Journal of Computational Mathematics, Vol.17, No.4, 1999, 353-368.

SEMI-DISCRETE AND FULLY DISCRETE PARTIAL PROJECTION FINITE ELEMENT METHODS FOR THE VIBRATING TIMOSHENKO BEAM*1)

Min-fu Feng Xiao-ping Xie Hua-xing Xiong
(*Department of Mathematics, Sichuan University, Chengdu 610064, China.*)

Abstract

In this paper, the partial projection finite element method is applied to the time-dependent problem–the damped vibrating Timoshenko beam model. It is proved that this method allows some new combinations of interpolations for stress and displacement fields. When assuming that a smooth solution exists, we obtain optimal convergence rates with constants independent of the beam thickness.

Key words: Timoshenko beam, Finite element.

1. Introduction

The Timoshenko beam model is given by

$$\begin{cases} -\theta_{xx} + d^{-2}(\theta - \omega_x) = 0 & on \ \ I, \\ d^{-2}(\theta - \omega_x)_x = g(x) & on \ \ I, \\ \theta(0) = \theta(1) = \omega(0) = \omega(1) = 0 \end{cases}$$

where the beam is considered damped, d represents the beam thickness and $I = [0, 1]$. $\theta(x)$ is the rotation of vertical fibers in the beam and $\omega(x)$ is the vertical displacement of the beam's centerline(under a vertical load given by $g(x)$).

Analogous to the situation one would meet in studying the Reissner–Mindlin plate model, the standard finite element methods fail to give good approximation when the beam thickness is too small, owing to a "locking" phenomenon. Instead, mixed methods, based on the introduction of the shear term as a new variable, have proved to be successful([1],[8],etc.). D.N. Arnold[1] studied the discretization with emphasis on the effect of the beam thickness and used a mixed finite element method–reduced integration approach. He obtained optimal-order error estimates with constants independent of the beam thickness.

On the baisis of [11], B.Semper considered the following time-dependent vibrating beam equations

$$\begin{cases} \theta_{tt} + \delta\theta_t - \theta_{xx} + d^{-2}(\theta - \omega_x) = 0 & on \ \ I \times (0, T], \\ \omega_{tt} + \delta\omega_t + d^{-2}(\theta - \omega_x)_x = g(x, t) & on \ \ I \times (0, T], \\ \theta(0) = \theta(1) = \omega(0) = \omega(1) = 0, & \forall t \in [0, T], \\ \theta(x, 0) = \theta_0(x), \theta_t(x, 0) = \theta_1(x), & \forall x \in I, \\ \omega(x, 0) = \omega_0(x), \omega_t(x, 0) = \omega_1(x), & \forall x \in I. \end{cases} \tag{1.1}$$

* Received December 4, 1995.
1) This work was sponsored by State Major Key Project for Basic Research.

Where δ represents a damping constant.

B.Semper discussed some semi-discrete and fully discrete approximations for this model.Following Arnold's idea, he also obtained optimal-order error estimates with constants independent of the beam thickness under the assumption of the regularity of the solution of (1.1), which we will derive in this paper(see Theorem 3.1).

As we have known, in studying of the Reissner–Mindlin plate model, Prof. Zhou Tianxiao [15] proposed a new mixed method:PPM–Partial projection method of finite element discretizations, which has attracted more and more researchers' interest[2,7].In comparison with Galerkin formulations, this method enhanced stability and is promising for the plate and shell problems. In this paper, we extend the idea of PPM to the time-dependent problem. Semi-discrete and fully discrete schemes are proposed for the vibrating beam model (1.1). As desired, this method allows some new combinations of interpolations for stress and displacement fields, and, when assuming a smooth solution, we obtain optimal-order error estimates with constants independent of the beam thickness .

We now give the arrangements of this paper.In section 2 some notations are collected and variational formulations are presented.In section 3 a prioi estimates are derived 3.In the following section new variational formulations are given. In the last two sections semi-discrete approximations and fully discrete approximations are considered and their convergence are analysed .

Throughout this paper we denote by C a constant independent of h and d, which may be different at its each occurrence.

2. Notations and the Original Variational Formulations

At first we introduce some notations. We will use the standard notations for the Sobolev spaces H^r and H_0^r with norm $||\cdot||_r$, with $H^0 = L^2$.The L^2-inner product is denoted by

$$(f,g) = \int_0^1 f(x)g(x)dx.$$

Furthermore we denote the dual space of H^{-r} by H^r. For any vectors $\Psi =<\psi_1, \psi_2>, \Phi =<\phi_1,\phi_2>\in [H^r]^2$, we interpret

$$(\Psi,\Phi) = (\psi_1,\phi_1) + (\psi_2,\phi_2),$$

$$||\Psi||_r^2 = (\Psi,\Psi)_r \equiv (\psi_1,\psi_1)_r + (\psi_2,\psi_2)_r = ||\psi_1||_r^2 + ||\psi_2||_r^2,$$

(here $(\cdot,\cdot)_r$ represents the $[H^r]^2-$inner product).We also define the following bilinear forms (for abbreviation, we denote $H_0^1(I) = H_0^1, L^2(I) = L^2$ in what follows):
For $<\Psi,\Phi>\in [H_0^1]^2 \times [H_0^1]^2, a(\Psi,\Phi) = ((\psi_1)_x,(\phi_1)_x)$,
For $<\Psi,\eta>\in [H_0^1]^2 \times L^2, b(\eta,\Psi) = (\eta,\psi_1 - (\psi_2)_x)$,
For $<\Psi,\Phi>\in [H_0^1]^2 \times [H_0^1]^2, c(\Psi,\Phi) = (\psi_1 - (\psi_2)_x,\phi_1 - (\phi_2)_x)$,

Given any Banach space V with norm $||\cdot||_V$, for any $v : [0,T] \to V$ which is Lebesgue integrable, we define the norms

$$||v||_{L^p(0,T;V)} = (\int_0^T ||v(\cdot,t)||_V^p dt)^{1/p},\ p = 1,2$$

$$||v||_{L^\infty(0,T;V)} = \sup_{0\le t\le T} ||v(\cdot,t)||_V$$

We also use the notation

$$||v||_r(t) = ||v(\cdot,t)||_r.$$

We define the following Banach spaces:

$$L^p(0,T;V) = \{v : [0,T] \to V, ||v||_{L^p(0,T;V)} < \infty\},\ p = 1,2,\infty$$

The standard variational formulations of (1.1) can now be written as:
Given $g(x,t) \in L^2(0,T;H^{-1})$, find $\Phi(x,t) =< \theta(x,t), \omega(x,t) >\in L^2(0,T;[H_0^1]^2)$ such that $\Phi_t, \Phi_{tt} \in L^2(0,T;[H^{-1}]^2)$ and

$$\begin{cases} (\Phi_{tt},\Psi) + \delta(\Phi_t,\Psi) + a(\Phi,\Psi) + d^{-2}c(\Phi,\Psi) = G(\Psi), & \forall\Psi \in [H_0^1]^2 \\ (\Phi(\cdot,0),\Psi) = (\Phi_0,\Psi), & \forall\Psi \in [H_0^1]^2 \\ (\Phi_t(\cdot,0),\Psi) = (\bar{\Phi}_0,\Psi), & \forall\Psi \in [H_0^1]^2 \\ \Phi_0 =< \theta_0,\omega_0 >, \bar{\Phi}_0 =< \theta_1,\omega_1 >, & \end{cases} \tag{2.1}$$

where $G(\Psi) \equiv (g, \psi_2)$.

If we introduce the auxiliary shear variable

$$\lambda(x,t) := d^{-2}(\phi_1 - (\phi_2)_x)$$

which is related to the shear stress,we get the mixed variational formulations of (1.1):
Given $g(x,t) \in L^2(0,T;H^{-1})$, find $< \Phi, \lambda >\in L^2(0,T;[H_0^1]^2 \times L^2)$ such that $\Phi_t, \Phi_{tt} \in L^2(0,T;[H^{-1}]^2)$ and

$$\begin{cases} (\Phi_{tt},\Psi) + \delta(\Phi_t,\Psi) + a(\Phi,\Psi) + b(\lambda,\Psi) = G(\Psi), & \forall\Psi \in [H_0^1]^2 \\ b(\eta,\Phi) - d^2(\lambda,\eta) = 0, & \forall\eta \in L^2 \\ (\Phi(\cdot,0),\Psi) = (\Phi_0,\Psi), & \forall\Psi \in [H_0^1]^2 \\ (\Phi_t(\cdot,0),\Psi) = (\bar{\Phi}_0,\Psi), & \forall\Psi \in [H_0^1]^2 \end{cases} \tag{2.2}$$

3. Regularity Result

To assert that the problem (2.1)(also (2.2)) is well posed, we will prove the following regularity result:

Theorem 3.1. *Let $\Phi =< \phi_1(x,t), \phi_2(x,t) >$ be the solution of the problem* (2.1)*,Φ_t, $\Phi_{tt} \in L^\infty(0,T;L^2)$.Then for any $t \in (0,T]$,there exists a constant C independent of h and d such that:*

$$||\phi_1||_2 + ||\phi_2||_2 + d^{-2}||\phi_1 - (\phi_2)_x||_1 \le C \tag{3.1}$$

To prove this theorem, we first cite a lemma proved by D.N.Arnold [1]:

Lemma 3.1. *Let $F, G \in H^{-1}, 0 < d \le 1$. Then there exists a unique pair of functions $< \phi_1, \phi_2 >\in [H_0^1]^2$ such that*

$$((\phi_1)_x, (\psi_1)_x) + d^{-2}(\phi_1 - (\phi_2)_x, \psi_1 - (\psi_2)_x) =< F, \psi_1 > + < G, \psi_2 >,$$

*for any $< \psi_1, \psi_2 >\in V^2 \subseteq [H_0^1]^2$.*Moreover,for $p = 0,1,...,$ there exists a constant C dependent on p such that

$$||\phi_1||_{p+1} + ||\phi_2||_{p+1} + d^{-2}||\phi_1 - (\phi_2)_x||_p \le C(||F||_{p-1} + ||G||_{p-1}). \tag{3.2}$$

Now we take two steps to prove Theorem 3.1:
Step 1:to bound $||\Phi_t||_{L^\infty(0,T;L^2)}$ and $||\Phi_{tt}||_{L^\infty(0,T;L^2)}$.

Let $\Psi = \Phi_t$ in (2.1),then we have

$$(\Phi_{tt}, \Phi_t) + \delta(\Phi_t, \Phi_t) + a(\Phi, \Phi_t) + d^{-2}c(\Phi, \Phi_t) = G(\Psi_t)$$

or equivalently

$$\frac{d}{dt}||\Phi_t||_0^2 + 2\delta||\Phi_t||_0^2 + \frac{d}{dt}||(\phi_1)_x||_0^2 + d^{-2}\frac{d}{dt}||\phi_1 - (\phi_2)_x||_0^2 = 2 < g, (\phi_2)_t > \tag{3.3}$$

Integrate (3.3) from 0 to t, we have

$$\begin{array}{l} ||\Phi_t||_0^2(t) + 2\delta \int_0^t ||\Phi_t||_0^2 + ||(\phi_1)_x||_0^2(t) \\ +d^{-2}||\phi_1 - (\phi_2)_x||_0^2(t) \\ = ||\Phi_t||_0^2(0+) + ||(\phi_1)_x||_0^2(0+) + 2\int_0^t < g, (\phi_2)_t > \\ \quad +d^{-2}||\phi_1 - (\phi_2)_x||_0^2(0+) \end{array} \tag{3.4}$$

Now we estimate the last term of (3.4).The B–B condition(Lemma 5.3)and (2.1) imply that

$$\begin{array}{l} d^{-2}||\phi_1 - (\phi_2)_x||_0^2(t) = ||\lambda||_0(t) \\ \qquad \le \frac{1}{\beta_0}\sup_\Psi \frac{b(\lambda,\Psi)}{||\Psi||_1} = \frac{1}{\beta_0}\sup_\Psi \frac{1}{||\Psi||_1}[-(\Phi_{tt}, \Psi) - \delta(\Phi_t, \Psi) - a(\Phi, \Psi) + G(\Psi)] \\ \qquad \le \frac{1}{\beta_0}[||\Phi_{tt}||_0(t) + \delta||\Phi_t||_0(t) + ||(\phi_1)_x||_0(t) + ||g||_{L^\infty(0,T;H^{-1})}] \end{array}$$

Then we have

$$d^{-2}||\phi_1 - (\phi_2)_x||_0^2(0+) \le \frac{1}{\beta_0}[||\Phi_{tt}||_0(0+) + \delta||\Phi_t||_0(0+) + ||(\phi_1)_x||_0(0+) + ||g||_{L^\infty(0,T;H^{-1})}]$$

Note that $\delta > 0, d > 0$, by (3.4) we have(assuming $d \in (0,1)$)

$$||\Phi_t||_0^2(t) \le C + 2T||g||_{L^\infty(0,T;H^{-1})}||\Phi_t||_{L^\infty(0,T;L^2)}$$

where C is dependent on $\delta, ||\Phi_t||_0(0+), ||(\phi_1)_x||_0(0+), ||g||_{L^\infty(0,T;H^{-1})}, ||\Phi_{tt}||_0(0+)$.

so we get

$$||\Phi_t||_{L^\infty(0,T;L^2)} \le C, \tag{3.5}$$

where C is independent of h and d.

Now differentiate the first equation of (2.1) with respect to t and let $\Psi = \Phi_{tt}$, Similarly we have

$$||\Phi_{tt}||_{L^\infty(0,T;L^2)} \le C, \tag{3.6}$$

where C is also independent of h and d.

Step 2: from (2.1) we have

$$\begin{array}{l} ((\phi_1)_x, (\psi_1)_x) + d^{-2}(\phi_1 - (\phi_2)_x, \psi_1 - (\psi_2)_x) \\ = -((\phi_1)_{tt} + \delta(\phi_1)_t, \psi_1) - ((\phi_2)_{tt} + \delta(\phi_2)_t, \psi_2) + < g, \psi_2 > \end{array}$$

Denote

$$-((\phi_1)_{tt} + \delta(\phi_1)_t, \psi_1) =< F, \psi_1 >$$

and

$$-((\phi_2)_{tt} + \delta(\phi_2)_t, \psi_2)+ < g, \psi_2 >=< G, \psi_2 >$$

then by virtue of triangle inequality and Lemma 3.1 we get(taking $p = 1$)

$$\begin{aligned} & ||\phi_1||_2(t) + ||\phi_2||_2(t) + d^{-2}||\phi_1 - (\phi_2)_x||_1(t) \\ & \leq C(||F||_0 + ||G||_0) \\ & \leq C(||\Phi_{tt}||_0 + \delta||\Phi_t||_0 + ||g||_0) \\ & \leq C(||\Phi_t||_{L^\infty(0,T;L^2)} + \delta||\Phi_{tt}||_{L^\infty(0,T;L^2)} + ||g||_{L^\infty(0,T;L^2)}) \\ & \leq C(\delta, ||\bar{\Phi}_0||_1, ||\Phi_0||_1, ||g||_{L^\infty(0,T;L^2)}, ||g_t||_{L^\infty(0,T;H^{-1})}, ||g||_{L^\infty(0,T;H^{-1})}) \end{aligned}$$

This completes the proof.□

4. New Variational Formulations of the Problem

To begin with, we introduce a weighted factor α which is generally assumed to be in (0,1].The new variational formulations equivalent to (2.2) are presented as follows: Find $< \Phi, \lambda >\in L^2(0, T; [H_0^1]^2 \times L^2)$ such that

$$(\Phi_{tt}, \Psi) + \delta(\Phi_t, \Psi) + B_1(\Phi, \lambda; \Psi, \eta) = G(\Psi),\ \forall < \Psi, \eta >\in [H_0^1]^2 \times L^2, \tag{4.1}$$

where

$$\begin{aligned} & B_1(\Phi, \lambda; \Psi, \eta) \\ & = a(\Phi, \Psi) + \alpha c(\Phi, \Psi) + (1 - \alpha d^2)d^2(\lambda, \eta) + (1 - \alpha d^2)[b(\lambda, \Psi) - b(\eta, \Phi)] \\ & = ((\phi_1)_x, (\psi_1)_x) + \alpha(\phi_1 - (\phi_2)_x, \psi_1 - (\psi_2)_x) + (1 - \alpha d^2)d^2(\lambda, \eta) \\ & \qquad -(1 - \alpha d^2)[(\eta, \phi_1 - (\phi_2)_x) - (\lambda, \psi_1 - (\psi_2)_x)] \end{aligned}$$

or equivalently

$$\begin{cases} (\Phi_{tt}, \Psi) + \delta(\Phi_t, \Psi) + a(\Phi, \Psi) + \alpha c(\Phi, \Psi) \\ \qquad +(1 - \alpha d^2)b(\lambda, \Psi) = G(\Psi), \\ -(1 - \alpha d^2)[b(\eta, \Phi) - d^2(\lambda, \eta)] = 0 \end{cases} \tag{4.2}$$

with conditioons

$$\begin{cases} (\Phi(\cdot, 0), \Psi) = (\Phi_0, \Psi), & \forall \Psi \in [H_0^1]^2 \\ (\Phi_t(\cdot, 0), \Psi) = (\bar{\Phi}_0, \Psi), & \forall \Psi \in [H_0^1]^2 \end{cases} \tag{4.3}$$

5. Semi-Discrete Approximations and Convergence Analysis

For our numerical methods, we assume I is partitioned quasiuniformly by a sequence of grids Γ_h:

$$0 = x_0 \leq x_1 \leq x_2 \leq ... \leq x_n = 1$$

such that for any grid

$$C \min_i(x_{i+1} - x_i) \geq h = \max_i(x_{i+1} - x_i),$$

where C is independent of h and the partition.On such a partition,we define $T_i = [x_i, x_{i+1}]$ and

$$(*)\qquad \begin{aligned} V_{1,h} &= \{v \in H_0^1(I) : v|_{T_i} \in P_K, i = 0,1,...,n-1\} \\ V_{2,h} &= \{v \in H_0^1(I) : v|_{T_i} \in P_L, i = 0,1,...,n-1\} \\ W_h &= \{w \in L^2(I) : w|_{T_i} \in P_J, i = 0,1,...,n-1\} \end{aligned}$$

where P_K, P_L and P_J are polynomials of degrees at most K, L and J, respectively.

Based on (4.1) or (4.2), we present a semi-discrete approximation scheme: For a fixed $t \in (0,T]$, find $< \Phi^h(\cdot,t), \lambda^h(\cdot,t) >\in [V_{1,h} \times V_{2,h}] \times W_h$ such that:

$$\begin{aligned} (\Phi^h_{tt}, \Psi) + \delta(\Phi^h_t, \Psi) + B_1(\Phi^h, \lambda^h; \Psi, \eta) = G(\Psi), \\ \forall < \Psi, \eta >\in [V_{1,h} \times V_{2,h}] \times W_h \end{aligned} \tag{5.1}$$

with the conditions

$$\begin{cases} (\Phi(\cdot,0) - \Phi_0, \Psi) = 0, & \forall \Psi \in V_{1,h} \times V_{2,h}, \\ (\Phi_t(\cdot,0) - \bar{\Phi}_0, \Psi) = 0, & \forall \Psi \in V_{1,h} \times V_{2,h} \end{cases} \tag{5.2}$$

or equivalently

$$\begin{aligned} (\Phi^h_{tt}, \Psi) + \delta(\Phi^h_t, \Psi) + a(\Phi^h, \Psi) + \alpha c(\Phi^h, \Psi) \\ +(1-\alpha d^2) b(\lambda^h, \Psi) = G(\Psi), \end{aligned} \tag{5.3a}$$

$$-(1-\alpha d^2)[b(\eta, \Phi^h) - d^2(\lambda^h, \eta) = 0 \tag{5.3b}$$

The existence and uniqueness of the finite element solution of (5.1) are trivial.Most of our work is about the error estimates.

At first we introduce an auxiliary steady-state Galerkin problem: Find $< \hat{\Phi}, \hat{\lambda} >\in [V_{1,h} \times V_{2,h}] \times W_h$ such that:

$$B_1(\hat{\Phi}, \hat{\lambda}; \Psi, \eta) = B_1(\Phi, \lambda; \Psi, \eta), \ \ \forall < \Psi, \eta >\in [V_{1,h} \times V_{2,h}] \times W_h \tag{5.4}$$

where $< \Phi, \lambda >$ is the exact solution of (4.1).

The equivalent form of (5.4) is given by:

$$\begin{cases} A(\hat{\Phi} - \Phi, \Psi) + (1-\alpha d^2) b(\hat{\lambda} - \lambda, \Psi) = 0, & \forall \Psi \in V_{1,h} \times V_{2,h} \\ -[b(\eta, \hat{\Phi} - \Phi) - d^2(\hat{\lambda} - \lambda, \eta)] = 0, & \forall \eta \in W_h \end{cases} \tag{5.5}$$

where $A(\Phi, \Psi) = ((\phi_1)_x, (\psi_1)_x) + \alpha(\phi_1 - (\phi_2)_x, \psi_1 - (\psi_2)_x)$.

Lemma 5.1. *There exists a constant C_1 independent of d such that: for any $\Phi \in [H_0^1]^2$*

$$A(\Phi, \Phi) \geq \alpha C_1 ||\Phi||_1^2$$

Proof. In $[H_0^1]^2$ we define a new norm $|| \cdot ||_B$ by:

$$||\Phi||_B^2 = ((\phi_1)_x, (\phi_1)_x) + (\phi_1 - (\phi_2)_x, \phi_1 - (\phi_2)_x)$$

It is easy to see that the norm $|| \cdot ||_B$ is equivalent to the norm $|| \cdot ||_1$ and then we have

$$C_1 ||\Phi||_1 \leq ||\Phi||_B$$

so we get

$$A(\Phi, \Phi) \geq \alpha ||\Phi||_B^2 \geq \alpha C_1 ||\Phi||_1^2. \ \square$$

Lemma 5.2.

$$A(\Phi, \Psi) \leq 2 ||\Phi||_1 ||\Psi||_1$$

$$b(\eta, \Psi) \leq ||\eta||_0||\Psi||_1$$

for any $\Phi, \Psi \in [H_0^1]^2, \eta \in L^2$.

Proof. Omitted.

Lemma 5.3. *(Babuska–Bressi condition)*$\forall \eta \in L^2$, *there exists a* $\Phi =< \phi_1, \phi_2 >\in [H_0^1]^2$, *such that*

$$b(\eta, \Phi) \geq \beta_0||\eta||_0||\Phi||_1$$

(in fact, $\beta_0 = 1/\sqrt{22}$)

Proof. We solve

$$-(\phi_2)_x + \phi_1 = \eta,$$

$$\phi_i(0) = \phi_i(1) = 0,$$

(for $i = 1, 2$) to obtain

$$\phi_2 = \int_0^x (\phi_1 - \eta)dx$$

Let $\phi_1 = ax(1-x)$, then $\phi_2(1) = 0$ implies

$$\int_0^1 \eta dx = \int_0^1 \phi_1 dx = a\int_0^1 x(1-x)dx = 1/6\ a$$

thus

$$\phi_1 = 6x(1-x)\int_0^1 \eta dx,$$

$$\phi_2 = \int_0^x (\phi_1 - \eta)dx$$

To prove $b(\eta, \Phi) \geq \beta_0||\eta||_0||\Phi||_1$ is to prove

$$(\eta, \eta) \geq \beta_0||\eta||_0||\Phi||_1$$

or

$$||\eta||_0 \geq \beta_0||\Phi||_1$$

Obviously,

$$||\phi_1||_0^2 = 36\int_0^1 x^2(1-x)^2dx(\int_0^1 \eta dx)^2 \leq 6/5||\eta||_0^2$$

Similarly we get

$$|\phi_1|_1^2 = ||(\phi_1)_x||_0^2 \leq 12||\eta||_0^2$$

$$||\phi_2||_0^2 \leq 22/5||\eta||_0^2$$

$$|\phi_2|_1^2 \leq 22/5||\eta||_0^2$$

Combine the above four inequalities we obtain

$$\begin{aligned} ||\Phi||_1^2 &= ||\phi_1||_1^2 + ||\phi_2||_1^2 \\ &= ||\phi_1||_0^2 + |\phi_1|_1^2 + ||\phi_2||_0^2 + |\phi_2|_1^2 \\ &\leq 22||\eta||_0^2 \end{aligned}$$

This gives the desired conclusion (with $\beta_0 = 1/\sqrt{22}$).□

Lemma 5.4. *(Discrete Babuska–Bressi condition)*

Let $V_{1,h}, V_{2,h}$ and W_h be defined as (*). Assume that there exists a function $\phi_1 \in V_{1,h}$ such that $\int_0^1 \phi_1 dx = 1$ and $|\phi_1|_1 \leq C_0$, and that $L-1 \geq J$, then for any $\eta \in W_h$, there exists a $\Phi =< \phi_1, \phi_2 >\in V_{1,h} \times V_{2,h}$, such that

$$b(\eta, \Phi) \geq \beta ||\eta||_0 ||\Phi||_1$$

where C_0 and β are constants independent of d.

The proof is similar to that of Lemma 5.3.

By virtue of the last four Lemma(5.1–5.4), we easily get the following important result:

Theorem 5.1. *Under the same assumption as in Lemma 5.4, problem* (5.4) *or its equivalent form* (5.5) *has a unique solution $< \hat{\Phi}, \hat{\lambda} >$. Moreover we have*

$$\begin{aligned} &||\hat{\Phi} - \Phi||_1 + (1-\alpha d^2)||\hat{\lambda} - \lambda||_0 \\ &\leq C(\inf_{\psi_{1,h}\in V_{1,h}} ||\phi_1 - \psi_{1,h}||_1 + \inf_{\psi_{2,h}\in V_{2,h}} ||\phi_2 - \psi_{2,h}||_1 + \inf_{\eta_h \in W_h} ||\lambda - \eta_h||_0) \\ &\leq Ch(||\phi_1||_2 + ||\phi_2||_2 + ||\lambda||_1) \end{aligned} \tag{5.6}$$

where C is dependent on α, C_1, β.

Proof. Existence and uniqueness are trivial. To analyse the estimate $||\hat{\Phi} - \Phi||_1 + (1-\alpha d^2)||\hat{\lambda} - \lambda||_0$, we rewrite (5.5) as follows:

$$\left\{ \begin{aligned} &A(\hat{\Phi} - \Psi_h, \Psi) + (1-\alpha d^2) b(\hat{\lambda} - \eta_h, \Psi) &&= A(\Phi - \Psi_h, \Psi) \\ & &&\quad +(1-\alpha d^2) b(\lambda - \eta_h, \Psi), \\ & &&\quad \forall \Psi, \Psi_h \in V_{1,h} \times V_{2,h} \\ &b(\eta, \hat{\Phi} - \Psi_h) - d^2(\hat{\lambda} - \eta_h, \eta) &&= b(\eta, \Phi - \Psi_h) - d^2(\lambda - \eta_h, \eta), \\ & &&\quad \forall \eta, \eta_h \in W_h \end{aligned} \right. \tag{5.7}$$

Take $\Psi = \hat{\Phi} - \Psi_h, \eta = \hat{\lambda} - \eta_h$, then we have

$$\begin{aligned} &A(\hat{\Phi} - \Psi_h, \hat{\Phi} - \Psi_h) + (1-\alpha d^2) d^2 (\hat{\lambda} - \eta_h, \hat{\lambda} - \eta_h) \\ &= A(\Phi - \Psi_h, \hat{\Phi} - \Psi_h) + (1-\alpha d^2) b(\lambda - \eta_h, \hat{\Phi} - \Psi_h) \\ &\quad +(1-\alpha d^2) d^2 (\lambda - \eta_h, \hat{\lambda} - \eta_h) - (1-\alpha d^2) b(\hat{\lambda} - \eta_h, \Phi - \Psi_h) \\ &\leq 2||\Phi - \Psi_h||_1 ||\hat{\Phi} - \Psi_h||_1 + (1-\alpha d^2)||\lambda - \eta_h||_0 ||\hat{\Phi} - \Psi_h||_1 \\ &\quad +(1-\alpha d^2) d^2 ||\lambda - \eta_h||_0 ||\hat{\lambda} - \eta_h||_0 + (1-\alpha d^2)||\hat{\lambda} - \eta_h||_0 ||\Phi - \Psi_h||_1 \\ &\leq [2||\Phi - \Psi_h||_1 + (1-\alpha d^2)||\lambda - \eta_h||_0] ||\hat{\Phi} - \Psi_h||_1 \\ &\quad +(1-\alpha d^2)[d^2 ||\lambda - \eta_h||_0 + ||\Phi - \Psi_h||_1] ||\hat{\lambda} - \eta_h||_0 \end{aligned} \tag{5.8}$$

In light of LEMMA 5.4, we have

$$\begin{aligned} \beta ||\hat{\lambda} - \eta_h||_0 &\leq \sup_{\Psi \in V_{1,h} \times V_{2,h}} \frac{b(\hat{\lambda} - \eta_h, \Psi)}{||\Psi||_1} \\ &= \sup_{\Psi \in V_{1,h} \times V_{2,h}} \frac{b(\hat{\lambda} - \lambda + \lambda - \eta_h, \Psi)}{||\Psi||_1} \\ &\leq 1/(1-\alpha d^2) \sup_{\Psi \in V_{1,h} \times V_{2,h}} \frac{A(\hat{\Phi} - \Psi_h + \Psi_h - \Phi, \Psi)}{||\Psi||_1} + ||\lambda - \eta_h||_0 \\ &\leq 2/(1-\alpha d^2)[||\hat{\Phi} - \Psi_h||_1 + ||\Phi - \Psi_h||_1] + ||\lambda - \eta_h||_0 \end{aligned} \tag{5.9}$$

Substitute (5.9) into (5.8) and employ Lemma 5.1 we get

$$\begin{aligned}
&\alpha C_1||\hat{\Phi}-\Psi_h||_1^2\\
&\le A(\hat{\Phi}-\Psi_h,\hat{\Phi}-\Psi_h)\\
&\le [2||\Phi-\Psi_h||_1+(1-\alpha d^2)||\lambda-\eta_h||_0]||\hat{\Phi}-\Psi_h||_1\\
&\qquad +(1-\alpha d^2)[d^2||\lambda-\eta_h||_0+||\Phi-\Psi_h||_1]\\
&\qquad\quad \cdot 1/\beta\{2/(1-\alpha d^2)[||\hat{\Phi}-\Psi_h||_1+||\Phi-\Psi_h||_1]+||\lambda-\eta_h||_0\}\\
&\le (assuming\ \ d\in(0,1), 1-\alpha d^2\in(0,1))\\
&\le [(2+2/\beta)||\Phi-\Psi_h||_1+(1+2/\beta)||\lambda-\eta_h||_0]||\hat{\Phi}-\Psi_h||_1\\
&\qquad +1/\beta[||\lambda-\eta_h||_0+2||\Phi-\Psi_h||_1]^2
\end{aligned}\tag{5.10}$$

A series of simple operations on the inequality (5.10) show us that

$$\begin{aligned}
||\hat{\Phi}-\Psi_h||_1 &\le 1/(\alpha C_1)[(2+2/\beta)||\Phi-\Psi_h||_1+(1+2/\beta)||\lambda-\eta_h||_0]\\
&\qquad +2/\sqrt{\alpha\beta C_1}[||\lambda-\eta_h||_0+2||\Phi-\Psi_h||_1]\\
&\le C[||\lambda-\eta_h||_0+||\Phi-\Psi_h||_1]
\end{aligned}$$

where C is dependent on α,β,C_1.

Employing triangle inequality we obtain

$$\begin{aligned}
||\hat{\Phi}-\Phi||_1 &\le ||\hat{\Phi}-\Psi_h||_1+||\Phi-\Psi_h||_1\\
&\le C[||\lambda-\eta_h||_0+||\Phi-\Psi_h||_1]+||\Phi-\Psi_h||_1\\
&\le C[||\lambda-\eta_h||_0+||\phi_1-\psi_{1,h}||_1+||\phi_2-\psi_{2,h}||_1]
\end{aligned}\tag{5.11}$$

Since $\Psi\in V_{1,h}\times V_{2,h}$ is arbitrary, (5.11) together with standard results from finite element interpolation theory yields

$$\begin{aligned}
||\hat{\Phi}-\Phi||_1 &\le C(\inf_{\psi_{1,h}\in V_{1,h}}||\phi_1-\psi_{1,h}||_1+\inf_{\psi_{2,h}\in V_{2,h}}||\phi_2-\psi_{2,h}||_1+\inf_{\eta_h\in W_h}||\lambda-\eta_h||_0)\\
&\le Ch(||\phi_1||_2+||\phi_2||_2+||\lambda||_1)
\end{aligned}$$

Similarly , from (5.9) we get

$$(1-\alpha d^2)||\hat{\lambda}-\lambda||_0\le Ch(||\phi_1||_2+||\phi_2||_2+||\lambda||_1)$$

The sum of the last two inequalities is (5.6).□

Remark 5.1. In applications we usually take

$$V_{1,h}=V_{2,h}=\{v\in H_0^1(I): v|_{T_i}\in P_L, i=0,1,...,n-1\}$$

Remark 5.2. Note that the shear term $\lambda=d^{-2}(\phi_1-(\phi_2)_x)$, and that theorem 3.1 has given the bound of $(||\phi_1||_2+||\phi_2||_2+d^{-2}||\phi_1-(\phi_2)_x||_1)$.

Remark 5.3. With an easy duality argument we also have the estimate

$$||\hat{\Phi}-\Phi||_0\le Ch^2\tag{5.12}$$

Remark 5.4. Similarly,we get the estimate

$$||(\hat{\Phi}-\Phi)_t||_0\le Ch^2$$

by differentiating (5.4) with respect to t.

Now we come to estimate the error $||\Phi - \Phi_h||_0$. The following theorem is the main result of this section.

Theorem 5.2. *Let* $<\Phi_h, \lambda_h>$ *be the semi-discrete Galerkin approximation given by* (5.1) *and* (5.2), *and let* $<\phi, \lambda>$ *be the solution of* (4.1) *and* (4.3). *Then if* $\Phi, \Phi_t \in L^\infty(0,T;H^2)$, *there exists a constant* C *independent of* h *and* d *such that*

$$||\Phi - \Phi_h||_{L^\infty(0,T;H^r)} \le Ch^{2-r}[||\Phi||_{L^\infty(0,T;H^2)} + ||\Phi_t||_{L^\infty(0,T;H^2)}] \tag{5.13}$$

for $r = 0, 1$.

Proof. From (4.1),(5.1) and (5.4) we get

$$\begin{aligned} &((\Phi - \hat{\Phi})_{tt}, \Psi) + \delta((\Phi - \hat{\Phi})_t, \Psi) \\ &= (-\hat{\Phi}_{tt}, \Psi) + \delta(-\hat{\Phi}_t, \Psi) - B_1(\Phi, \lambda; \Psi, \eta) + G(\Psi) \\ &= (-\hat{\Phi}_{tt}, \Psi) + \delta(-\hat{\Phi}_t, \Psi) - B_1(\hat{\Phi}, \hat{\lambda}; \Psi, \eta) \\ &\qquad + (\Phi_{tt,h}, \Psi) + \delta(\Phi_{t,h}, \Psi) + B_1(\Phi_h, \lambda_h; \Psi, \eta) \\ &= ((\Phi_h - \hat{\Phi})_{tt}, \Psi) + \delta((\Phi_h - \hat{\Phi})_t, \Psi) + B_1(\Phi_h - \hat{\Phi}, \lambda_h - \hat{\lambda}; \Psi, \eta), \end{aligned}$$

for any $<\Psi, \eta> \in [V_{1,h} \times V_{2,h}] \times W_h$.

Let $E = \Phi_h - \hat{\Phi}, \hat{e} = \Phi - \hat{\Phi}, E^\lambda = \lambda_h - \hat{\lambda}$, we have

$$(\hat{e}_{tt}, \Psi) + \delta(\hat{e}_t, \Psi) = (E_{tt}, \Psi) + \delta(E_t, \Psi) + B_1(E, E^\lambda; \Psi, \eta) \tag{5.14}$$

Taking $\eta = 0$ in (5.14) we get

$$\begin{aligned} &(\hat{e}_{tt}, \Psi) + \delta(\hat{e}_t, \Psi) \\ &= (E_{tt}, \Psi) + \delta(E_t, \Psi) + a(E, \Psi) + \alpha c(E, \Psi) + (1 - \alpha d^2) b(E^\lambda, \Psi) \end{aligned} \tag{5.15}$$

Taking $\Psi = 0$ in (5.14) we get

$$(1 - \alpha d^2)[d^2(E^\lambda, \eta) - b(\eta, E)] = 0 \tag{5.16}$$

Integrate (5.15) from 0 to t ,we then get

$$\begin{aligned} &(\hat{e}_t, \Psi) + \delta(\hat{e}, \Psi) + (E_t(\cdot, 0), \Psi) + \delta(E(\cdot, 0), \Psi) \\ &= (E_t, \Psi) + \delta(E, \Psi) + (\textstyle\int_0^t (E_1)_x, (\psi_1)_x) \\ &\qquad + \alpha(\textstyle\int_0^t [E_1 - (E_2)_x], \psi_1 - (\psi_2)_x) + (1 - \alpha d^2) b(\textstyle\int_0^t E^\lambda, \Psi) \end{aligned} \tag{5.17}$$

Now let $\Psi = E, \eta = \int_0^t E^\lambda$ in (5.16) and (5.17), and we easily get

$$\begin{aligned} &\tfrac{d}{dt}||E||_0^2 + 2\delta||E||_0^2 + \tfrac{d}{dt}||\textstyle\int_0^t (E_1)_x||_0^2 \\ &\qquad + \alpha \tfrac{d}{dt}||\textstyle\int_0^t [E_1 - (E_2)_x]||_0^2 + (1 - \alpha d^2) d^2 \tfrac{d}{dt}||\textstyle\int_0^t E^\lambda||_0^2 \\ &= 2[(E_t, E) + \delta(E, E) + (\textstyle\int_0^t (E_1)_x, (E_1)_x) \\ &\qquad + \alpha(\textstyle\int_0^t [E_1 - (E_2)_x], E_1 - (E_2)_x) + (1 - \alpha d^2) d^2 (\textstyle\int_0^t E^\lambda, E^\lambda)] \\ &= 2[(\hat{e}_t, E) + \delta(\hat{e}, E) + (E_t(\cdot, 0), E) + \delta(E(\cdot, 0), E)] \end{aligned}$$

If we integrate this equation from 0 to t ,we also get

$$\begin{aligned} &||E||_0^2 + ||\textstyle\int_0^t (E_1)_x||_0^2 + \alpha||\textstyle\int_0^t [E_1 - (E_2)_x]||_0^2 \\ &\qquad + (1 - \alpha d^2) d^2 ||\textstyle\int_0^t E^\lambda||_0^2 \\ &\le ||E||_0^2(0) + 2\textstyle\int_0^t (\hat{e}_t, E) + 2\delta \textstyle\int_0^t (\hat{e}, E) \\ &\qquad + 2\textstyle\int_0^t (E_t(\cdot, 0), E) + 2\delta \textstyle\int_0^t (E(\cdot, 0), E) \\ &\le C[||E||_0(0) + ||\hat{e}_t||_{L^\infty(0,T;L^2)} + ||\hat{e}||_{L^\infty(0,T;L^2)} + ||E_t||_0(0)]||E||_{L^\infty(0,T;L^2)} \end{aligned} \tag{5.18}$$

Note that by Remark 5.3 and Remark 5.4 and (4.3)

$$\begin{aligned} ||E||_0(0) &= ||\Phi_h - \hat{\Phi}||_0(0) \\ &\le ||\Phi - \Phi_h||_0(0) + ||\Phi - \hat{\Phi}||_0(0) \\ &\le Ch^2||\Phi||_{L^\infty(0,T;H^2)} \\ ||\hat{e}_t||_{L^\infty(0,T;L^2)} &= ||(\Phi - \hat{\Phi})_t||_{L^\infty(0,T;L^2)} \\ &\le Ch^2||\Phi_t||_{L^\infty(0,T;H^2)} \\ ||\hat{e}||_{L^\infty(0,T;L^2)} &= ||\Phi - \hat{\Phi}||_{L^\infty(0,T;L^2)} \\ &\le Ch^2||\Phi||_{L^\infty(0,T;H^2)} \\ ||E_t||_0(0) &= ||(\Phi_h - \hat{\Phi})_t||_0(0) \\ &\le ||(\Phi - \Phi_h)_t||_0(0) + ||(\Phi - \hat{\Phi})_t||_0(0) \\ &\le Ch^2||\Phi_t||_{L^\infty(0,T;H^2)} \end{aligned}$$

We have from (5.18)

$$\begin{aligned} ||\Phi_h - \hat{\Phi}||_{L^\infty(0,T;L^2)} &= ||E||_{L^\infty(0,T;L^2)} \\ &\le Ch^2[||\Phi||_{L^\infty(0,T;H^2)} + ||\Phi_t||_{L^\infty(0,T;H^2)}] \end{aligned} \tag{5.19}$$

Also we get from (5.18) and (5.19)

$$\begin{aligned} &|| \textstyle\int_0^t (E_1)_x||_0^2 + \alpha|| \int_0^t [E_1 - (E_2)_x]||_0^2 \\ &\le Ch^2[||\Phi||_{L^\infty(0,T;H^2)} + ||\Phi_t||_{L^\infty(0,T;H^2)}]^2 \end{aligned} \tag{5.20}$$

It is not difficult to prove that in $H_0^1 \times H_0^1$, the norm $|| \cdot ||_*$ defined by

$$||\Phi||_* = \sup_t [|| \int_0^t (\phi_1)_x||_0^2 + || \int_0^t [\phi_1 - (\phi_2)_x||_0^2]^{1/2}$$

is equivalent to $|| \cdot ||_{L^\infty(0,T;H^1)}$, so we have

$$||\Phi||_{L^\infty(0,T;H^1)} \le C||\Phi||_* \tag{5.21}$$

Now (5.20) and (5.21) lead to

$$\begin{aligned} ||\Phi_h - \hat{\Phi}||_{L^\infty(0,T;H^1)} &= ||E||_{L^\infty(0,T;H^1)} \\ &\le C||E||_* \\ &\le Ch^2[||\Phi||_{L^\infty(0,T;H^2)} + ||\Phi_t||_{L^\infty(0,T;H^2)}] \end{aligned} \tag{5.22}$$

The error estimate (5.22) is over-optimal.

Finally, by means of triangle inequality, (5.6),(5.12),(5.19) and (5.22) imply the conclusion (5.13).□

6. Fully Discrete Approximations and Convergence Analysis

We now consider a fully discrete Galerkin scheme for (1.1). Suppose [0,T] is partitioned into equal subinternals of size τ. The following Crank–Nicholson scheme may be used for approximating (4.2) and (4.3). For $n = 0, 1, 2, ..., J$, find $< \Phi^n, Q^n, \lambda^n > \in [V_{1,h} \times V_{2,h}] \times [V_{1,h} \times V_{2,h}] \times W_h$ such that:

$$\begin{cases} (\partial_\tau Q^n, \Psi) + \delta(\partial_\tau \Phi^n, \Psi) + a(\Phi^{n+1/2}, \Psi) \\ \qquad +\alpha c(\Phi^{n+1/2}, \Psi) + (1-\alpha d^2) b(\lambda^{n+1/2}, \Psi) \\ \qquad\qquad = G^{n+1/2}(\Psi), & \forall \Psi \in V_{1,h} \times V_{2,h} \\ -(1-\alpha d^2)[b(\eta, \Phi^{n+1/2}) - d^2(\lambda^{n+1/2}, \eta)] = 0, & \forall \eta \in W_h \\ \partial_\tau \Phi^n = Q^{n+1/2} \\ (\Phi^0, \Psi) = (\Phi_0, \Psi), & \forall \Psi \in V_{1,h} \times V_{2,h} \\ (Q^0, \Psi) = (\bar{\Phi}_0, \Psi), & \forall \Psi \in V_{1,h} \times V_{2,h} \\ (\lambda^0, \eta) = d^{-2} b(\eta, \Phi_0), & \forall \eta \in W_h \end{cases} \tag{6.1}$$

where for any $\Psi \in V_{1,h} \times V_{2,h}$,

$$\partial_\tau \Psi^n = 1/\tau[\Psi^{n+1} - \Psi^n]$$

$$\Psi^{n+1/2} = 1/2[\Psi^{n+1} + \Psi^n]$$

$$G^{n+1/2}(\Psi) = (1/2[g(\cdot, (n+1)\tau) + g(\cdot, n\tau)], \psi_2)$$

(Similar expressions hold for functions in W_h).

It is not difficult to show that (6.1) has a unique solution for each time step. Using techniques of baker[3], we can easily prove the following result:

Theorem 6.1. *Suppose $\Phi(x,t)$ is a solution of (4.2) and (4.3) such that $\Phi, \Phi_t \in L^\infty(0,T;H^2)$, $\frac{\partial^3 \Phi}{\partial t^3}, \frac{\partial^4 \Phi}{\partial t^4} \in L^2(0,T;L^2)$. If Φ^n is the solution generated by (6.1), there exists a constant C independent of τ, d and h, such that for $n = 0, 1, ..., J$*

$$||\Phi(\cdot, n\tau) - \Phi^n|| \le C[h^2 + \tau^2(||\frac{\partial^3 \Phi}{\partial t^3}||_{L^2(0,T;L^2)} + ||\frac{\partial^4 \Phi}{\partial t^4}||_{L^2(0,T;L^2)})] \tag{6.2}$$

We give a rough proof for this result. Some triffling calculating are omitted. For the sake of simplicity, we denote $\Psi(\cdot, k\tau) = \Psi(k)$, for any $\Psi \in [H_0^1]^2$ and $||\cdot||_0 = ||\cdot||$.

Proof. Let

$$E^n = \Phi^n - \hat{\Phi}(n),$$

$$P^n = Q^n - \hat{\Phi}_t(n),$$

$$\hat{e} = \Phi - \hat{\Phi},$$

$$M^n = \lambda^n - \hat{\lambda}(n),$$

$$\rho_1^n = \partial_\tau(\Phi_t(n)) - \Phi_{tt}(n+1/2),$$

$$\rho_2^n = \partial_\tau(\Phi(n)) - \Phi_t(n+1/2),$$

then we have from (6.1) and (4.1)

$$\begin{aligned} &(\partial_\tau P^n, \Psi) + \delta(\partial_\tau E^n, \Psi) + B_1(E^{n+1/2}, M^{n+1/2}; \Psi, \eta) \\ &= (-\rho_1^n + \partial_\tau \hat{e}_t(n) + \delta \rho_2^n - \partial_\tau \hat{e}(n), \Psi) \end{aligned} \tag{6.3}$$

and

$$\partial_\tau E^n = P^{n+1/2} + \partial_\tau \hat{e}(n) - \hat{e}_t(n+1/2) - \rho_2^n \tag{6.4}$$

then we get

$$\partial_\tau E^0 = P^0 + \tau/2\partial_\tau P^0 + \partial_\tau \hat{e}(0) - \hat{e}_t(1/2) - \rho_2^0 \tag{6.5}$$

$$\partial_\tau E^n = P^0 + \tau/2 \sum_0^n \partial_\tau P^k + \tau/2 \sum_0^{n-1} \partial_\tau P^k + \partial_\tau \hat{e}(n) - \hat{e}_t(n+1/2) - \rho_2^n \tag{6.6}$$

Now combining (6.5) ,(6.6) with (6.3), we have

$$\begin{aligned}(\partial_\tau E^0, \Psi) + \tau/2\delta(\partial_\tau E^0, \Psi) + \tau/2B_1(E^{1/2}, M^{1/2}; \Psi, \eta)\\ = \tau/2(-\rho_1^0 + \partial_\tau \hat{e}_t(0) + \delta\rho_2^0 - \partial_\tau \hat{e}(0), \Psi)\\ +(P^0 + \partial_\tau \hat{e}(0) - \hat{e}_t(1/2) - \rho_2^0, \Psi)\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}&(\partial_\tau E^n, \Psi) + \tau/2\delta(\sum_0^n \partial_\tau E^k + \sum_0^{n-1} \partial_\tau E^k, \Psi)\\ &+\tau/2B_1(\sum_0^n E^{k+1/2} + \sum_0^{n-1} E^{k+1/2}, \sum_0^n M^{k+1/2} + \sum_0^{n-1} M^{k+1/2}; \Psi, \eta)\\ &= \tau/2(\sum_0^n + \sum_0^{n-1})(-\rho_1^k + \partial_\tau \hat{e}_t(k) + \delta(\rho_2^k - \partial_\tau \hat{e}(k)), \Psi)\\ &+(P^0 + \partial_\tau \hat{e}(n) - \hat{e}_t(n+1/2) - \rho_2^n, \Psi)\end{aligned} \tag{6.8}$$

Let

$$\Psi = 2E^{n+1/2} = E^{n+1} + E^n = 2(\sum_0^n E^{k+1/2} - \sum_0^{n-1} E^{k+1/2}),$$

$$\eta = 2(\sum_0^n M^{k+1/2} - \sum_0^{n-1} M^{k+1/2}),$$

so we have from (6.8)

$$\begin{aligned}&1/\tau(||E^{n+1}||^2 - ||E^n||^2) + \delta/2||E^{n+1} + E^n||^2\\ &+\tau B_1(\sum_0^n E^{k+1/2}, \sum_0^n M^{k+1/2}; \sum_0^n E^{k+1/2}, \sum_0^n M^{k+1/2})\\ &-\tau B_1(\sum_0^{n-1} E^{k+1/2}, \sum_0^{n-1} M^{k+1/2}; \sum_0^{n-1} E^{k+1/2}, \sum_0^{n-1} M^{k+1/2})\\ &= 2\delta(E^0, E^{n+1/2})\\ &+\tau(\sum_0^n + \sum_0^{n-1})(-\rho_1^k + \partial_\tau \hat{e}_t(k) + \delta(\rho_2^k - \partial_\tau \hat{e}(k)), E^{n+1/2})\\ &+2(P^0 + \partial_\tau \hat{e}(n) - \hat{e}_t(n+1/2) - \rho_2^n, E^{n+1/2})\end{aligned} \tag{6.9}$$

Sum them from 0 to $l-1, (1 \le l \le J)$ then we get

$$\begin{aligned}&||E^l||^2 \le ||E^0||^2 + 2\delta\tau(E^0, \sum_0^{l-1} E^{n+1/2})\\ &+\tau^2 \sum_0^{l-1}(\sum_0^n + \sum_0^{n-1})(-\rho_1^k + \partial_\tau \hat{e}_t(k) + \delta(\rho_2^k - \partial_\tau \hat{e}(k)), E^{n+1/2})\\ &+2\tau \sum_0^{l-1}(P^0 + \partial_\tau \hat{e}(n) - \hat{e}_t(n+1/2) - \rho_2^n, E^{n+1/2})\end{aligned} \tag{6.10}$$

Let

$$\epsilon_1^n = (\sum_0^n + \sum_0^{n-1})(-\rho_1^k + \partial_\tau \hat{e}_t(k) + \delta(\rho_2^k - \partial_\tau \hat{e}(k))$$

$$\epsilon_2^n = P^0 + \partial_\tau \hat{e}(n) - \hat{e}_t(n+1/2) - \rho_2^n$$

Using the inequality $2ab \le \epsilon a^2 + 1/\epsilon b^2, (\epsilon > 0)$, we have

$$\begin{aligned} 2\delta\tau(E^0, \sum_0^{l-1} E^{n+1/2}) &\le 2\delta\tau l(|E^0|, \max_{0\le n\le J} |E^n|) \\ &\le \delta T(4\delta T||E^0||^2 + 1/(4\delta T) \max_{0\le n\le J} ||E^n||^2) \\ &= 4\delta^2 T^2 ||E^0||^2 + 1/4 \max_{0\le n\le J} ||E^n||^2 \end{aligned} \tag{6.11a}$$

$$\tau^2 \sum_0^{l-1} (\epsilon_1^n, E^{n+1/2}) \le \tau^3 T \sum_0^{l-1} ||\epsilon_1^n||^2 + 1/4 \max_{0\le n\le J} ||E^n||^2 \tag{6.11b}$$

$$2\tau \sum_0^{l-1} (\epsilon_2^n, E^{n+1/2}) \le 4\tau T \sum_0^{l-1} ||\epsilon_2^n||^2 + 1/4 \max_{0\le n\le J} ||E^n||^2 \tag{6.11c}$$

Substitute (6.11a),(6.11b) and (6.11c) into (6.10), we have

$$\begin{aligned} &1/4 \max_{0\le n\le J} ||E^n||^2 \\ &\le ||E^0||^2 + 4\delta^2 T^2 ||E^0||^2 + \tau^3 T \sum_0^{l-1} ||\epsilon_1^n||^2 + 4\tau T \sum_0^{l-1} ||\epsilon_2^n||^2 \\ &\le ||E^0||^2 + 4\delta^2 T^2 ||E^0||^2 + \tau^2 T^2 \max_{0\le n\le J} ||\epsilon_1^n||^2 + 4\tau T \sum_0^{l-1} ||\epsilon_2^n||^2 \end{aligned} \tag{6.12}$$

Now the work left to us is to estimate each term on the right side of (6.12). It is easy to prove that

$$\rho_1^k = 1/(2\tau) \int_{k\tau}^{(k+1)\tau} [(k+1)\tau - s][k\tau - s] \frac{\partial^4 \Phi}{\partial t^4}(\cdot, s) ds,$$

$$\rho_2^k = 1/(2\tau) \int_{k\tau}^{(k+1)\tau} [(k+1)\tau - s][k\tau - s] \frac{\partial^3 \Phi}{\partial t^3}(\cdot, s) ds,$$

we have

$$||\rho_1^k||^2 \le C\tau^3 \int_{k\tau}^{(k+1)\tau} ||\frac{\partial^4 \Phi}{\partial t^4}||^2 ds,$$

$$||\rho_2^k||^2 \le C\tau^3 \int_{k\tau}^{(k+1)\tau} ||\frac{\partial^3 \Phi}{\partial t^3}||^2 ds,$$

and therefore

$$\tau(\sum_0^n + \sum_0^{n-1})||\rho_1^k||^2 \le 2\tau \sum_0^J ||\rho_1^k||^2 \le 2C\tau^4 ||\frac{\partial^4 \Phi}{\partial t^4}||^2_{L^2(0,T;L^2)} \tag{6.13a}$$

$$\tau(\sum_0^n+\sum_0^{n-1})||\rho_2^k||^2 \le 2\tau\sum_0^J||\rho_2^k||^2 \le 2C\tau^4||\frac{\partial^3\Phi}{\partial t^3}||^2_{L^2(0,T;L^2)} \tag{6.13b}$$

Thus we get

$$\begin{aligned}\tau^2||\epsilon_1^n||^2 &\le \tau^2[(\sum_0^n+\sum_0^{n-1})(||\rho_1^k||+\delta||\rho_2^k||)\\ &\quad +||(\sum_0^n+\sum_0^{n-1})(\partial_\tau\hat{e}_t(k)-\delta\partial_\tau\hat{e}(k))||]^2\\ &\le 8\tau^2J(\sum_0^J||\rho_1^k||^2+\delta\sum_0^J||\rho_2^k||^2)+2||\hat{e}_t(n+1)+\hat{e}_t(n)\\ &\quad -2\hat{e}_t(0)-\delta\hat{e}(n+1)-\delta\hat{e}(n)+2\delta\hat{e}(0)||^2\\ &\le C\tau^4[||\tfrac{\partial^4\Phi}{\partial t^4}||^2_{L^2(0,T;L^2)}+||\tfrac{\partial^3\Phi}{\partial t^3}||^2_{L^2(0,T;L^2)}]+Ch^4\end{aligned} \tag{6.14}$$

and

$$\begin{aligned}\tau\sum_0^{l-1}||\epsilon_2^n||^2 &\le \tau\sum_0^{l-1}(||P^0||+||\partial_\tau\hat{e}(n)||+||\hat{e}_t(n+1/2)||+||\rho_2^n||)^2\\ &\le 4\tau(\sum_0^{l-1}||P^0||^2+\sum_0^{l-1}||\partial_\tau\hat{e}(n)||^2\\ &\quad +\sum_0^{l-1}||\hat{e}_t(n+1/2)||^2+\sum_0^{l-1}||\rho_2^n||^2)\\ &\le 4TCh^4+4C\tau^4[||\tfrac{\partial^4\Phi}{\partial t^4}||^2_{L^2(0,T;L^2)}+||\tfrac{\partial^3\Phi}{\partial t^3}||^2_{L^2(0,T;L^2)}]\\ &\quad +Ch^4+Ch^4\\ &\le Ch^4+C\tau^4[||\tfrac{\partial^4\Phi}{\partial t^4}||^2_{L^2(0,T;L^2)}+||\tfrac{\partial^3\Phi}{\partial t^3}||^2_{L^2(0,T;L^2)}]\end{aligned} \tag{6.15}$$

where the following relations are used:

$$\begin{aligned}\tau\sum_0^{l-1}||\partial_\tau\hat{e}(n)||^2 &= \tau\sum_0^{l-1}||1/\tau\int_{k\tau}^{(k+1)\tau}\hat{e}_t(\cdot,s)ds||^2\\ &\le \tau\sum_0^{l-1}1/\tau\int_{k\tau}^{(k+1)\tau}||\hat{e}_t||^2ds\\ &\le ||\hat{e}_t||^2_{L^2(0,T;L^2)}\le Ch^4\end{aligned}$$

$$||P^0||^2=||Q^0-\hat{\Phi}_t(0)||^2=||\Phi_t(0)-\hat{\Phi}_t(0)||^2\le Ch^4$$

Also we have

$$||E^0||^2=||\Phi^0-\hat{\Phi}(0)||^2=||\Phi(0)-\hat{\Phi}(0)||^2\le Ch^4 \tag{6.16}$$

Combining (6.12) with (6.14),(6.15),(6.16), we have

$$\max_{0\le n\le J}||E^n||^2\le C[h^4+\tau^4(||\frac{\partial^4\Phi}{\partial t^4}||^2_{L^2(0,T;L^2)}+||\frac{\partial^3\Phi}{\partial t^3}||^2_{L^2(0,T;L^2)})]$$

or equivalently

$$\max_{0\le n\le J}||E^n||\le C[h^2+\tau^2(||\frac{\partial^4\Phi}{\partial t^4}||^2_{L^2(0,T;L^2)}+||\frac{\partial^3\Phi}{\partial t^3}||^2_{L^2(0,T;L^2)})] \tag{6.17}$$

Finally by employing the triangle inequality, we obtain from (6.17) and Remark 5.3 the desired conclusion.□

Acknowledgement. The authors are grateful to prof. Zhou Tianxiao for his helpful comments.The authors would also like to thank the anonymous referees for their valuable suggestion.

References

[1] D.N. Arnold, Discretization by finite elements of a model parameter dependent problem,*Numer. Math.,* **37**(1981), 405-421.
[2] D.N. Arnold, F. Brezzi, Some new elements for the Reissner-Mindlin plate model, PENNSTATE, Report No. AM 129.
[3] G.A. Baker, Error estimates for finite element methods for second order hyperbolic equations, *SIAM J. Numer. Anal.* **13**(1976), 564-576.
[4] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, 1992.
[5] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[6] R. Duran, E. Liberman, On mixed finite element methods for the Reissner-Mindlin plate model,*Math. Comput.* **198**(1992), 561-573.
[7] Minfu Feng, Rongkui Yang, The partial projection finite element method for the simply soft Reissner-Mindlin plate.
[8] L.P. Franca, A.F.D. Loula, A new mixed finite element method for the Timoshenko beam problem, RAIRO **25**(1991) 561-578.
[9] A.F.D. Loula, J. Hughes, L.P. Franca, I. Miranda, Mixed Petrov-Galerkin methods for the Timoshenko beam problem, *Comput. Methods Appl. Mech. Engrg,* **63**(1987), 133-154.
[10] J.T. Oden, J.N. Reddy, An introduction to the mathematic theory of finite element. A Wiley-Interscience Publication(1976).
[11] B. Semper, Semi-discrete and fully discrete Galerkin methods for the vibrating Timoshenko beam, *Comput. Methods Appl. Mech. Engrg,* **117**(1994), 353-360.
[12] B. Semper, A fully discrete Galerkin-Crank-Nicholson scheme for the vibrating Timoshenko beam equations, Report, TA Math. Dept. 1994.
[13] V. Thomée, Galerkin methods for hyperbolic problem 1984.
[14] S.P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philos. Mag. Ser.* **6:41**(1921), 744-746.
[15] Tianxiao Zhou, The partial projection method in the finite elements discretization of the Reissner-Mindlin plate model, *J. Comput. Math.* **13:2**(1995).