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STABILITY ANALYSIS OF FINITE ELEMENT METHODS FOR THE ACOUSTIC WAVE EQUATION WITH ABSORBING BOUNDARY CONDITIONS (PART I)[*1)]

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Abstract

In Part I and Part II of this paper initial-boundary value problems of the acoustic wave equation with absorbing boundary conditions are considered. Their finite element-finite difference computational schemes are proposed. The stability of the schemes is discussed and the corresponding stability conditions are given. Part I and Part II concern the first- and the second-order absorbing boundary conditions, respectively. Finally, numerical results are presented in Part II to show the correctness of theoretical analysis.

Key words: Stability, Finite element methods, Wave equation, Absorbing boundary conditions

1. Introduction

In the numerical simulation of wave propagation in unbounded or semi-unbounded medium it is necessary to introduce artificial boundaries to obtain finite computational regions. Then some boundary conditions have to be imposed on these boundaries, which should eliminate the reflection of waves at artificial boundaries, so that the obtained solutions rather accurately simulate the solutions in the unbounded domains. (That is why they are called absorbing boundary conditions). The conditions on the artificial boundaries should also guarantee the well-posedness of solutions to the differential equations, which is a necessary condition for the stability of the finite difference or the finite element approximations.

In recent thirty years, a variety of absorbing boundary conditions for wave equations have been developed (see [1]). What is most widely used was given by Clayton and Engquist[2], Engquist and Majda [3,4], based on the pseudodifferential operator theory. A hierarchy of differential boundary conditions was derived to approximate the boundary conditions of the pseudodifferential operator forms. Let the artificial boundary be $x = 0$, and the domain be $t \geq 0, x \leq 0$. For the acoustic wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.1}$$

the mentioned conditions are the followings:

$$\mathcal{B}_1 u|_{x=0} = \Big(\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\Big)\Big|_{x=0} = 0,$$

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$$\mathcal{B}_2 u|_{x=0} = \Big(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial t \partial x} - \frac{1}{2}\frac{\partial^2 u}{\partial y^2}\Big)\Big|_{x=0} = 0, \tag{1.2}$$

$$\mathcal{B}_{N+1} u|_{x=0} = \Big(\frac{\partial}{\partial t}\mathcal{B}_N u - \frac{1}{4}\frac{\partial^2}{\partial y^2}\mathcal{B}_{N-1} u\Big)\Big|_{x=0} = 0.$$

The corresponding conditions for the elastic wave equations are complicated, and we are not going to write them here.

In [3], the well-posedness of (1.2) (i.e., the Clayton-Engquist-Majda conditions for the acoustic wave equation) when $N \le 3$ has been proved. In [5], the authors of this paper generalize (1.2) to the anisotropic elastic wave equations and have proved that the Clayton-Engquist-Majda conditions for the elastic wave equations are ill posed when $N \ge 2$.

In this paper, only the acoustic wave equation

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{C^2(x,y)}\frac{\partial^2 u}{\partial t^2} = f(x,y,t) \tag{1.3}$$

is discussed. But some conclusions are significant also for other wave equations.

In numerical computations, the equation (1.3) with absorbing boundary conditions is approximated usually by finite difference schemes, and seldom by finite element approaches. The author of [6] affirmed that the main difficulty comes from the order of the boundary conditions for which it is not easy to derive a weak formulation which provides a suitable energy estimate. In [6], therefore, a third-order energy is introduced, and a first-order hyperbolic system of 7 unknowns is derived, for which finite element methods can be applied. Obviously, this approach is not desirable for practical computation.

In this paper, finite element-finite difference schemes for the equation (1.3) with the first and second order absorbing boundary conditions of (1.2) are proposed. Their stability is discussed, and the stability conditions are given. The Part I is devoted to the first order absorbing boundary condition, and the Part II to the second order boundary condition. The numerical results are presented in the Part II, which show the correctness of the theoretical conclusions.

For the sake of simplicity, we shall restrict ourselves to the two-dimensional case. The three-dimensional case can be discussed similarly without any difficulty.

2. Finite Element-Finite Difference Schemes

Let the computational domain be $\overline{\Omega}, \Omega = \{(x,y) : -a < x < a, 0 < y < b\}$; $\Gamma_1 = \{(x,y)\colon -a \le x \le a, y = 0\}$ be a natural boundary, and $\partial\Omega' = \partial\Omega/\Gamma_1$ be the artificial boundary.

Introduce the inner product notations

$$(u,v) = \int\int_\Omega uv dx dy, \quad \langle u,v\rangle = \int_{\partial\Omega'} uv ds.$$

Define the space $H^{1,0}(\Omega) = \{v(x,y) \in H^1(\Omega) : v|_{\Gamma_1} = 0\}$. It is obvious that $H^{1,0}(\Omega)$ is a closed subspace of $H^1(\Omega)$.

In the following discussion, let n denote outer normal direction, and s tangential direction of the boundary $\partial\Omega'$. Suppose that in (1.3), $C(x,y) \in L^\infty(\Omega)$ and $C(x,y) > 0$;

$f(x,y,t) \in H_0^1(\Omega) \times C([0,T])$. Consequently, $f(x,y,t) \in L^2(\Omega) \times C([0,T])$. Hence $f(x,y,t) \in L^\infty(\Omega) \times C([0,T])$. For convenience of computation, we write $f(x,y,t)$ as $\overline{f}(x,y,t)/C^2(x,y)$, where $\overline{f}(x,y,t) = C^2(x,y)f(x,y,t)$.

Corresponding to the first-order absorbing boundary condition in (1.2), consider the following generalized solution of initial-boundary value problem of the equation (1.3).

Problem I. Find a function $u(x,y,t)$ which is second-order continuously differentiable with respect to t when $(x,y) \in \Omega$ and belongs to $H^{1,0}(\Omega)$ for any fixed $t \in [0,T]$, and satisfies the following equations

$$\begin{cases} \Big(\dfrac{1}{C^2}\dfrac{\partial^2 u}{\partial t^2}, v\Big) + (\nabla u, \nabla v) - \langle\dfrac{\partial u}{\partial n}, v\rangle + \Big(\dfrac{1}{C^2}\overline{f}, v\Big) = 0 \\ \Big\langle\dfrac{\partial u}{\partial n} + \dfrac{1}{C}\dfrac{\partial u}{\partial t}, v\Big\rangle = 0 \\ (u,v)|_{t\le 0} = 0 \\ \Big(\dfrac{\partial u}{\partial t}, v\Big)|_{t\le 0} = 0 \end{cases} \tag{2.1}$$

for every $v(x,y) \in H^{1,0}(\Omega)$.

Remark. Replacing the homogeneous Dirichlet boundary condition on Γ_1 in the problem I, we can consider the boundary condition $u|_{\Gamma_1} = g(x,t)$. In this case, u should be replaced by $u - u_0$, f by $f + L(u_0)$, where $u_0(x,y,t)$ is a function such that

1) $u_0 \in H^1(\Omega)$ with respect to x and y;
2) $u_0|_{\Gamma_1} = g(x,t)$.

Discretise the spatial variables x and y by using the finite element method. Denote the nodes by P_i $(i = 1,\cdots,m)$. Suppose that S^h is a finite element space, $S^h \in H^1(\Omega)$, and its basis functions are φ_i $(i = 1,\cdots,m)$ which possess the feature $\varphi_i(P_j) = \delta_{ij}$. Take φ_i as the function v in (2.1) (except those φ_i which correspond to the nodes on Γ_1). Find the solution of the problem I in the subspace S^h. Then the problem is reduced to the following initial value problem of ODEs

$$I' : \begin{cases} M\ddot{U} + SU - W = MG(t) \\ W + M_B\dot{U} = 0 \\ U(0) = U_t(0) = W(0) = 0 \end{cases} \tag{2.2}$$

where U is the nodal unknown vector, M the mass matrix, S the stiffness matrix, M_B the boundary mass matrix, and W the vector related with the normal derivative on the artificial boundary. The elements of the matrices M, S, M_B and the vector W are, respectively,

$$M_{ij} = \int\int_\Omega \frac{1}{C^2}\varphi_i\varphi_j dxdy, \quad S_{ij} = \int\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j dxdy,$$

$$(M_B)_{ij} = \int_{\partial\Omega'} \frac{1}{C}\varphi_i\varphi_j ds \quad (i,j = 1,2\cdots,m), \tag{2.3}$$

$$W_i = \int_{\partial\Omega'} \frac{\partial u}{\partial n}\varphi_i dS \quad (i = 1,\cdots,m), \tag{2.4}$$

and $G(t)$ is a vector which consists of the values of the function $\overline{f}(x,y,t)$ at nodes.

In the next section, the following finite element spaces will be concerned:

1) Triangle elements and linear basis functions: Let (x_i, y_i) $(i = 1,2,3)$ be the vertices of the triangle. Then the basis functions $\varphi_i(x,y)$ $(i = 1,2,3)$ of the element are the followings:

$$\begin{aligned}
\xi(x,y) = \varphi_1(x,y) &= \frac{1}{2\triangle}\{(y_2-y_3)x-(x_2-x_3)y+x_2y_3-x_3y_2\},\\
\eta(x,y) = \varphi_2(x,y) &= \frac{1}{2\triangle}\{(y_3-y_1)x-(x_3-x_1)y+x_3y_1-x_1y_3\},\\
\zeta(x,y) = \varphi_3(x,y) &= \frac{1}{2\triangle}\{(y_1-y_2)x-(x_1-x_2)y+x_1y_2-x_2y_1\},
\end{aligned} \tag{2.5}$$

where $\triangle$ is the directive area of triangle, ξ, η and ζ are usually called barycentric coordinates. The mapping (2.5) transforms the element e in (x,y) coordinates into the canonical right triangle $\hat{e}$ in (ξ,η) coordinates.

2) Isoparametric quadrilateral elements and bilinear basis functions: Let

$$\begin{aligned}
&\varphi_i(\xi,\eta) = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \quad (i = 1,2,3,4)\\
&\xi_1 = \xi_4 = \eta_1 = \eta_2 = 1, \quad \xi_2 = \xi_3 = \eta_3 = \eta_4 = -1.
\end{aligned} \tag{2.6}$$

The same polynomials $\varphi_i(\xi,\eta)$ $(i = 1,2,3,4)$ of second degree are used for the transformation of coordinates as for the basis functions within each element. That is,

$$x = \sum_{i=1}^{4} x_i\varphi_i(\xi,\eta), \quad y = \sum_{i=1}^{4} y_i\varphi_i(\xi,\eta), \quad u = \sum_{i=1}^{4} u_i\varphi_i(\xi,\eta). \tag{2.7}$$

Then (2.7) maps the square $\hat{e}$ with the vertices $(1,1), (-1,1), (-1,-1)$ and $(1,-1)$ in the (ξ,η) plane onto the quadrilateral element e with the vertices (x_i,y_i) $(i = 1,2,3,4)$ in the (x,y) plane.

We consider the wave speed C as a constant in each element. Through a lumping process the matrices M and M_B are replaced by diagonal matrices. In the case of triangle elements mentioned above, the entrices of element mass matrix are

$$M_{ij}^e = \begin{cases} \dfrac{\triangle}{3C^2} & i = j \\ 0 & i \neq j, \end{cases} \tag{2.8}$$

and the entrices of element boundary mass matrix are

$$(M_B)_{ij}^e = \begin{cases} \dfrac{h}{2C} & i = j \\ 0 & i \neq j, \end{cases} \tag{2.9}$$

where h is the mesh size of a boundary element. In the case of isoparametric quadrilateral elements, the corresponding entrices are

$$M_{ij}^e = \begin{cases} \dfrac{\square}{4C^2} & i = j \\ 0 & i \neq j \end{cases} \tag{2.10}$$

and (2.9), respectively.

Discretise the time variable in (2.3) by using the finite difference scheme

$$\begin{cases} \dfrac{U^{n+1}-U^n}{\triangle t}=V^{n+1} \\ M\dfrac{V^{n+1}-V^n}{\triangle t}+SU^n-W^n=MG^n \\ W^{n+1}+M_BV^{n+1}=0. \end{cases} \tag{2.11}$$

That is,

$$\begin{cases} MV^{n+1}=MV^n-\triangle tSU^n+\triangle tW^n+\triangle tMG^n \\ U^{n+1}=U^n+\triangle tV^{n+1} \\ W^{n+1}=-M_BV^{n+1}. \end{cases} \tag{2.11$'$}$$

The initial values U^0, V^0 and W^0 are zero. Utilizing the formulas $(2.11)'$, U^{n+1}, V^{n+1} and W^{n+1} can be obtained successively from U^n, V^n and W^n.

3. Stability of the Schemes (2.11)

In this section we will give the stability condition of the scheme (2.11) for the general finite element spaces and then for two concrete finite element spaces mentioned in the last section.

When finite element methods are used, the choice of mesh size h mainly depends on the accuracy requirement. For example, ten or more grid points per wavelength are necessary. After h has been determined, the stability conditions are, in fact, conditions which $\triangle t$ should satisfy.

In the following discussions, $\lambda_{\max}$ and $\lambda_{\min}$ denote the maximum and minimum eigenvalues of a matrix, $h_{\max}$ and $h_{\min}$ the maximum and minimum element sizes, $C_{\max}$ and $C_{\min}$ the maximum and minimum wave speeds, respectively. $F:(\xi,\eta)\in\hat{e}\to(x,y)\in e$ is a mapping from a canonical element $\hat{e}$ onto a element e in (x,y)-coodinates, which has the formula (2.5) and (2.7) for the triangle and isoparametrix quadrilateral elements, respectively. J_F is the Jacobian of F, and $|J_F|=\det(J_F)$. We use the following notations as usual:

$$|J_F|_{0,\infty,\hat{e}}=\sup_{(\xi,\eta)\in\hat{e}}|J_F|,$$

$$|F|_{1,\infty,\hat{e}}=\sup_{(\xi,\eta)\in\hat{e}}\max\Big\{\frac{\partial x}{\partial\xi},\frac{\partial x}{\partial\eta},\frac{\partial y}{\partial\xi},\frac{\partial y}{\partial\eta}\Big\}.$$

Similar notations $|J_{F^{-1}}|_{0,\infty,e}$ and $|F^{-1}|_{1,\infty,e}$ will be used for the inverse of F. Denote $u(x(\xi,\eta),y(\xi,\eta))$ by $\hat{u}(\xi,\eta)$.

Lemma 1. *For the finite element space of triangle elements and linear basis functions, the following inequalities*

$$\lambda_{\max}(S)\le\frac{6qh_{\max}}{h_{\min}\sin\theta},\quad \lambda_{\min}(M)\ge\frac{h_{\min}^2\sin\theta}{6C_{\max}^2},\quad \lambda_{\max}(M_B)\le\frac{h_{\max}}{C_{\min}} \tag{3.1}$$

are valid, where q is the maximum number of elements which meet at any node, and θ is the minimum interior angle of triangle elements.

Proof. First, consider the maximum eigenvalue of the stiffness matrix S. As is well known,

$$\lambda_{\max}(S) = \max_{U\in R^m} \frac{U^TSU}{U^TU}. \tag{3.2}$$

For the unknown function $u(x,y,t)$ and the nodal unknown vector U, we have

$$U^TSU = \sum_e |u|^2_{1,e}. \tag{3.3}$$

It is easy to prove that $|u|^2_{1,e} \le 4|F^{-1}|^2_{1,\infty,e}|J_F|_{0,\infty,\hat{e}}|\hat{u}|^2_{1,\hat{e}}$. From (2.5),

$$|F^{-1}|_{1,\infty,e} \le \frac{h^e_{\max}}{2\triangle}.$$

Since $|J_F| = 2\triangle \ge h^e_{\max}h^e_{\min}\sin\theta$, we obtain

$$\begin{aligned} |u|^2_{1,e} &\le \frac{4h^e_{\max}}{h^e_{\min}\sin\theta}|\hat{u}|^2_{1,\hat{e}} = \frac{4h^e_{\max}}{h^e_{\min}\sin\theta}(U^{\hat{e}})^TS^{\hat{e}}U^{\hat{e}} \\ &\le \frac{4h^e_{\max}}{h^e_{\min}\sin\theta}\lambda_{\max}(S^{\hat{e}})(U^{\hat{e}})^TU^{\hat{e}} = \frac{4h^e_{\max}}{h^e_{\min}\sin\theta}\lambda_{\max}(S^{\hat{e}})(U^e)^TU^e, \end{aligned}$$

where U^e is the nodal unknown vector of element e, $S^{\hat{e}}$ is the stiffness matrix of canonical element $\hat{e}$,

$$S^{\hat{e}} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{3.4}$$

(or the matrices which are obtained from the above matrix by permulations). It is easy to get that $\lambda_{\max}(S^{\hat{e}}) = 3/2$. Therefore,

$$|u|^2_{1,e} \le \frac{6h^e_{\max}}{h^e_{\min}\sin\theta}(U^e)^TU^e.$$

From (3.2) and (3.3), we can get

$$\lambda_{\max}(S) \le \frac{6qh_{\max}}{h_{\min}\sin\theta},$$

which is the first inequality of (3.1).

The mass matrix M and the boundary mass matrix M_B are both diagonal matrices. A node is at least a vertex of one element and a boundary node is at most vertex of two elements. Considering this fact and utilizing the expressions (2.8) and (2.9) for the entries of M and M_B, we have the last two inequalities of (3.1), which completes the proof.

Lemma 2. *For the finite element space of isoparametric quadrilateral elements and bilinear basis functions, the following inequalities*

$$\lambda_{\max}(S) \le \frac{4\sigma^2q}{\sin\theta}, \quad \lambda_{\min}(M) \ge \frac{h^2_{\min}}{4C^2_{\max}}, \quad \lambda_{\max}(M_B) \le \frac{h_{\max}}{C_{\min}} \tag{3.5}$$

are valid, where q is the maximum number of elements which meet at any node, θ is the minimum interior angle of elements, and

$$\sigma = \max_e \sigma_e = \max_e \frac{h^e_{\max}}{h^e_{\min}}, \tag{3.6}$$

$h^e_{\max}$ and $h^e_{\min}$ are the maximum and minimum mesh sizes of the element e, respectively.

Proof. First, discuss the maximum eigenvalue of the stiffness matrix S. For each element e, we have

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{1}{4}(b_{11} + b_{12}\eta), \quad \frac{\partial y}{\partial \xi} = \frac{1}{4}(b_{21} + b_{22}\eta), \\ \frac{\partial x}{\partial \eta} &= \frac{1}{4}(b_{31} + b_{32}\xi), \quad \frac{\partial y}{\partial \eta} = \frac{1}{4}(b_{41} + b_{42}\xi), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} b_{11} &= x_1 - x_2 - x_3 + x_4, \qquad b_{12} = x_1 - x_2 + x_3 - x_4, \\ b_{21} &= y_1 - y_2 - y_3 + y_4, \qquad b_{22} = y_1 - y_2 + y_3 - y_4, \\ b_{31} &= x_1 + x_2 - x_3 - x_4, \qquad b_{32} = x_1 - x_2 + x_3 - x_4, \\ b_{41} &= y_1 + y_2 - y_3 - y_4, \qquad b_{42} = y_1 - y_2 + y_3 - y_4 \end{aligned} \tag{3.8}$$

from (2.6) and (2.7). The direct computation by using (3.7) and (3.8) can derive that the Jacobian determinant $|J_F|$ of the transformation F is a linear function of ξ and η, so that its maximum and minimum must be at the vertices of the quadrilateral. Denote the vertices of the quadrilateral element by $P_i (i = 1, 2, 3, 4)$ and the corresponding interior angles by $\theta_i (i = 1, 2, 3, 4)$. Let $P_5 = P_1$. It can be obtained from direct computation that

$$|J_F(P_i)| = \frac{1}{4}|\overline{P_i P_{i-1}}| \cdot |\overline{P_i P_{i+1}}| \sin\theta_i \quad (i = 1, 2, 3, 4). \tag{3.9}$$

Therefore,

$$|J_F| \geq \frac{1}{4}(h^e_{\min})^2 \sin\theta. \tag{3.10}$$

The Jacobian of the transformation F^{-1} is

$$J_{F^{-1}} = \frac{1}{4|J_F|} \begin{pmatrix} b_{41} + b_{42}\xi & -(b_{31} + b_{32}\xi) \\ -(b_{21} + b_{22}\eta) & b_{11} + b_{12}\eta \end{pmatrix},$$

which implies

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{4|J_F|}\Big\{(b_{41} + b_{42}\xi)\frac{\partial u}{\partial \xi} - (b_{31} + b_{32}\eta)\frac{\partial u}{\partial \eta}\Big\}, \\ \frac{\partial u}{\partial y} &= \frac{1}{4|J_F|}\Big\{ - (b_{21} + b_{22}\xi)\frac{\partial u}{\partial \xi} + (b_{11} + b_{12}\eta)\frac{\partial u}{\partial \eta}\Big\}. \end{aligned} \tag{3.11}$$

It is obvious that

$$\begin{aligned} |b_{i1} + b_{i2}\xi| &\leq 2h^e_{\max} \quad (i = 3, 4), \\ |b_{i1} + b_{i2}\eta| &\leq 2h^e_{\max} \quad (i = 1, 2) \end{aligned} \tag{3.12}$$

when $-1 \le \xi, \eta \le 1$. From (3.10)–(3.12),

$$
\begin{aligned}
|u|_{1,e}^2 &\le \frac{4\sigma_e^2}{\sin\theta}|\hat{u}|_{1,\hat{e}}^2 = \frac{4\sigma_e^2}{\sin\theta}(U^{\hat{e}})^T S^{\hat{e}} U^{\hat{e}} \\
&\le \frac{4\sigma_e^2}{\sin\theta}\lambda_{\max}(S^{\hat{e}})(U^{\hat{e}})^T U^{\hat{e}} = \frac{4\sigma_e^2}{\sin\theta}\lambda_{\max}(S^{\hat{e}})(U^e)^T U^e,
\end{aligned}
$$

where the notation σ_e is defined by (3.6), and $S^{\hat{e}}$ is the stiffness matrix of canonical element $\hat{e}$,

$$
S^{\hat{e}} = \frac{1}{6}\begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix} \tag{3.13}
$$

It is easy to find that $\lambda_{\max}(S^{\hat{e}}) = 1$, so that

$$
|u|_{1,e}^2 \le \frac{4\sigma_e^2}{\sin\theta}(U^e)^T U^e.
$$

It follows that

$$
U^T S U = \sum_e |u|_{1,e}^2 \le \frac{4q\sigma^2}{\sin\theta} U^T U,
$$

where $\sigma = \max_e \sigma_e$. Finally, we have

$$
\lambda_{\max}(S) \le \frac{4q\sigma^2}{\sin\theta}.
$$

From (2.10) and (2.9) we can get the estimates of the minimum and maximum eigenvalues of the mass matrix M and the maximum eigenvalue of the boundary mass matrix M_B. They are the second and third inequalities in (3.5). This completes the proof.

For the uniform right-angled triangle and square elements, by using direct proofs, rather than considering them as special cases of the Lemma 1 and 2, we can obtain better results than those in the Lemma 1 and 2.

Lemma 3. *For the finite element space of right-angled triangle elements and linear basis functions, the following inequalities*

$$
\lambda_{\max}(S) \le 9, \quad \lambda_{\min}(M) \ge \frac{h^2}{6C^2}, \quad \lambda_{\max}(M_B) \le \frac{h}{C} \tag{3.14}
$$

are valid.

Proof. For the nodal unknown vector U, we have

$$
U^T S U = \sum_e (U^e)^T S^e U^e \le q\lambda_{\max}(S^e) U^T U,
$$

where q is the maximum number of elements which meet at a node, and S^e the element stiffness matrix, as in the Lemma 1. In the present case, $q = 6$ and S^e is the matrix (3.4), so that $\lambda_{\max}(S^e) = 3/2$. From (3.2), the first inequality of (3.14) can be obtained.

The second and third inequalities are obvious, which completes the proof.

Lemma 4. *For the finite element space of square elements and bilinear basis functions, the following inequalities*

$$\lambda_{\max}(S) \le 4, \quad \lambda_{\min}(M) \ge \frac{h^2}{4C^2}, \quad \lambda_{\max}(M_B) \le \frac{h}{C} \tag{3.15}$$

are valid.

Proof. The approach is similar to the Lemma 1. We have only to notice that in the present case, $q = 4$ and S^e is the matrix (3.13), so that $\lambda_{\max}(S^e) = 1$. The proof is completed.

Now we are on the position to investigate the stability properties of the schemes (2.11).

Define the following inner products and norms of vectors:

$$(U,V)_2 = U^T V, \quad \|U\|_2 = \sqrt{(U,U)_2},$$
$$(U,V)_M = U^T M V, \quad \|U\|_M = \sqrt{(U,U)_M},$$
$$(U,V)_S = U^T S V, \quad \|U\|_S = \sqrt{(U,U)_S}.$$

The latter two definitions of norm are reasonable because of the positive-definiteness of the matrices S and M. Since M is a diagonal matrix, we have

$$\frac{1}{\lambda_{\max}(M)}\|U\|_M^2 \le \|U\|_2 \le \frac{1}{\lambda_{\min}(M)}\|U\|_M^2.$$

Theorem 1. *If the condition*

$$\lambda_{\max}(S) \triangle t^2 + 2\lambda_{\max}(M_B) \triangle t - 4\lambda_{\min}(M) \le -\varepsilon < 0 \tag{3.16}$$

or

$$\triangle t \le \frac{-\lambda_{\max}(M_B) + \sqrt{\lambda_{\max}^2(M_B) + 4\lambda_{\max}(S)\lambda_{\min}(M)}}{\lambda_{\max}(S)} - \varepsilon \tag{3.16$'$}$$

is satisfied, where ε is any given positive number small enough, then for any $T \in R$ big enough and $N \in Z^+$, $0 < (N+1) \triangle t < T$, there exists a constant $C_1(\varepsilon)$ such that the solution of (2.11) *is subject to the inequality*

$$\max_{0\le n\le N} \|U^{n+1}\|_M + \max_{0\le n\le N} \|V^{n+1}\|_M \le C_1(\varepsilon)\|f\|_\infty, \tag{3.17}$$

i.e., the scheme (2.11) *is stable.*

Proof. From (2.11),

$$M\frac{U^{n+1} - 2U^n + U^{n-1}}{\triangle t^2} + SU^n + M_B\frac{U^n - U^{n-1}}{\triangle t} = MG^n. \tag{3.18}$$

Multiply the equation (3.18) by $\triangle t\Big(\dfrac{U^{n+1} - U^{n-1}}{\triangle t}\Big)^T$, and sum the obtained equation from 0 to N with respect to n. Since M is a diagonal matrix, the first term of the left-hand side can be reduced to

$$\triangle t\Big(\frac{U^{n+1} - U^{n-1}}{\triangle t}\Big)^T M\frac{U^{n+1} - 2U^n + U^{n-1}}{\triangle t^2}$$

$$=\Big(\frac{U^{n+1}-U^n}{\triangle t}\Big)^T M\frac{U^{n+1}-U^n}{\triangle t} - \Big(\frac{U^n-U^{n-1}}{\triangle t}\Big)^T M\frac{U^n-U^{n-1}}{\triangle t}.$$

After the summation, we obtain

$$\triangle t\sum_{n=0}^{N}\Big(\frac{U^{n+1}-U^{n-1}}{\triangle t}\Big)^T M\frac{U^{n+1}-2U^n+U^{n-1}}{\triangle t^2} = \Big(\frac{U^{N+1}-U^N}{\triangle t}\Big)^T M\frac{U^{N+1}-U^N}{\triangle t}.$$

Here the condition $U^0 = U^{-1} = 0$ has been used. For the second term of the left-hand side we have

$$\begin{aligned}\triangle t\left(\frac{U^{n+1}-U^{n-1}}{\triangle t}\right)^T SU^n =&\frac{1}{2}\{(U^{n+1})^T SU^{n+1} - (U^{n+1}-U^n)^T S(U^{n+1}-U^n)\\ &- (U^{n-1})^T SU^{n-1} + (U^n-U^{n-1})^T S(U^n-U^{n-1})\}.\end{aligned}$$

After the summation it becomes

$$\begin{aligned}\triangle t\sum_{n=0}^{N}\Big(\frac{U^{n+1}-U^{n-1}}{\triangle t}\Big)^T SU^n =&\frac{1}{2}(U^{N+1})^T SU^{N+1} + \frac{1}{2}(U^N)^T SU^N\\ &- \frac{1}{2}(U^{N+1}-U^N)^T S(U^{N+1}-U^N)\\ =&(U^{N+1})^T SU^{N+1} - (U^{N+1})^T S(U^{N+1}-U^N).\end{aligned}$$

For the third term, it can be proved that

$$\begin{aligned}\triangle t\Big(\frac{U^{n+1}-U^{n-1}}{\triangle t}\Big)^T M_B\frac{U^n-U^{n-1}}{\triangle t} &= \frac{1}{2\triangle t}(U^{n+1}-U^{n-1})^T M_B(U^{n+1}-U^{n-1})\\ + \frac{1}{2\triangle t}(U^n-U^{n-1})^T M_B(U^n-U^{n-1}) &- \frac{1}{2\triangle t}(U^{n+1}-U^n)^T M_B(U^{n+1}-U^n).\end{aligned}$$

It follows that

$$\begin{aligned}&\triangle t\sum_{n=0}^{N}\Big(\frac{U^{n+1}-U^{n-1}}{\triangle t}\Big)^T M_B\frac{U^n-U^{n-1}}{\triangle t}\\ &\quad=\frac{\triangle t}{2}\sum_{n=0}^{N}\Big(\frac{U^{n+1}-U^{n-1}}{\triangle t}\Big)^T M_B\Big(\frac{U^{n+1}-U^{n-1}}{\triangle t}\Big)\\ &\quad\quad-\frac{\triangle t}{2}\Big(\frac{U^{N+1}-U^N}{\triangle t}\Big)^T M_B\Big(\frac{U^{N+1}-U^N}{\triangle t}\Big).\end{aligned}$$

Thus the equation (3.18) is reduced to

$$\begin{aligned}&\Big(\frac{U^{N+1}-U^N}{\triangle t}\Big)^T\Big(M-\frac{\triangle t}{2}M_B\Big)\frac{U^{N+1}-U^N}{\triangle t} + (U^{N+1})^T SU^{N+1}\\ &\quad- (U^{N+1})^T S(U^{N+1}-U^N) + \frac{\triangle t}{2}\sum_{n=0}^{N}\end{aligned}$$

$$\times \Big(\frac{U^{n+1}-U^{n-1}}{\triangle t}\Big)^T M_B \Big(\frac{U^{n+1}-U^{n-1}}{\triangle t}\Big)$$
$$= \triangle t \sum_{n=0}^{N} \Big(\frac{U^{n+1}-U^{n-1}}{\triangle t}\Big)^T M G^n. \tag{3.19}$$

It is well known that

$$\Big(\frac{U^{N+1}-U^{N}}{\triangle t}\Big)^T \Big(M - \frac{\triangle t}{2} M_B\Big) \frac{U^{N+1}-U^{N}}{\triangle t}$$
$$\geq \Big\{1 - \frac{\triangle t}{2}\frac{\lambda_{\max}(M_B)}{\lambda_{\min}(M)}\Big\} \Big\| \frac{U^{N+1}-U^{N}}{\triangle t}\Big\|_M^2,$$
$$(U^{N+1})^T S (U^{N+1}-U^N) = (U^{N+1}, U^{N+1}-U^N)_S \leq \|U^{N+1}\|_S \cdot \|U^{N+1}-U^N\|_S,$$

and

$$\triangle t \sum_{n=0}^{N} \left(\frac{U^{n+1}-U^{n-1}}{\triangle t}\right)^T M_B \left(\frac{U^{n+1}-U^{n-1}}{\triangle t}\right) \geq 0.$$

If we take $\triangle t$ such that

$$\alpha = 1 - \frac{\triangle t}{2}\frac{\lambda_{\max}(M_B)}{\lambda_{\min}(M)} > 0, \tag{3.20}$$

then

$$\begin{aligned}
\text{left of (3.19)} \geq & \alpha \left\| \frac{U^{N+1}-U^{N}}{\triangle t}\right\|_M^2 + \|U^{N+1}\|_S^2 - \|U^{N+1}\|_S \cdot \|U^{N+1}-U^N\|_S \\
\geq & \alpha \left\| \frac{U^{N+1}-U^{N}}{\triangle t}\right\|_M^2 + \|U^{N+1}\|_S^2 \\
& - \sqrt{\frac{\lambda_{\max}(S)}{\lambda_{\min}(M)}} \triangle t \|U^{N+1}\|_S \left\| \frac{U^{N+1}-U^{N}}{\triangle t}\right\|_M \\
\geq & \left\{1 - \frac{\triangle t}{2\sqrt{\alpha}}\sqrt{\frac{\lambda_{\max}(S)}{\lambda_{\min}(M)}}\right\} \cdot \left\{\alpha \left\| \frac{U^{N+1}-U^{N}}{\triangle t}\right\|_M^2 + \|U^{N+1}\|_S^2\right\} \\
& + \frac{\triangle t}{2\sqrt{\alpha}}\sqrt{\frac{\lambda_{\max}(S)}{\lambda_{\min}(M)}} \left\{\sqrt{\alpha}\left\| \frac{U^{N+1}-U^{N}}{\triangle t}\right\|_M - \|U^{N+1}\|_S\right\}^2 \\
\geq & \left\{1 - \frac{\triangle t}{2\sqrt{\alpha}}\sqrt{\frac{\lambda_{\max}(S)}{\lambda_{\min}(M)}}\right\} \alpha \left\| \frac{U^{N+1}-U^{N}}{\triangle t}\right\|_M^2 .
\end{aligned} \tag{3.21}$$

Take $\triangle t$ once again such that

$$1 - \frac{\triangle t}{2\sqrt{\alpha}}\sqrt{\frac{\lambda_{\max}(S)}{\lambda_{\min}(M)}} \geq \varepsilon > 0,$$

where ε is any given positive number small enough. It implies

$$\lambda_{\max}(S) \triangle t^2 + 2(1-\varepsilon)^2 \lambda_{\max}(M_B) \triangle t - 4(1-\varepsilon)^2 \lambda_{\min}(M) \leq 0, \tag{3.22}$$

i.e.,

$$\triangle t \leq \frac{-(1-\varepsilon)^2\lambda_{\max}(M_B) + (1-\varepsilon)\sqrt{(1-\varepsilon)^2\lambda_{\max}^2(M_B) + 4\lambda_{\max}(S)\lambda_{\min}(M)}}{\lambda_{\max}(S)}. \quad (3.23)$$

It is easy to see that (3.22) (consequently, (3.23)) contains (3.20). (3.23) can be replaced by the following stronger condition:

$$\triangle t \leq (1-\varepsilon)^2 \frac{-\lambda_{\max}(M_B) + \sqrt{\lambda_{\max}^2(M_B) + 4\lambda_{\max}(S)\lambda_{\min}(M)}}{\lambda_{\max}(S)}. \quad (3.24)$$

Introduce the notation

$$D = \frac{-\lambda_{\max}(M_B) + \sqrt{\lambda_{\max}^2(M_B) + 4\lambda_{\max}(S)\lambda_{\min}(M)}}{\lambda_{\max}(S)}.$$

(3.24) can be rewritten into

$$\triangle t \leq D - \varepsilon' = \frac{-\lambda_{\max}(M_B) + \sqrt{\lambda_{\max}^2(M_B) + 4\lambda_{\max}(S)\lambda_{\min}(M)}}{\lambda_{\max}(S)} - \varepsilon', \quad (3.25)$$

where $\varepsilon' = (2\varepsilon - \varepsilon^2)D$. It is equivalent to

$$\lambda_{\max}(S) \triangle t^2 + 2\lambda_{\max}(M_B) \triangle t - 4\lambda_{\min}(M) \leq -\varepsilon'' < 0, \quad (3.26)$$

where

$$\varepsilon'' \doteq 2\varepsilon' \sqrt{\lambda_{\max}^2(M_B) + 4\lambda_{\max}(S)\lambda_{\min}(M)}$$

(in which the terms of orders higher than 1 of ε' in the Toylor expansion are neglected). (3.26) and (3.25) are just the conditions (3.16) and (3.16)', respectively. Denote

$$\alpha_0 = 1 - \frac{D - \varepsilon'}{2} \frac{\lambda_{\max}(M_B)}{\lambda_{\min}(M)}.$$

Then from (3.21),

$$\text{left of (3.19)} \quad \geq \varepsilon\alpha_0 \left\| \frac{U^{N+1} - U^N}{\triangle t} \right\|_M^2. \quad (3.27)$$

For the right-hand side of (3.19) we have

$$\triangle t \sum_{n=0}^{N} \left(\frac{U^{n+1} - U^{n-1}}{\triangle t} \right)^T MG^n \leq 2 \triangle t \sum_{n=0}^{N} \left(\frac{U^{n+1} - U^n}{\triangle t} \right)_+^T MG_+^n,$$

where the notation $(\quad)_+$ denotes a vector the entries of which are absolute values of entries of the original vector. Obviously, $|G_i^n| \leq C_{\max}^2 \|f\|_\infty$ $(i = 1, \cdots, m)$. Introduce the m-dimensional vector $E = \{1, 1, \cdots, 1\}^T$. Then

$$\triangle t \sum_{n=0}^{N} \left(\frac{U^{n+1} - U^{n-1}}{\triangle t} \right)^T MG^n \leq 2 \triangle t \sum_{n=0}^{N} \left(\frac{U^{n+1} - U^n}{\triangle t} \right)_+^T \cdot C_{\max}^2 \|f\|_\infty ME$$

$$
\begin{aligned}
&= 2 \triangle t \sum_{n=0}^{N} \left(\left(\frac{U^{n+1} - U^n}{\triangle t} \right)_+ , C_{\max}^2 \|f\|_\infty E \right)_M \\
&\leq 2 \triangle t \sum_{n=0}^{N} \left\| \frac{U^{n+1} - U^n}{\triangle t} \right\|_M \cdot C_{\max}^2 \|f\|_\infty \|E\|_M \\
&\leq \varepsilon \alpha_0 \triangle t \sum_{n=0}^{N} \left\| \frac{U^{n+1} - U^n}{\triangle t} \right\|_M^2 + \frac{1}{\varepsilon \alpha_0} C_{\max}^4 \|f\|_\infty^2 \triangle t \sum_{n=0}^{N} \|E\|_M^2.
\end{aligned} \tag{3.28}
$$

From (3.27) and (3.28), it is obtained that

$$
\begin{aligned}
&\left\| \frac{U^{N+1} - U^N}{\triangle t} \right\|_M^2 \leq \triangle t \sum_{n=0}^{N} \left\| \frac{U^{n+1} - U^n}{\triangle t} \right\|_M^2 + K \|f\|_\infty^2, \\
&K = \frac{C_{\max}^2}{\varepsilon^2 \alpha_0^2} \triangle t \sum_{n=0}^{N} \|E\|_M^2.
\end{aligned}
$$

From the discrete Gronwall's inequality[7] we get

$$
\max_{0 \leq n \leq N} \|V^{n+1}\|_M^2 = \max_{0 \leq n \leq N} \left\| \frac{U^{n+1} - U^n}{\triangle t} \right\|_M \leq K \|f\|_\infty^2 \exp\{2T\} = C(\varepsilon) \|f\|_\infty^2.
$$

That is

$$
\max_{0 \leq n \leq N} \|V^{n+1}\|_M \leq \sqrt{C(\varepsilon)} \|f\|_\infty. \tag{3.29}
$$

Moreover, $\|U^{n+1}\|_M \leq \triangle t \{ \|V^{n+1}\|_M + \|V^n\|_M + \cdots + \|V^1\|_M \}$. It follows

$$
\max_{0 \leq n \leq N} \|U^{n+1}\|_M \leq T \sqrt{C(\varepsilon)} \|f\|_\infty. \tag{3.30}
$$

Adding (3.29) and (3.30), we obtain (3.17), which completes the proof.

Theorem 2. *Consider the finite element space of triangle elements and linear basis functions. If the condition*

$$
\frac{C_{\max} \triangle t}{h_{\min}} \leq \frac{\sin\theta}{6q} \frac{C_{\max}}{C_{\min}} \left\{ -1 + \sqrt{1 + 4q \frac{h_{\min}}{h_{\max}} \left(\frac{C_{\min}}{C_{\max}} \right)^2} \right\} - \varepsilon \tag{3.31}
$$

is satisfied, then the scheme (2.11) *is stable in the sense of* (3.17) . *Here* θ *is the minimum interior angle of elements,* q *is the maximum number of elements at nodes, and* ε *is any given positive number small enough.*

Proof. The conclusion of theorem can be obtained directly from Theorem 1 and Lemma 1.

Corollary. *If* $C(x,y)$ *is constant and the domain is subdivided into uniform right-angled triangle elements with mesh size* h*, then the stability condition of the scheme* (2.11) *becomes*

$$
\frac{C \triangle t}{h} \leq 0.18. \tag{3.32}
$$

Proof. Repeat the proof of Theorem 1 and utilize the inequalities of the Lemma 3. Then (3.32) can be obtained.

Theorem 3. *Consider the finite element space of isoparametric quadrilateral elements and bilinear basis functions. If the condition*

$$\frac{C_{\max}\triangle t}{h_{\min}} \le \beta\frac{C_{\max}}{C_{\min}}\frac{h_{\max}}{h_{\min}}\left\{-1+\sqrt{1+\frac{1}{\beta}\left(\frac{C_{\min}}{C_{\max}}\right)^2\left(\frac{h_{\min}}{h_{\max}}\right)^2}\right\}-\varepsilon,$$
$$\beta = \frac{\sin\theta}{4\sigma^2 q} \tag{3.33}$$

is satisfied, then the scheme (2.11) *is stable in the sense of* (3.17). *Here θ is the minimum interior angle of elements, q is the maximum number of elements at nodes, σ is defind as in* (3.6), *and ε is any given positive number small enough.*

Proof. It is a direct consequence of Theorem 1 and Lemma 2.

Corollary. *If $C(x,y)$ is constant and the elements are all squares with mesh size h, then the stability condition of the schemes* (2.11) *becomes*

$$\frac{C\triangle t}{h} \le 0.3. \tag{3.34}$$

Proof. If the Lemma 4 is utilized in the proof of Theorem 1, then (3.34) can be obtained.

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