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GENERAL DIFFERENCE SCHEMES WITH INTRINSIC PARALLELISM FOR SEMILINEAR PARABOLIC SYSTEMS OF DIVERGENCE TYPE*1)

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Abstract

In this paper the general finite difference schemes with intrinsic parallelism for the boundary value problem of the semilinear parabolic system of divergence type with bounded coefficients are constructed, and the existence and uniqueness of the difference solution for the general schemes are proved. And the convergence of the solutions of the difference schemes to the generalized solution of the original boundary value problem of the semilinear parabolic system is obtained. The multi-dimensional problems are also studied.

Key words: Difference scheme, Intrinsic parallelism, Semilinear parabolic system, Convergence.

1. Introduction

In [1] and [2] the general finite difference schemes having the intrinsic character of parallelism for the boundary value problems of the nonlinear parabolic system of general form (i.e., non-divergence type) are discussed under the assumption that there is an unique smooth solution for the original problem. In [3] and [4] the boundary value problems of the one-dimensional quasilinear parabolic system and multi-dimensional semilinear parabolic system of non-divergence type with bounded measurable coefficients are solved by the finite difference methods of general schemes with intrinsic parallelism. In these papers the general difference schemes with intrinsic parallelism are constructed by taking the difference approximations for the derivatives of second order to be in general the various linear combinations of the four kinds of difference quotients: the scheme ahead, the backward scheme, the scheme on the top cover of the grid, and the downward scheme. Since the parameters in the construction of the general difference schemes have large degree of freedom, in [5] some practical difference schemes are obtained by suitablely choicing these parameters for the nonlinear parabolic systems of non-divergence type. The time steplength for these difference

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schemes can be taken at least $8k$ times the time steplength for the fully explicit finite difference schemes (k can be any positive integer).

In this paper we solve the boundary value problems of the semilinear parabolic system of divergence type with bounded measurable coefficients by the finite difference methods of general schemes with intrinsic parallelism. The existence, uniqueness and convergence of the discrete vector solution for the general schemes with intrinsic parallelism are proved. Moreover, we can get some practical schemes with intrinsic parallelism by suitablely choicing the parameters in the general schemes. For these difference schemes, the time steplength can be taken at least $8k$ times the time steplength for the fully explicit finite difference schemes (k can be any positive integer), provided that the discontinuity of the coefficient matrix of the parabolic system does not occur at the interface of the domain decomposition. In the sections 2–6, we consider the case of one-dimensional problems. In the section 7 the multi-dimensional problems are discussed.

2. Difference Schemes with Intrinsic Parallelism

Let us now consider the boundary value problems for the semilinear parabolic systems of second order of the form

$$u_t = (A(x,t)u_x)_x + B(x,t,u)u_x + f(x,t,u) \tag{1}$$

where $u(x,t) = (u_1(x,t), \cdots, u_m(x,t))$ is the m-dimensional vector unknown function ($m \geq 1$), $u_t = \dfrac{\partial u}{\partial t}$, $u_x = \dfrac{\partial u}{\partial x}$ are the corresponding vector derivatives. The matrix $A(x,t)$ is an $m \times m$ positive definite coefficient matrix, and $B(x,t,u)$ is the $m \times m$ matrix, and $f(x,t,u)$ is the m-dimensional vector function. Let us consider in the rectangular domain $Q_T = \{0 \leq x \leq l,\ 0 \leq t \leq T\}$ with $l > 0$ and $T > 0$, the problem for the systems (1) with the boundary value condition

$$u(0,t) = u(l,t) = 0 \tag{2}$$

and the initial value condition

$$u(x,0) = \varphi(x) \tag{3}$$

where $\varphi(x)$ is a given m-dimensional vector function of variable $x \in [0,l]$.

Suppose that the following conditions are fulfilled.

(I) For any fixed $u \in R^m$, $A(x,t)$, $B(x,t,u)$ and $f(x,t,u)$ are bounded measurable functions with respect to $(x,t) \in Q_T$; for any fixed $(x,t) \in Q_T$, $B(x,t,u)$ and $f(x,t,u)$ are continuous with respect to $u \in R^m$; for any fixed $x \in [0,l]$, $A(x,t)$ is $m \times m$ symmetric matrix and is Lipschtiz continuous with respect to $t \in [0,T]$; and $|A(x,t)| \leq A_0$, where A_0 is a constant; and there are constants $A_1 > 0$, $B_0 > 0, C > 0$ such that $|A_t(x,t)| \leq A_1$, $|B(x,t,u)| \leq B_0$, $|f(x,t,u)| \leq |f(x,t,0)| + C|u|$.

(II) There is a constant $\sigma_0 > 0$, such that, for any vector $\xi \in R^m$, and for $(x,t) \in Q_T$,

$$(\xi, A(x,t)\xi) \geq \sigma_0 |\xi|^2$$

(III) The initial value m-dimensional vector function $\varphi(x) \in H_0^1(0,l)$.

Let us divide the rectangular domain Q_T into small grids by the parallel lines $x = x_j$ $(j = 0,1,\cdots,J)$ and $t = t^n$ $(n = 0,1,\cdots,N)$ with $x_j = jh$ and $t^n = n\tau$, where $Jh = l$ and $N\tau = T$, J and N are integers, and h and τ are steplengths of the grids. Denote $Q_j^n = \{x_j < x \le x_{j+1},\ t^n < t \le t^{n+1}\}$, where $j = 0,1,\cdots,J-1$; $n = 0,1,\cdots,N-1$. Denote $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\cdots,J;\ n = 0,1,\cdots,N\}$ the m-dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta = \{(x_j,t^n)|\ j = 0,1,\cdots,J; n = 0,1,\cdots,N\}$ of the grid points.

Let us now construct the finite difference scheme with intrinsic parallelism for the mentioned semilinear parabolic system (1)–(3) as follows:

$$\frac{v_j^{n+1} - v_j^n}{\tau} = \frac{1}{h}(A_j^n \delta v_j^{n+\xi_j^n} - A_{j-1}^n \delta v_{j-1}^{n+\eta_j^n}) + B_j^{n+\alpha_j^n} \bar{\delta}^1 v_j^{n+\alpha_j^n} + f_j^{n+\alpha_j^n}, \tag{4}$$
$$(j = 1,2,\cdots,J-1; n = 0,1,\cdots,N-1);$$
$$v_0^n = v_J^n = 0, \quad (n = 0,1,\cdots,N), \tag{5}$$
$$v_j^0 = \varphi_j, \quad (j = 0,1,\cdots,J), \tag{6}$$

where $\varphi_j = \varphi(x_j)$, $(j = 0,1,\cdots,J)$, and there are $\varphi_0 = \varphi_J = 0$; and

$$A_j^n = \frac{1}{h\tau}\int\int_{Q_j^n} A(x,t)\omega\Big(\frac{x - x_{j+\frac12}}{h}, \frac{t - t^{n+\frac12}}{\tau}\Big)dxdt,$$
$$B_j^{n+\alpha_j^n} = \frac{1}{h\tau}\int\int_{Q_j^n} B(x,t,\hat{\delta}^0 v_j^{n+\alpha_j^n})\omega\Big(\frac{x - x_{j+\frac12}}{h}, \frac{t - t^{n+\frac12}}{\tau}\Big)dxdt,$$
$$f_j^{n+\alpha_j^n} = \frac{1}{h\tau}\int\int_{Q_j^n} f(x,t,\tilde{\delta}^0 v_j^{n+\alpha_j^n})\omega\Big(\frac{x - x_{j+\frac12}}{h}, \frac{t - t^{n+\frac12}}{\tau}\Big)dxdt, \tag{7}$$

where $\omega(x,t) \in C_0^\infty(R^2)$, $\omega(x,t) \ge 0$, $\operatorname{supp}\omega \subset B_{\frac12} \equiv \Big\{|x| < \frac12,\ |t| < \frac12\Big\}$ and $\int\int_{R^2} \omega(x,t)\ dxdt = 1$, and $x_{j+\frac12} = \Big(j + \frac12\Big)h$, $t^{n+\frac12} = \Big(n + \frac12\Big)\tau$. If $\omega(x,t) = \chi_{B_{\frac12}}$, the results in this paper also hold. And

$$\alpha_j^n = \frac{\xi_j^n + \eta_j^n}{2}, \quad (j = 1,2,\cdots,J-1; n = 0,1,\cdots,N-1).$$

Define the difference approximations $\hat{\delta}^0 v_j^{n+\alpha_j^n}$ and $\bar{\delta}^1 v_j^{n+\alpha_j^n}$ in the following forms:

$$\hat{\delta}^0 v_j^{n+\alpha_j^n} = [\xi_j^n \hat{\beta}_{1j}^n v_{j+1}^{n+1} + \alpha_j^n \hat{\beta}_{2j}^n v_j^{n+1} + \eta_j^n \hat{\beta}_{3j}^n v_{j-1}^{n+1}] + [\hat{\beta}_{4j}^n v_{j+1}^n + \hat{\beta}_{5j}^n v_j^n + \hat{\beta}_{6j}^n v_{j-1}^n],$$
$$\bar{\delta}^1 v_j^{n+\alpha_j^n} = \Big[\xi_j^n \gamma_{1j}^n \frac{\Delta_+ v_j^{n+1}}{h} + \eta_j^n \gamma_{2j}^n \frac{\Delta_- v_j^{n+1}}{h}\Big] + \Big[\gamma_{3j}^n \frac{\Delta_+ v_j^n}{h} + \gamma_{4j}^n \frac{\Delta_- v_j^n}{h}\Big]. \tag{8}$$

Similarly we can define the difference approximations $\tilde{\delta}^0 v_j^{n+\alpha_j^n}$.

Define the piecewise constant function

$$\xi_h^\tau(x,t) = \xi_j^n, \quad \text{for} \quad (x,t) \in Q_j^n, \quad (j = 0,1,\cdots,J-1; n = 0,1,\cdots,N-1).$$

Similarly we can define the piecewise constant functions $\eta_h^\tau(x,t)$, $\hat{\beta}_{kh}^\tau(x,t)$, $\tilde{\beta}_{kh}^\tau(x,t)$, $(1 \le k \le 6), \gamma_{mh}^\tau(x,t)$, $(1 \le m \le 4)$, $\alpha_h^\tau(x,t)$. For these functions, we assume the following condition holds.

(IV) The piecewise constant functions defined above are bounded functions uniformly with repsect to h and τ. There are functions $\hat{\beta}_k(x,t), \tilde{\beta}_k(x,t) \in L^\infty(Q_T)$, $(1 \le k \le 6)$, and $\gamma_m(x,t) \in L^\infty(Q_T)$, $(1 \le m \le 4)$, such that, as $h \to 0, \tau \to 0$, there hold

$$\begin{aligned}
&\|\xi_h^\tau(x,t) - \xi(x,t)\|_{L^2(Q_T)} + \|\eta_h^\tau(x,t) - \eta(x,t)\|_{L^2(Q_T)} \to 0,\\
&\sum_{k=1}^{6} \|\hat{\beta}_{kh}^\tau(x,t) - \hat{\beta}_k(x,t)\|_{L^2(Q_T)} \to 0, \quad \sum_{k=1}^{6} \|\tilde{\beta}_{kh}^\tau(x,t) - \tilde{\beta}_k(x,t)\|_{L^2(Q_T)} \to 0,\\
&\sum_{m=1}^{4} \|\gamma_{mh}^\tau(x,t) - \gamma_m(x,t)\|_{L^2(Q_T)} \to 0, \quad \alpha_h^\tau = \frac{\xi_h^\tau + \eta_h^\tau}{2},\\
&\xi_h^\tau \hat{\beta}_{1h}^\tau + \alpha_h^\tau \hat{\beta}_{2h}^\tau + \eta_h^\tau \hat{\beta}_{3h}^\tau + \hat{\beta}_{4h}^\tau + \hat{\beta}_{5h}^\tau + \hat{\beta}_{6h}^\tau \equiv 1,\\
&|\xi_h^\tau \hat{\beta}_{1h}^\tau| + |\alpha_h^\tau \hat{\beta}_{2h}^\tau| + |\eta_h^\tau \hat{\beta}_{3h}^\tau| + |\hat{\beta}_{4h}^\tau| + |\hat{\beta}_{5h}^\tau| + |\hat{\beta}_{6h}^\tau| \le \hat{\delta}_0,\\
&\xi_h^\tau \tilde{\beta}_{1h}^\tau + \alpha_h^\tau \tilde{\beta}_{2h}^\tau + \eta_h^\tau \tilde{\beta}_{3h}^\tau + \tilde{\beta}_{4h}^\tau + \tilde{\beta}_{5h}^\tau + \tilde{\beta}_{6h}^\tau \equiv 1,\\
&\left|\xi_h^\tau \tilde{\beta}_{1h}^\tau\right| + \left|\alpha_h^\tau \tilde{\beta}_{2h}^\tau\right| + \left|\eta_h^\tau \tilde{\beta}_{3h}^\tau\right| + \left|\tilde{\beta}_{4h}^\tau\right| + \left|\tilde{\beta}_{5h}^\tau\right| + \left|\tilde{\beta}_{6h}^\tau\right| \le \tilde{\delta}_0,\\
&\xi_h^\tau \gamma_{1h}^\tau + \eta_h^\tau \gamma_{2h}^\tau + \gamma_{3h}^\tau + \gamma_{4h}^\tau \equiv 1,\\
&|\xi_h^\tau \gamma_{1h}^\tau| + |\eta_h^\tau \gamma_{2h}^\tau| + |\gamma_{3h}^\tau| + |\gamma_{4h}^\tau| \le \gamma_0,
\end{aligned}$$

where $\hat{\delta}_0$, $\tilde{\delta}_0$ and γ_0 are given constants.

3. Existence

In this section we prove the existence of the discrete vector solutions for the finite difference system (4)–(6). Let us now at first turn to the *a priori* estimates of these solutions.

Making the scalar product of the vector $\dfrac{v_j^{n+1} - v_j^n}{\tau} h$ with the vector finite difference equation (4) and summing up the resulting products for $j = 1,2,\cdots,J-1$, we have

$$\begin{aligned}
\sum_{j=1}^{J-1} \left|\frac{v_j^{n+1} - v_j^n}{\tau}\right|^2 h = &\sum_{j=1}^{J-1} \left(\frac{v_j^{n+1} - v_j^n}{\tau}, A_j^n \delta v_j^{n+\xi_j^n} - A_{j-1}^n \delta v_{j-1}^{n+\eta_j^n}\right)\\
&+ \sum_{j=1}^{J-1} \left(\frac{v_j^{n+1} - v_j^n}{\tau}, B_j^{n+\alpha_j^n} \bar{\delta}^1 v_j^{n+\alpha_j^n} + f_j^{n+\alpha_j^n}\right) h. \qquad (9)
\end{aligned}$$

Since $A(x,t)$ is $m\times m$ symmetric matrix,

$$\sum_{j=1}^{J-1}\Big(\frac{v_j^{n+1}-v_j^n}{\tau},A_j^n\delta v_j^{n+\xi_j^n}-A_{j-1}^n\delta v_{j-1}^{n+\eta_j^n}\Big)=-\frac{1}{2\tau}\sum_{j=0}^{J-1}[(A_j^{n+1}\delta v_j^{n+1},\delta v_j^{n+1})$$
$$-(A_j^n\delta v_j^n,\delta v_j^n)]h+\frac{1}{2\tau}\sum_{j=0}^{J-1}((A_j^{n+1}-A_j^n)\delta v_j^{n+1},\delta v_j^{n+1})h$$
$$+\sum_{j=0}^{J-1}\Big(A_j^n(\delta v_j^{n+1}-\delta v_j^n),\frac{1}{2\tau}(\delta v_j^{n+1}-\delta v_j^n)h+\xi_j^n\frac{v_j^{n+1}-v_j^n}{\tau}-\eta_{j+1}^n\frac{v_{j+1}^{n+1}-v_{j+1}^n}{\tau}\Big),$$

where the last sum at the right hand of above equality is equal to

$$I\equiv\frac{\tau}{h^2}\sum_{j=0}^{J-1}\Big(A_j^n\Big(\frac{v_{j+1}^{n+1}-v_{j+1}^n}{\tau}-\frac{v_j^{n+1}-v_j^n}{\tau}\Big),\Big(\frac{1}{2}-\eta_{j+1}^n\Big)\frac{v_{j+1}^{n+1}-v_{j+1}^n}{\tau}$$
$$-\Big(\frac{1}{2}-\xi_j^n\Big)\frac{v_j^{n+1}-v_j^n}{\tau}\Big)h=\frac{\tau}{2h^2}\sum_{j=0}^{J-1}\Big\{(1-\eta_{j+1}^n-\xi_j^n)$$
$$\times\Big(A_j^n\Big(\frac{v_{j+1}^{n+1}-v_{j+1}^n}{\tau}-\frac{v_j^{n+1}-v_j^n}{\tau}\Big),\frac{v_{j+1}^{n+1}-v_{j+1}^n}{\tau}-\frac{v_j^{n+1}-v_j^n}{\tau}\Big)h$$
$$+\Big([(\xi_{j-1}^n-\eta_j^n)A_{j-1}^n-(\xi_j^n-\eta_{j+1}^n)A_j^n]\frac{v_j^{n+1}-v_j^n}{\tau},\frac{v_j^{n+1}-v_j^n}{\tau}\Big)h\Big\}$$
$$=\frac{\tau}{2h^2}\sum_{j=0}^{J-1}\Big\{(1-\eta_{j+1}^n-\xi_j^n)\Big(A_j^n\Big(\frac{v_{j+1}^{n+1}-v_{j+1}^n}{\tau}-\frac{v_j^{n+1}-v_j^n}{\tau}\Big),\frac{v_{j+1}^{n+1}-v_{j+1}^n}{\tau}$$
$$-\frac{v_j^{n+1}-v_j^n}{\tau}\Big)h+\frac{1}{2}\Big([(\xi_{j-1}^n-\eta_j^n+\xi_j^n-\eta_{j+1}^n)(A_{j-1}^n-A_j^n)$$
$$+(\xi_{j-1}^n-\eta_j^n-\xi_j^n+\eta_{j+1}^n)(A_j^n+A_{j-1}^n)]\frac{v_j^{n+1}-v_j^n}{\tau},\frac{v_j^{n+1}-v_j^n}{\tau}\Big)h\Big\}. \tag{10}$$

By using the Cauchy inequality and the condition **(I)** and the interpolation formula (see [6]), we get

$$\frac{1}{2\tau}\sum_{j=0}^{J-1}((A_j^{n+1}-A_j^n)\delta v_j^{n+1},\delta v_j^{n+1})h+\sum_{j=1}^{J-1}\Big(\frac{v_j^{n+1}-v_j^n}{\tau},B_j^{n+\alpha_j^n}\bar{\delta}^1 v_j^{n+\alpha_j^n}+f_j^{n+\alpha_j^n}\Big)h$$
$$\le\frac{A_1}{2}\|\delta v_h^{n+1}\|_2^2+\varepsilon_1\Big\|\frac{v_h^{n+1}-v_h^n}{\tau}\Big\|_2^2+\frac{C}{\varepsilon_1}(\|\delta v_h^{n+1}\|_2^2+\|\delta v_h^n\|_2^2+1). \tag{11}$$

Combining (9)–(11) gives the following inequality

$$\Big\|\frac{v_h^{n+1}-v_h^n}{\tau}\Big\|_2^2+\frac{1}{2\tau}\sum_{j=0}^{J-1}[(A_j^{n+1}\delta v_j^{n+1},\delta v_j^{n+1})-(A_j^n\delta v_j^n,\delta v_j^n)]h$$
$$\le\frac{A_1}{2}\|\delta v_h^{n+1}\|_2^2+\varepsilon_1\Big\|\frac{v_h^{n+1}-v_h^n}{\tau}\Big\|_2^2+\frac{C}{\varepsilon_1}(\|\delta v_h^{n+1}\|_2^2+\|\delta v_h^n\|_2^2+1)$$

$$+\frac{\tau}{2h^2}\Big\{\sum_{j=1}^{J-1}[2(1-\eta_j^n-\xi_{j-1}^n)^+|A_{j-1}^n|$$

$$+2(1-\eta_{j+1}^n-\xi_j^n)^+|A_j^n|]\Big|\frac{v_j^{n+1}-v_j^n}{\tau}\Big|^2h$$

$$+\frac{1}{2}\sum_{j=1}^{J-1}[|\xi_{j-1}^n-\eta_j^n+\xi_j^n-\eta_{j+1}^n||A_{j-1}^n-A_j^n|$$

$$+(\xi_{j-1}^n-\eta_j^n-\xi_j^n+\eta_{j+1}^n)^+|A_{j-1}^n+A_j^n|]\Big|\frac{v_j^{n+1}-v_j^n}{\tau}\Big|^2h\Big\}, \tag{12}$$

where ξ_0^n and η_J^n can be taken any constants, $A_0^n=A_1^n$ and $|A|$ is defined as

$$|A|=\sup_{x\in R^m}\frac{|Ax|}{|x|}.$$

Introduce the restriction condition for the choice of steplengths τ and h.

(V) Suppose that the steplengths τ and h are so chosen that, there is the relation of restriction

$$\frac{\tau}{h^2}\max_{j=1,2,\cdots,J-1}\Lambda_j\le 1-\varepsilon,$$

where $0<\varepsilon\le 1$, and Λ_j is given by

$$\Lambda_j=\Lambda_j^n=(1-\eta_j^n-\xi_{j-1}^n)^+|A_{j-1}^n|+(1-\eta_{j+1}^n-\xi_j^n)^+|A_j^n|$$
$$+\frac{1}{4}|\xi_{j-1}^n-\eta_j^n+\xi_j^n-\eta_{j+1}^n||A_{j-1}^n-A_j^n|+\frac{1}{4}(\xi_{j-1}^n-\eta_j^n-\xi_j^n+\eta_{j+1}^n)^+|A_{j-1}^n+A_j^n|.$$

From (12) and the restriction **(V)** it follows that

$$(\varepsilon-\varepsilon_1)\Big\|\frac{v_h^{n+1}-v_h^n}{\tau}\Big\|_2^2+\frac{1}{2\tau}\sum_{j=0}^{J-1}[(A_j^{n+1}\delta v_j^{n+1},\delta v_j^{n+1})-(A_j^n\delta v_j^n,\delta v_j^n)]h$$
$$\le\frac{A_1}{2}\|\delta v_h^{n+1}\|_2^2+\frac{C}{\varepsilon_1}(\|\delta v_h^{n+1}\|_2^2+\|\delta v_h^n\|_2^2+1).$$

Taking $\varepsilon_1=\dfrac{\varepsilon}{2}$ and summing up the inequalities for $n=0,1,\cdots,N-1$, we get

$$\varepsilon\sum_{n=0}^{N-1}\Big\|\frac{v_h^{n+1}-v_h^n}{\tau}\Big\|_2^2\tau+\|\delta v_h^N\|_2^2\le C\Big(\sum_{n=0}^{N}\|\delta v_h^n\|_2^2\tau+1\Big),$$

then the Gronwall inequality yields that

$$\sum_{n=0}^{N-1}\Big\|\frac{v_h^{n+1}-v_h^n}{\tau}\Big\|_2^2\tau+\max_{n=0,1,\cdots,N}\|\delta v_h^n\|_2^2\le C. \tag{13}$$

Using the interpolation formula, we obtain

$$\max_{n=0,1,\cdots,N}(\|v_h^n\|_2,\|v_h^n\|_\infty)\le C, \tag{14}$$

where C is a constant independent of the steplengths τ and h.

By means of Leray-Schauder's fixed point theorem in finite dimensional space and the estimates (13)–(14), there is the existence theorem for the discrete solution $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ of the general finite differnce scheme (4)–(6) with intrinsic parallelism corresponding to the boundary value problem (2) and (3) for the semilinear parabolic system (1) as follows:

Theorem 1. *Suppose that the conditions* **(I)**, **(II)** *and* **(III)** *are fulfilled. And assume that the steplengths τ and h satisfy the conditions of restriction* **(V)**. *Then the general finite difference scheme* (4)–(6) *with intrinsic parallelism corresponding to the original problem* (1), (2) *and* (3) *has at least one discrete solution $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N\}$.*

4. Uniqueness

To prove the uniqueness of the solution for the difference scheme (4)–(6) we further assume the following condition holds.

(I)′ $B(x,t,u), f(x,t,u)$ are locally Lipschtiz continuous with respect to $u \in R^m$.

Given the values $\{v_j^n | j = 0, 1, \cdots, J\}$ of the difference scheme (4)–(6) on the n-th layer. Let $\{v_j^{n+1} | j = 0, 1, \cdots, J\}$ and $\{\bar{v}_j^{n+1} | j = 0, 1, \cdots, J\}$ be the two solutions of the difference scheme (4)–(6) on the $(n+1)$-th layer, i.e.,

$$\begin{aligned} &\frac{v_j^{n+1} - v_j^n}{\tau} = \frac{1}{h}(A_j^n \delta v_j^{n+\xi_j^n} - A_{j-1}^n \delta v_{j-1}^{n+\eta_j^n}) + B_j^{n+\alpha_j^n} \bar{\delta}^1 v_j^{n+\alpha_j^n} + f_j^{n+\alpha_j^n}, \\ &\qquad (j = 1, 2, \cdots, J-1) \\ &v_0^{n+1} = v_J^{n+1} = 0; \end{aligned}$$

and

$$\begin{aligned} &\frac{\overline{v}_j^{n+1} - v_j^n}{\tau} = \frac{1}{h}(A_j^n \delta \bar{v}_j^{n+\xi_j^n} - A_{j-1}^n \delta \bar{v}_{j-1}^{n+\eta_j^n}) + \overline{B}_j^{n+\alpha_j^n} \bar{\delta}^1 \overline{v}_j^{n+\alpha_j^n} + \overline{f}_j^{n+\alpha_j^n}, \\ &\qquad (j = 1, 2, \cdots, J-1) \\ &\overline{v}_0^{n+1} = \overline{v}_J^{n+1} = 0, \end{aligned}$$

where $\overline{B}_j^{n+\alpha_j^n}$ and $\overline{f}_j^{n+\alpha}$ $(j = 1, 2, \cdots, J-1)$ are obtained from $B_j^{n+\alpha_j^n}$ and $f_j^{n+\alpha_j^n}$ $(j = 1, 2, \cdots, J-1)$ respectively by replacing v_j^{n+1} $(j = 0, 1, \cdots, J)$ with the corresponding $\overline{v}_j^{n+1}$ $(j = 0, 1, \cdots, J)$. The difference $w_j = v_j^{n+1} - \overline{v}_j^{n+1}$ satisfies

$$w_j = \frac{\tau}{h}(A_j^n \xi_j^n \delta w_j - A_{j-1}^n \eta_j^n \delta w_{j-1}^{n+1}) + \tau R_j^n, \quad (j = 1, 2, \cdots, J-1) \tag{15}$$

$$w_0 = w_J = 0, \tag{16}$$

where

$$R_j^n = (B_j^{n+\alpha_j^n} - \bar{B}_j^{n+\alpha_j^n}) \bar{\delta}^1 \bar{v}_j^{n+\alpha_j^n} + B_j^{n+\alpha_j^n} (\bar{\delta}^1 v_j^{n+\alpha_j^n} - \bar{\delta}^1 \overline{v}_j^{n+\alpha_j^n}) + (f_j^{n+\alpha_j^n} - \bar{f}_j^{n+\alpha_j^n}).$$

Now firstly making the scalar product of the vectors w_jh with the vector equation (15) and summing up the resulting products for $j=1,2,\cdots,J-1$, and proceeding the same calculation as that in section 3, we have

$$\begin{aligned}\|w_h\|_2^2+\frac{\tau}{2}\sum_{j=0}^{J-1}(\xi_j^n+\eta_{j+1}^n)(A_j^n\delta w_j,\delta w_j)h-\frac{\tau}{2}\sum_{j=0}^{J-1}(\xi_j^n-\eta_{j+1}^n)(A_j^n\delta w_j,w_j+w_{j+1})\\ =\tau\sum_{j=1}^{J-1}(R_j^n,w_j)h.\end{aligned}$$

It follows that

$$\begin{aligned}\|w_h\|_2^2+\frac{\tau}{2}\sum_{j=1}^{J-1}(A_j^n\delta w_j,\delta w_j)h=&\frac{\tau}{2}\sum_{j=1}^{J-1}(1-\xi_j^n-\eta_{j+1}^n)(A_j^n\delta w_j,\delta w_j)h\\ &+\frac{\tau}{2h^2}\sum_{j=0}^{J-1}(\xi_j^n-\eta_{j+1}^n)(A_j^n(w_{j+1}-w_j),w_{j+1}+w_j)h+\tau\sum_{j=1}^{J-1}(R_j^n,w_j)h\\ \le&\frac{\tau}{2h^2}\Big\{\sum_{j=1}^{J-1}[2(1-\xi_{j-1}^n-\eta_j^n)^+|A_{j-1}^n|+2(1-\xi_j^n-\eta_{j+1}^n)^+|A_j^n|]|w_j|^2h\\ &+\sum_{j=1}^{J-1}\Big[\frac{1}{2}|\xi_{j-1}^n-\eta_j^n+\xi_j^n-\eta_{j+1}^n||A_{j-1}^n-A_j^n|\\ &+\frac{1}{2}(\xi_{j-1}^n-\eta_j^n-\xi_j^n+\eta_{j+1}^n)^+(|A_{j-1}^n+A_j^n|)\Big]|w_j|^2h\Big\}+\tau\sum_{j=1}^{J-1}(R_j^n,w_j)h.\end{aligned}$$

By the restriction condition **(V)** we have

$$\varepsilon\|w_h\|_2^2+\frac{\tau}{2}\sigma_0\|\delta w_h\|_2^2\le\tau\sum_{j=1}^{J-1}(R_j^n,w_j)h. \tag{17}$$

Since

$$\begin{aligned}&|B_j^{n+\alpha_j^n}-\bar{B}_j^{n+\alpha_j^n}|\le C(|w_{j+1}|+|w_j|+|w_{j-1}|),\\ &|f_j^{n+\alpha_j^n}-\bar{f}_j^{n+\alpha_j^n}|\le C(|w_{j+1}|+|w_j|+|w_{j-1}|),\end{aligned}$$

and

$$\bar{\delta}^1 v_j^{n+\alpha_j^n}-\bar{\delta}^1\bar{v}_j^{n+\alpha_j^n}=\xi_j^n\gamma_{1j}^n\delta w_j+\eta_j^n\gamma_{2j}^n\delta w_{j-1},$$

there holds

$$\begin{aligned}\tau\sum_{j=1}^{J-1}(R_j^n,w_j)h\le&C\tau[(\|\delta\bar{v}_h^{n+1}\|_2+\|\delta v_h^n\|_2)\|w_h\|_4^2+\|\delta w_h\|_2\|w_h\|_2+\|w_h\|_2^2]\\ \le&C\tau(\varepsilon_1\|\delta w_h\|_2^2+C(\varepsilon_1)\|w_h\|_2^2),\end{aligned}$$

where the estmates (13) and the interplation inequality are used. It reduce that

$$(\varepsilon - \tau C(\varepsilon_1))\|w_h\|_2^2 + \tau\Big(\frac{\sigma_0}{2} - C\varepsilon_1\Big)\|\delta w_h\|_2^2 \le 0,$$

and then taking ε_1 and τ small, we get $w_h \equiv 0$. The uniqueness theorem is proved.

Theorem 2. *Suppose that the conditions* **(I)–(III), (V)** *and* **(I)$'$** *are satisfied. As the meshstep τ is sufficiently small, the discrete solution $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ of the difference scheme* (4)–(6) *is unique.*

5. Convergence

In this section we shall prove the convergence of the solution for difference scheme (4)–(6) with intrinsic parallelism to the genernalized solution of the problem (1)–(3) on the basis of the obtained estimates and the convergence properties of the discrete solutions $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N\}$.

From (13) it is easy to obtained the following lemma.

Lemma 1. *For the discrete solution of the difference scheme* (4)–(6)*, there are the estimates*

$$\Big(\max_{n=0,1,\cdots,N-1} \sum_{j=0}^{J-m} |v_{j+m}^{n+1} - v_j^{n+1}|^2 h\tau\Big)^{\frac{1}{2}} \le Cmh, \tag{18}$$

$$\Big(\sum_{n=0}^{N-s-1} \sum_{j=0}^{J} |v_j^{n+s+1} - v_j^{n+1}|^2 h\tau\Big)^{\frac{1}{2}} \le Cs\tau. \tag{19}$$

Let us define the piecewise constant functions

$$v_h^\tau(x,t) = v_j^n, \quad v_h^{\eta\tau}(x,t) = v_j^{n+\eta_{j+1}^n}, \quad v_{xh}^\tau(x,t) = \delta v_j^n,$$
$$v_{xh}^{\xi\tau}(x,t) = \delta v_j^{n+\xi_j^n}, \quad v_{th}^\tau(x,t) = \frac{v_j^{n+1} - v_j^n}{\tau}$$

for $(x,t) \in Q_j^n$ $(j = 0, 1, \cdots, J-1; n = 0, 1, \cdots, N-1)$.

Lemma 2. *Assume that* **(I)–(V)** *and* **(IV)** *hold. When $h \to 0, \tau \to 0$ (for some subsequence), there is a function $u(x,t) \in L^\infty(0,T; H_0^1(0,l)) \cap H^1(0,T; L^2(0,l))$ such that*

(i) $v_h^\tau(x,t) \to u(x,t)$ *strongly in* $L^2(Q_T)$ *and a.e. in* Q_T*;*
(ii) $v_h^{\eta\tau}(x,t) \to u(x,t)$ *strongly in* $L^2(Q_T)$ *and a.e. in* Q_T*;*
(iii) $v_{xh}^\tau(x,t) \to u_x(x,t)$ *weakly in* $L^2(Q_T)$*;*
(iv) $v_{xh}^{\xi\tau}(x,t) \to u_x(x,t)$ *weakly in* $L^2(Q_T)$*;*
(v) $v_{th}^\tau(x,t) \to u_t(x,t)$ *weakly in* $L^2(Q_T)$*.*

Proof. Using Lemma 1 and the well–known method in [6], we can prove (i) and (iii).

From the condition **(IV)** and the conclusion (i), we see that (ii) and (iv) hold. (v) can be proved as the usual way. Now we note that $u(x,t) \in L^\infty(0,T; H_0^1(0,l))$. Define the bilinear function: for $(x,t)\in Q_j^n$ $(j=0,1,\cdots,J-1;n=0,1,\cdots,N-1)$

$$\overset{*}{v}{}_h^\tau(x,t)=\frac{(x-x_j)(t-t^n)}{h\tau}v_{j+1}^{n+1}+\frac{(x_{j+1}-x)(t-t^n)}{h\tau}v_j^{n+1}+\frac{(x-x_j)(t^{n+1}-t)}{h\tau}v_{j+1}^{n}+\frac{(x_{j+1}-x)(t^{n+1}-t)}{h\tau}v_j^{n}.$$

Obviously there holds $\overset{*}{v}{}_h^\tau(x,t)\in L^\infty(0,T;H_0^1(0,l))$ and as $h\to,\tau\to 0$,

$$\max_{0\le t\le T}\|\overset{*}{v}{}_h^\tau-v_h^\tau\|_{L^2(0,l)}+\max_{0\le t\le T}\|(\overset{*}{v}{}_h^\tau)_x-v_{xh}^\tau\|_{L^2(0,l)}\to 0.$$

Then the conclusion $u(x,t)\in L^\infty(0,T;H_0^1(0,l))$ is true.

Define the piecewise constant functions, for $(x,t)\in Q_j^n$ $(j=0,1,\cdots,J-1;n=0,1,\cdots,N-1)$,

$$\hat v_h^\tau(x,t)=\hat\delta^0 v_j^{n+\alpha_j^n},\quad \tilde v_h^\tau(x,t)=\tilde\delta^0 v_j^{n+\alpha_j^n},\bar v_{xh}^\tau(x,t)=\bar\delta^1 v_j^{n+\alpha_j^n},$$
$$A_h^\tau(x,t)=A_j^n,\ B_h^\tau(x,t)=B_j^{n+\alpha_j^n},\quad f_h^\tau(x,t)=f_j^{n+\alpha_j^n}.$$

Lemma 3. *Assume that the same conditions as those in Lemma 2 and* **(I)**$'$ *hold. When $h\to 0,\tau\to 0$, there are*

(i) $\hat v_h^\tau(x,t)\to u(x,t)$ *and* $\tilde v_h^\tau(x,t)\to u(x,t)$ *strongly in* $L^2(Q_T)$ *and a.e. in* Q_T*;*

(ii) $\bar v_{xh}^\tau(x,t)\to u_x(x,t)$ *weakly in* $L^2(Q_T)$*;*

(iii) $A_h^\tau(x,t)\to A(x,t)$, $B_h^\tau(x,t)\to B(x,t,u(x,t))$ *and* $f_h^\tau(x,t)\to f(x,t,u(x,t))$ *strongly in* $L^2(Q_T)$ *and a.e. in* Q_T.

Proof. (i) and (ii) can be proved easily by using Lemma 2 (i) and (ii) and the condition **(IV)**. Now we prove (iii). Note that $B(x,t,u(x,t))\in L^\infty(Q_T)$. There are

$$\begin{aligned}
&\|B_h^\tau(x,t)-B(x,t,u(x,t))\|^2_{L^2(Q_T)}\\
&=\sum_{n=0}^{N-1}\sum_{j=0}^{J-1}\int\int_{Q_j^n}\Big|\frac{1}{h\tau}\int\int_{Q_j^n}(B(y,s,\hat\delta^0 v_j^{n+\alpha_j^n})-B(x,t,u(x,t)))\\
&\quad\times\omega\Big(\frac{y-x_{j+\frac12}}{h},\frac{s-t^{n+\frac12}}{\tau}\Big)dyds\Big|^2dxdt\\
&\le\sum_{n=0}^{N-1}\sum_{j=0}^{J-1}\frac{1}{h\tau}\int_{|x'|\le h}\int_{|t'|\le\tau}dx'dt'\\
&\quad\times\int\int_{Q_j^n}|B(x+x',t+t',\hat\delta^0 v_j^{n+\alpha_j^n})-B(x,t,u(x,t))|^2dxdt\\
&\le 2\max_{|x'|\le h,|t'|\le\tau}\int\int_{Q_T}|B(x+x',t+t',\hat v_h^\tau(x,t))\\
&\quad-B(x+x',t+t',u(x+x',t+t'))|^2dxdt\\
&\quad+2\max_{|x'|\le h,|t'|\le\tau}\int\int_{Q_T}|B(x+x',t+t',u(x+x',t+t'))-B(x,t,u(x,t))|^2dxdt\\
&\equiv I_1+I_2.
\end{aligned}$$

From $(\mathbf{I})'$ and the assertion (i), it follows that $\lim_{h\to 0,\tau\to 0} I_1 = 0$; and since $B(x,t,u(x,t)) \in L^\infty(Q_T) \subset L^2(Q_T)$, the continuity of translation of functions in $L^2(Q_T)$ yields that $\lim_{h\to 0,\tau\to 0} I_2 = 0$. Then we have proved that $B_h^\tau(x,t) \to B(x,t,u(x,t))$ strongly in $L^2(Q_T)$ and a.e. in Q_T. The other assertions in (iii) can be proved by the same way.

Let $\Phi(x,t) \in C^\infty(Q_T)$ and $\Phi(x,t) = 0$ near $x = 0$ and $x = l$. Denote $\Phi_j^n = \Phi(x_j, t^n)$. Define the piecewise constant functions $\Phi_h^\tau(x,t) = \Phi_j^n, \Phi_{xh}^\tau(x,t) = \delta\Phi_j^n$, for $(x,t) \in Q_j^n$. Assume that

$(\mathbf{V})'$ for any given constant $M > 0$, $\dfrac{\tau}{h^2} \le M$.

There holds

$$\begin{aligned}
&\int\!\!\int_{Q_T} [A_h^\tau(x,t) v_{xh}^{\xi\tau} \Phi_{xh}^\tau + (v_{th}^\tau(x,t) - B_h^\tau(x,t)\bar{v}_{xh}^\tau - f_h^\tau(x,t))\Phi_h^\tau(x,t)]dxdt \\
&\quad= \sum_{n=0}^{N-1}\sum_{j=1}^{J-1} \Big[A_j^n \delta v_j^{n+\xi_j^n} \delta\Phi_j^n + \Big(\frac{v_j^{n+1} - v_j^n}{\tau} - B_j^{n+\alpha_j^n} \bar{\delta}^1 v_j^{n+\alpha_j^n} - f_j^{n+\alpha_j^n}\Big)\Phi_j^n\Big] h\tau \\
&\quad= \sum_{n=0}^{N-1}\sum_{j=1}^{J-1} \Big[\frac{v_j^{n+1} - v_j^n}{\tau} - \frac{1}{h}(A_j^n \delta v_j^{n+\xi_j^n} - A_{j-1}^n \delta v_{j-1}^{n+\eta_j^n}) \\
&\qquad - B_j^{n+\alpha_j^n} \bar{\delta}^1 v_j^{n+\alpha_j^n} - f_j^{n+\alpha_j^n}\Big]\Phi_j^n h\tau + I,
\end{aligned}$$

where

$$\begin{aligned}
I &= \sum_{n=0}^{N-1}\sum_{j=0}^{J-1} A_j^n \Phi_{j+1}^n (\xi_j^n - \eta_{j+1}^n)(\delta v_j^{n+1} - \delta v_j^n)\tau \\
&= \frac{\tau}{h^2} \sum_{n=0}^{N-1}\sum_{j=0}^{J-1} [(\xi_{j-1}^n - \eta_j^n) A_{j-1}^n \Phi_j^n - (\xi_j^n - \eta_{j+1}^n) A_j^n \Phi_{j+1}^n] \frac{v_j^{n+1} - v_j^n}{\tau} h\tau \\
&\le \frac{\tau}{h^2} \Big(\sum_{n=0}^{N-1}\sum_{j=0}^{J-1} [(\xi_{j-1}^n - \eta_j^n) A_{j-1}^n \Phi_j^n - (\xi_j^n - \eta_{j+1}^n) A_j^n \Phi_{j+1}^n]^2 h\tau\Big)^{\frac{1}{2}} \Big\| \frac{v_h^{n+1} - v_h^n}{\tau} \Big\| \to 0,
\end{aligned}$$

where we have used the assumption $(\mathbf{V})'$ and as $h \to 0, \tau \to 0$,

$$\begin{aligned}
&\Big(\sum_{n=0}^{N-1}\sum_{j=1}^{J-1} (\xi_j^n - \xi_{j-1}^n)^2 h\tau\Big)^{\frac{1}{2}} \to 0, \quad \Big(\sum_{n=0}^{N-1}\sum_{j=1}^{J-1} (\eta_{j+1}^n - \eta_j^n)^2 h\tau\Big)^{\frac{1}{2}} \to 0, \\
&\Big(\sum_{n=0}^{N-1}\sum_{j=1}^{J-1} (A_j^n - A_{j-1}^n)^2 h\tau\Big)^{\frac{1}{2}} \to 0, \quad \Big(\sum_{n=0}^{N-1}\sum_{j=1}^{J-1} (\Phi_{j+1}^n - \Phi_j^n)^2 h\tau\Big)^{\frac{1}{2}} \to 0,
\end{aligned}$$

and

$$\Big(\sum_{n=0}^{N-1}\sum_{j=1}^{J-1} |\xi_{j-1}^n|^2 h\tau\Big)^{\frac{1}{2}} \le C(1 + \|\xi\|_{L^2(Q_T)}), \quad \Big(\sum_{n=0}^{N-1}\sum_{j=1}^{J-1} |\eta_j^n|^2 h\tau\Big)^{\frac{1}{2}} \le C(1 + \|\eta\|_{L^2(Q_T)}).$$

When letting $h \to 0, \tau \to 0$ (for some subsequences), we get, for any smooth test function $\Phi(x,t)$,

$$\int\!\!\int_{Q_T} [A(x,t)u_x(x,t)\Phi_x(x,t) + (u_t(x,t) - B(x,t,u)u_x(x,t) - f(x,t,u))\Phi(x,t)]dxdt = 0.$$

This means that the m-dimensional vector function $u(x,t) \in L^\infty(0,T;H_0^1(0,l)) \cap H^1(0,T;L^2(0,l))$ satisfies the semilinear parabolic system (1) of partial differential equations the homogeneous boundary conditions (2) and the initial condition (3) in a generalized sense.

The uniqueness of the generalized solution for the problem (1)–(3) can be justified by usual way. By means of the uniqueness of the generalized solution of the homogeneous boundary problem (1)–(3), we then can obtain the convergence theorem for the finite difference schem (4)–(6) with intrinsic parallelism as follows:

Theorem 3. *Under the conditions* **(I)**–**(V)***,* **(I)**$'$ *and* **(V)**$'$*, as the meshsteps* h *and* τ *tend to zero, the* m*-dimensional discrete vector solution* $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\cdots,J;\ n = 0,1,\cdots,N\}$ *of the finite difference scheme* (4)–(6) *with intrinsic parallelism converges to the unique generalized solution* $u(x,t) \in L^\infty(0,T;H_0^1(0,l)) \cap H^1(0,T;L^2(0,l))$ *of the boundary problem* (3) *and* (4) *for the semilinear parabolic system* (1) *of partial differential equations.*

6. Some Practical Schemes with Intrinsic Parallelism

Here analogous to the method in [5] we construct some the difference schemes with intrinsic parallelism satisfying **(V)** and **(V)**$'$. The time steplength for these difference schemes can be taken at least $8k$ times the time steplength for the fully explicit finite difference schemes (k can be any positive integer), if the coefficient matrix $A(x,t)$ is piecewise smooth, and the interface of the discrete subdomains for the suitable constructed schemes with intrinsic parallelism is not at the discontinuity points of the matrix function $A(x,t)$.

The discrete domain of the difference scheme is decomposed into two types of discrete segments "$\mathcal{AB}$" and "$\mathcal{BA}$" alternatively. On "$\mathcal{BA}$" we define

$$\begin{pmatrix}\xi_j^n\\ \eta_j^n\end{pmatrix}=\begin{pmatrix}\xi_j\\ \eta_j\end{pmatrix}=\Bigg(\overbrace{\begin{matrix}1 & 1 & \frac{2k-1}{2k} & \frac{2k-1}{2k} & \cdots & \frac{k+1}{2k} & \frac{k+1}{2k}\\ 0 & 0 & \frac{1}{2k} & \frac{1}{2k} & \cdots & \frac{k-1}{2k} & \frac{k-1}{2k}\end{matrix}}^{2k}\overbrace{\begin{matrix}\frac{1}{2} & \cdots & \frac{1}{2}\\ \frac{1}{2} & \cdots & \frac{1}{2}\end{matrix}}^{l}$$

$$\overbrace{\begin{matrix}\frac{k-1}{2k} & \frac{k-1}{2k} & \frac{k-2}{2k} & \frac{k-2}{2k} & \cdots & \frac{1}{2k} & \frac{1}{2k}\\ \frac{k+1}{2k} & \frac{k+1}{2k} & \frac{k+2}{2k} & \frac{k+2}{2k} & \cdots & \frac{2k-1}{2k} & \frac{2k-1}{2k}\end{matrix}}^{2k}\begin{matrix}0 & 0\\ 1 & 1\end{matrix}\Bigg).$$

On "$\mathcal{AB}$" we define

$$\begin{pmatrix}\xi_j^n\\ \eta_j^n\end{pmatrix}=\begin{pmatrix}\xi_j\\ \eta_j\end{pmatrix}=\Bigg(\overbrace{\begin{matrix}0 & 0 & \frac{1}{2k} & \frac{1}{2k} & \cdots & \frac{k-1}{2k} & \frac{k-1}{2k}\\ 1 & 1 & \frac{2k-1}{2k} & \frac{2k-1}{2k} & \cdots & \frac{k+1}{2k} & \frac{k+1}{2k}\end{matrix}}^{2k}\overbrace{\begin{matrix}\frac{1}{2} & \cdots & \frac{1}{2}\\ \frac{1}{2} & \cdots & \frac{1}{2}\end{matrix}}^{l}$$

$$\left.\begin{matrix} \overbrace{\frac{k+1}{2k} \quad \frac{k+1}{2k} \quad \frac{k+2}{2k} \quad \frac{k+2}{2k} \quad \cdots \quad \frac{2k-1}{2k} \quad \frac{2k-1}{2k}}^{2k} & 1 & 1 \\ \frac{k-1}{2k} \quad \frac{k-1}{2k} \quad \frac{k-2}{2k} \quad \frac{k-2}{2k} \quad \cdots \quad \frac{1}{2k} \quad \frac{1}{2k} & 0 & 0 \end{matrix}\right).$$

The discontinuous points of $A(x,t)$ are assumed to be in the interior of the sub-domains where $\xi_j = \eta_j = \dfrac{1}{2}$.

7. Multi-Dimensional Problems

The results obtained in sections 2–6 can be generalized to the boundary value problems for the multi-dimensional parabolic systems. For simplicity, we briefly describe only the results about the two dimensional problems.

Consider the boundary value problems for the two dimensional semilinear parabolic systems

$$u_t = (A(x,y,t)u_x)_x + (A(x,y,t)u_y)_y + B(x,y,t)u_x + C(x,y,t)u_y + f(x,y,t,u) \tag{20}$$

$$u(x,y,t) = u(x,y,t) = 0, \quad \text{on} \quad \partial\Omega \times [0,T] \tag{21}$$

$$u(x,y,0) = \varphi(x,y), \quad \text{on} \quad \Omega \tag{22}$$

where $u(x,y,t) = (u_1(x,y,t), \cdots, u_m(x,y,t))$ is the m-dimensional vector unknown function $(m \geq 1)$, $u_t = \dfrac{\partial u}{\partial t}$, $u_x = \dfrac{\partial u}{\partial x}$, $u_y = \dfrac{\partial u}{\partial y}$ are the corresponding vector derivatives. The matrix $A(x,y,t)$ is $m \times m$ positive definite coefficient matrix, $B(x,y,t)$ and $C(x,y,t)$ are the $m \times m$ matrices, and $f(x,y,t,u)$ and $\varphi(x,y)$ are given m–dimensional vector functions; $\Omega = \{0 \leq x \leq l_1, 0 \leq y \leq l_2\}$, $Q_T = \Omega \times [0,T]$.

Suppose that the following conditions are fulfilled.

($\mathbf{I}_2$) $A(x,y,t), B(x,y,t)$ and $C(x,y,t)$ and $f(x,y,t,u)$ (for any fixed $u \in R^m$) are bounded measurable functions with respect to $(x,y,t) \in Q_T$; for any fixed $(x,y,t) \in Q_T$, $f(x,y,t,u)$ is Lipschtiz continuous with respect to $u \in R^m$; $A(x,y,t)$ is $m \times m$ symmetric matrix and is Lipschtiz continuous with respect to $t \in [0,T]$ for any fixed $(x,y) \in \Omega$; and $|A(x,y,t)| \leq A_0$, where A_0 is a constant; and there are constants $A_1 > 0$, $B_0 > 0, C_0 > 0, C > 0$ such that $|A_t(x,y,t)| \leq A_1$, $|B(x,y,t)| \leq B_0$, $|C(x,y,t)| \leq C_0$, $|f(x,y,t,u)| \leq |f(x,y,t,0)| + C|u|$.

($\mathbf{II}_2$) There is a constant $\sigma_0 > 0$, such that, for any vector $\xi \in R^m$, and for $(x,y,t) \in Q_T$,

$$(\xi, A(x,y,t)\xi) \geq \sigma_0 |\xi|^2.$$

($\mathbf{III}_2$) The initial value m-dimensional vector function $\varphi(x,y) \in H_0^1(\Omega)$.

Let us divide the rectangular domain Q_T into small grids by the parallel lines $x = x_i, y = y_j$ $(i = 0,1,\cdots,I;\ j = 0,1,\cdots,J)$ and $t = t^n$ $(n = 0,1,\cdots,N)$ with $x_i = ih_1$, $y_j = jh_2$ and $t^n = n\tau$, where $Ih_1 = l_1$, $Jh_2 = l_2$ and $N\tau = T$, I, J and N are integers, and h_1, h_2 and τ are steplengths of the grids. Denote $Q_{ij}^n = \{x_i < x \le x_{i+1}, y_j < y \le y_{j+1}, t^n < t \le t^{n+1}\}$, where $i = 0, 1, \cdots, I-1; j = 0, 1, \cdots, J-1;\ n = 0, 1, \cdots, N-1$. Denote $v_\Delta = v_{h_1h_2}^\tau = \{v_{ij}^n | i = 0, 1, \cdots, I;\ j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ the m-dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta = \{(x_i, y_j, t^n) | i = 0, 1, \cdots, I;\ j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ of the grid points.

Construct the finite difference scheme with intrinsic parallelism for the problem (20)–(22) as follows:

$$\begin{aligned} \frac{v_{ij}^{n+1} - v_{ij}^n}{\tau} &= \frac{1}{h_1}(A_{ij}^n \delta_x v_{ij}^{n+\xi_{ij}^n} - A_{i-1j}^n \delta_x v_{i-1j}^{n+\eta_{ij}^n}) + \frac{1}{h_2}(A_{ij}^n \delta_y v_{ij}^{n+\dot{\xi}_{ij}^n} - A_{ij-1}^n \delta_y v_{ij-1}^{n+\dot{\eta}_{ij}^n}) \\ &\quad + B_{ij}^n \tilde{\delta}_x^1 v_{ij}^{n+\bar{\alpha}_{ij}^n} + C_{ij}^n \tilde{\delta}_y^1 v_{ij}^{n+\bar{\alpha}_{ij}^n} + f_{ij}^{n+\bar{\alpha}_{ij}^n}, \qquad (23) \\ &\qquad (i = 1, 2, \cdots, I-1; j = 1, 2, \cdots, J-1; n = 0, 1, \cdots, N-1); \end{aligned}$$

$$v_{0j}^n = v_{Ij}^n = v_{i0}^n = v_{iJ}^n = 0, \quad (i = 0, 1, \cdots, I; j = 0, 1, \cdots, J; n = 0, 1, \cdots, N), \qquad (24)$$

$$v_{ij}^0 = \varphi_{ij}, \quad (i = 0, 1, \cdots, I; j = 0, 1, \cdots, J), \qquad (25)$$

where $\varphi_{0j} = \varphi_{Ij} = \varphi_{i0} = \varphi_{iJ} = 0$, $(i = 0, 1, \cdots, I; j = 0, 1, \cdots, J)$,

$$\varphi_{ij} = \frac{1}{h_1 h_2} \int_\Omega \varphi(x, y) \bar{\omega}\Big(\frac{x - x_{i+\frac{1}{2}}}{h_1}, \frac{y - y_{i+\frac{1}{2}}}{h_2}\Big) dxdy,$$

where $\bar{\omega}(x, y) \in C_0^\infty(R^2), \bar{\omega}(x, y) \ge 0$, $\operatorname{supp} \bar{\omega} \subset B_{\frac{1}{2}} \equiv \Big\{|x| < \frac{1}{2},\ |y| < \frac{1}{2}\Big\}$, and $\int_{R^2} \bar{\omega}(x, y) dxdy = 1$. And

$$A_{ij}^n = \frac{1}{h_1 h_2 \tau} \int_{Q_{ij}^n} A(x, y, t) \omega\Big(\frac{x - x_{i+\frac{1}{2}}}{h_1}, \frac{y - y_{i+\frac{1}{2}}}{h_2}, \frac{t - t^{n+\frac{1}{2}}}{\tau}\Big) dxdydt,$$

$$f_{ij}^{n+\bar{\alpha}_{ij}^n} = \frac{1}{h_1 h_2 \tau} \int_{Q_{ij}^n} f(x, y, t, \bar{\delta}^0 v_{ij}^{n+\bar{\alpha}_{ij}^n}) \omega\Big(\frac{x - x_{i+\frac{1}{2}}}{h_1}, \frac{y - y_{i+\frac{1}{2}}}{h_2}, \frac{t - t^{n+\frac{1}{2}}}{\tau}\Big) dxdydt,$$

where B_{ij}^n and C_{ij}^n are defined similarly; and $\omega(x, y, t) \in C_0^\infty(R^3)$, $\omega(x, y, t) \ge 0$, $\operatorname{supp} \omega \subset B_{\frac{1}{2}} \equiv \Big\{|x| < \frac{1}{2}, |y| < \frac{1}{2}, |t| < \frac{1}{2}\Big\}$ and $\int_{R^3} \omega(x, t) dxdydt = 1$, and $x_{i+\frac{1}{2}} = \Big(i + \frac{1}{2}\Big) h_1$, $y_{j+\frac{1}{2}} = \Big(j + \frac{1}{2}\Big) h_2$, $t^{n+\frac{1}{2}} = \Big(n + \frac{1}{2}\Big)\tau$. If $\omega(x, y, t) = \chi_{B_{\frac{1}{2}}}$, also the results stated here hold.

The expression $\bar{\delta}^0 v_{ij}^{n+\bar{\alpha}_{ij}^n}$ can be taken as in the following ways:

$$\begin{aligned} \bar{\delta}^0 v_{ij}^{n+\bar{\alpha}_{ij}^n} =& \xi_{ij}^n \bar{\beta}_{1ij}^n v_{i+1j}^{n+1} + \alpha_{ij}^n \bar{\beta}_{2ij}^n v_{ij}^{n+1} + \eta_{ij}^n \bar{\beta}_{3ij}^n v_{i-1j}^{n+1} \\ &+ \dot{\xi}_{ij}^n \bar{\beta}_{4ij}^n v_{ij+1}^{n+1} + \dot{\alpha}_{ij}^n \bar{\beta}_{5ij}^n v_{ij}^{n+1} + \dot{\eta}_{ij}^n \bar{\beta}_{6ij}^n v_{ij-1}^{n+1} + \sum_{\bar{i}, \bar{j} = \pm 1, 0} \bar{\beta}_{ij, \bar{i}\bar{j}}^n v_{i+\bar{i}, j+\bar{j}}^n, \end{aligned}$$

where

$$\alpha_{ij}^n = \frac{\xi_{ij}^n + \eta_{ij}^n}{2}, \quad \dot{\alpha}_{ij}^n = \frac{\dot{\xi}_{ij}^n + \dot{\eta}_{ij}^n}{2}.$$

For the expressions $\tilde{\delta}_x^1 v_{ij}^{n+\bar{\alpha}_{ij}^n}$ and $\tilde{\delta}_y^1 v_{ij}^{n+\bar{\alpha}_{ij}^n}$ we can take

$$\begin{aligned}\tilde{\delta}_x^1 v_{ij}^{n+\bar{\alpha}_{ij}^n} &= \xi_{ij}^n \gamma_{1ij}^n \delta_x v_{ij}^{n+1} + \eta_{ij}^n \gamma_{2ij}^n \delta_x v_{i-1j}^{n+1} + \gamma_{3ij}^n \delta_x v_{ij}^n + \gamma_{4ij}^n \delta_x v_{i-1j}^n, \\ \tilde{\delta}_y^1 v_{ij}^{n+\bar{\alpha}_{ij}^n} &= \dot{\xi}_{ij}^n \dot{\gamma}_{1ij}^n \delta_y v_{ij}^{n+1} + \dot{\eta}_{ij}^n \dot{\gamma}_{2ij}^n \delta_y v_{ij-1}^{n+1} + \dot{\gamma}_{3ij}^n \delta_y v_{ij}^n + \dot{\gamma}_{4ij}^n \delta_x v_{ij-1}^n.\end{aligned}$$

Define the piecewise constant function $\xi_{h_1h_2}^\tau(x,y) = \xi_{ij}$ for $x_i < x < x_{i+1}$, $y_j < y < y_{j+1}$, $(i = 0, 1, \cdots, I-1; j = 0, 1, \cdots, J-1)$. By $(\mathbf{III}_2)$ and simple computation, we can obtain $\|\varphi_{h_1h_2}(x,y) - \varphi(x,y)\|_{L^2(\Omega)} \to 0$, as $h_1 \to 0, h_2 \to 0$, and $\|\delta_x \varphi_{h_1h_2}\|_2 \leq C\|\varphi_x(x,y)\|_{L^2(\Omega)}$, $\|\delta_y \varphi_{h_1h_2}\|_2 \leq C\|\varphi_y(x,y)\|_{L^2(\Omega)}$. Moreover, there are $\|\delta_x \varphi_{h_1h_2}(x,y) - \varphi_x(x,y)\|_{L^2(\Omega)} \to 0$, $\|\delta_y \varphi_{h_1h_2}(x,y) - \varphi_y(x,y)\|_{L^2(\Omega)} \to 0$, as $h_1 \to 0, h_2 \to 0$.

Define the piecewise constant function $\xi_{h_1h_2}^\tau(x,y,t) = \xi_{ij}^n$, for $(x,y,t) \in Q_{ij}^n$, $(i = 0, 1, \cdots, I-1; j = 0, 1, \cdots, J-1; n = 0, 1, \cdots, N-1)$. Similarly we can define the piecewise constant functions $\eta_{h_1h_2}^\tau(x,y,t), \bar{\beta}_{kh_1h_2}^\tau(x,y,t)$, $(1 \leq k \leq 6)$, $\bar{\beta}_{h_1h_2,\bar{i}\bar{j}}^\tau(x,y,t)$, $(\bar{i}, \bar{j} = \pm 1, 0)$, $\gamma_{mh_1h_2}^\tau(x,y,t), (1 \leq m \leq 4)$, $\alpha_{h_1h_2}^\tau(x,y,t) = \frac{\xi_{h_1h_2}^\tau + \eta_{h_1h_2}^\tau}{2}$, $\dot{\gamma}_{mh_1h_2}^\tau(x,y,t), (1 \leq m \leq 4)$, $\dot{\alpha}_{h_1h_2}^\tau(x,y,t), \dot{\xi}_{h_1h_2}^\tau, \dot{\eta}_{h_1h_2}^\tau$.

For these functions above, we assume the following condition holds.

(IV$_2$) The piecewise constant functions defined above are bounded functions uniformly with repsect to h_1, h_2 and τ. And as $h_1 \to 0, h_2 \to 0, \tau \to 0$, these piecewise constant functions converge strongly in $L^2(Q_T)$.

Remark. Note that in [1] and [2], no such condition **(IV$_2$)** is imposed, where the known data are assumed to be smooth. Here they are weaken to be bounded measurable. The assumption **(IV$_2$)** excludes those "morbid" functions appearing in the construction of the difference schemes (23)–(25). But, it is not a severe restriction, since almost all the well-known difference schemes with intrinsic parallelism, in particular, those discussed in [5], satisfy the condition.

Introduce the restriction condition of the steplengths:

$$(\mathbf{V}_2) \qquad \tau\Big(\frac{1}{h_1^2} + \frac{1}{h_2^2}\Big) \max_{1 \leq i \leq I-1; 1 \leq j \leq J-1} \Lambda_{ij} \leq 1 - \varepsilon,$$

where $0 < \varepsilon \leq 1$, and

$$\begin{aligned}\Lambda_{ij} =& (1 - \eta_{ij}^n - \xi_{i-1j}^n)^+ |A_{i-1j}^n| + (1 - \eta_{i+1j}^n - \xi_{ij}^n)^+ |A_{ij}^n| \\ &+ \frac{1}{4}|\xi_{i-1j}^n - \eta_{ij}^n + \xi_{ij}^n - \eta_{i+1j}^n||A_{i-1j}^n - A_{ij}^n| \\ &+ \frac{1}{4}|\xi_{i-1j}^n - \eta_{ij}^n - \xi_{ij}^n + \eta_{i+1j}^n||A_{i-1j}^n + A_{ij}^n| \\ &+ (1 - \dot{\eta}_{ij}^n - \dot{\xi}_{ij-1}^n)^+ |A_{ij-1}^n| + (1 - \dot{\eta}_{ij+1}^n - \dot{\xi}_{ij}^n)^+ |A_{ij}^n| \\ &+ \frac{1}{4}|\dot{\xi}_{ij-1}^n - \dot{\eta}_{ij}^n + \dot{\xi}_{ij}^n - \dot{\eta}_{ij+1}^n||A_{ij-1}^n - A_{ij}^n| \\ &+ \frac{1}{4}|\dot{\xi}_{ij-1}^n - \dot{\eta}_{ij}^n - \dot{\xi}_{ij}^n + \dot{\eta}_{ij+1}^n||A_{ij-1}^n + A_{ij}^n|.\end{aligned}$$

Under the conditions $(\mathbf{I}_2)$–$(\mathbf{V}_2)$, the existence, uniqueness and convergence theorems for the difference schemes (23)–(25) can be proved. Moreover, when $A(x,y,t)$ is piecewise smooth and the discontinuity surface of $A(x, y, t)$ satisfies certain necessary confinement, by choicing suitably the parameters (i.e., ξ's and η's) we can get some practical difference schemes (see [4] and [5]). The construction principle of the schemes is similar to that in the section 6.

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