathbf{u}^m_{-,N} - \nabla\mathbf{u}_{-,N}\|_{-,GL} + \|p^m_{-,N} - p_{-,N}\|_{-,G} \\ \leq\ & c\beta_N\|p^m_{+,N} - p_{+,N}\|_{0,\Gamma}, \end{aligned}$$

we derive from (4.37) that

$$\mathbf{u}^m_{-,N} \to \mathbf{u}_{-,N} \ \text{ in } \ X_- \ , \qquad p^m_{-,N} \to p_{-,N} \ \text{ in } \ M_- \ , \ \text{ as } \ m \to \infty \ .$$

$\square$

5. Generalization to the coupled Navier-Stokes/Euler equations

We generalize the coupled model (2.1) to the coupled problem between the Navier-Stokes equations and the Euler equations:

$$\begin{cases} \dfrac{\partial\mathbf{u}_-}{\partial t} + (\mathbf{u}_-\cdot\nabla)\mathbf{u}_- - \nu\triangle\mathbf{u}_- + \nabla p_- = \mathbf{f}_- \quad \text{in } Q_-, \\ \dfrac{\partial\mathbf{u}_+}{\partial t} + (\mathbf{u}_+\cdot\nabla)\mathbf{u}_+ + \nabla p_+ = \mathbf{f}_+ \quad \text{in } Q_+, \\ \mathbf{u}_-(0) = \mathbf{u}^0_- \ \text{ in } \Omega_-, \qquad \mathbf{u}_+(0) = \mathbf{u}^0_+ \ \text{ in } \Omega_+, \\ \mathbf{u}_-|_{\Sigma_-} = 0 \ , \qquad \mathbf{u}_+\cdot\mathbf{n}_+|_{\Sigma_+} = 0 \end{cases} \tag{5.1}$$

with the incompressibility $\nabla\cdot\mathbf{u} = 0$, where $Q_k = \Omega_k \times (0,T), \Sigma_k = \Gamma_k \times (0,T), k = -, +$, and $\mathbf{u}^0_-, \mathbf{u}^0_+$ are two functions given. The non-linear term is treated by the method of characteristics. That is, we rewrite (5.1) under the form

$$\begin{cases} \dfrac{D\mathbf{u}_-}{Dt} - \nu\triangle\mathbf{u}_- + \nabla p_- = \mathbf{f}_- \qquad \text{in } Q_-, \\ \dfrac{D\mathbf{u}_+}{Dt} + \nabla p_+ = \mathbf{f}_+ \qquad \text{in } Q_+, \\ \mathbf{u}_-(0) = \mathbf{u}^0_- \ \text{ in } \Omega_-, \qquad \mathbf{u}_+(0) = \mathbf{u}^0_+ \ \text{ in } \Omega_+, \\ \mathbf{u}_-|_{\Sigma_-} = 0 \ , \quad \mathbf{u}_+\cdot\mathbf{n}_+|_{\Sigma_+} = 0 \ , \end{cases} \tag{5.2}$$

where D/Dt is the total derivative in the direction $\mathbf{u}$. We discretize (5.2) in time by an implicit scheme:

$$\begin{cases} \alpha\mathbf{u}^{n+1}_- - \nu\triangle\mathbf{u}^{n+1}_- + \nabla p^{n+1}_- = \mathbf{f}^{n+1}_- + \alpha\mathbf{u}^n_-(\chi^n(\cdot)) \qquad \text{in } \Omega_-, \\ \alpha\mathbf{u}^{n+1}_+ + \nabla p^{n+1}_+ = \mathbf{f}^{n+1}_+ + \alpha\mathbf{u}^n_+(\chi^n(\cdot)) \qquad \text{in } \Omega_+, \\ \mathbf{u}^{n+1}_-|_{\Gamma_-} = 0 \ , \qquad \mathbf{u}^{n+1}_+\cdot\mathbf{n}_+|_{\Gamma_+} = 0 \ , \end{cases}$$

where $\alpha = \dfrac{1}{\triangle t}$ with $\triangle t$ the time step, and $\chi^n(\mathbf{x}) = \chi(\mathbf{x}, (n+1)\triangle t, n\triangle t)$ is the solution of

$$\frac{d\chi}{d\tau} = \mathbf{u}^n(\chi) \ , \quad \chi(\mathbf{x}, (n+1)t; (n+1)t) = \mathbf{x} \ . \tag{5.3}$$

The time scheme is unconditionally stable, and each time iteration requires a coupled viscous/inviscid resolution plus a transport of the previous solution on the characteristics.

We note that, on the interface Γ, we have $\mathbf{u}_- \cdot \mathbf{n}_- = \mathbf{u}_+ \cdot \mathbf{n}_-$. Thus (5.3) can be solved globally in all domain Ω without any additional interface conditions on Γ.

6. Concluding remarks

1. We have presented an efficient iteration-by-subdomain algorithm to solve numerically the viscous/inviscid coupled equations. We have given the detailed proof of the convergence results. The key to the success is the definitions of the interface iteration function φ_N^m in (4.10) and the scalar product $((\cdot,\cdot))_N$ given in (4.25). It is crucial to get from (4.27) that $((L_N\lambda, \lambda))_N$ is non-positive. We have also presented the idea to generalize the present coupled model and numerical algorithm to the full Navier-Stokes/Euler coupling.

2. It is seen, from the proof of theorem 4.3, that the contraction constant $k(\theta)$ is independent on the choices of the pressure discrete space $M_{-,N}$ in the viscous part (see remark 4.3). This means that the convergence rate of the iteration-by-subdomain procedure is independent on the choices of $M_{-,N}$. Hence the choice of $M_{-,N}$ can be made by its proper considerations.

3. We have obtained (see remark 4.4) the optimal value θ^* of the relaxation parameter. However the exact estimations of the constants c, c_1 and c_2 in (4.31) are not trivial. In a future work, we plan to investigate numerically the dependence of convergence rate on θ.

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