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FINITE ELEMENT ANALYSIS OF A LOCAL EXPONENTIALLY FITTED SCHEME FOR TIME-DEPENDENT CONVECTION-DIFFUSION PROBLEMS*1)

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Abstract

In [16], Stynes and O'Riordan(91) introduced a local exponentially fitted finite element (FE) scheme for a singularly perturbed two-point boundary value problem without turning-point. An ε-uniform $h^{1/2}$-order accuracy was obtain for the ε-weighted energy norm. And this uniform order is known as an optimal one for global exponentially fitted FE schemes (see [6, 7, 12]).

In present paper, this scheme is used to a parabolic singularly perturbed problem. After some subtle analysis, a uniformly in ε convergent order $h|\ln h|^{1/2}+\tau$ is achieved (h is the space step and τ is the time step), which sharpens the results in present literature. Furthermore, it implies that the accuracy order in [16] is actuallay $h|\ln h|^{1/2}$ rather than $h^{1/2}$.

Key words: Singularly perturbed, Exponentially fitted, Uniformly in ε convergent, Petrov-Galerkin finite element method.

1. Introduction

Consider the time-dependent convection-diffusion problem

$$u_t-\varepsilon u_{xx}+a(x,t)u_x+b(x,t)u=f(x,t),\quad (x,t)\in[0,1]\times[0,T] \tag{1.1}$$

$$u(0,t)=u(1,t)=0,\quad t\in[0,T], \tag{1.2}$$

$$u(x,0)=u_0(x),\quad x\in[0,1], \tag{1.3}$$

$$a(x,t)\geq\alpha>0, \tag{1.4}$$

$$b(x,t)-a_x(x,t)/2\geq\beta>0, \tag{1.5}$$

where $0\leq\varepsilon\ll 1$. (1.1)-(1.5) can be regarded as a parabolic singularly perturbed problem. In general, the solution has a boundary layer at the outflow boundary $x=1$. See [1] and [15] for discusss of the properties of $u(x,t)$.

Such problems are all pervasive in applications of mathematics to problems in the science and engineering. Among these are the Navier-Stokes equation of fluid flow

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at high Reynolds number, the drift-diffusion of semiconductor, the mass conservation law in porous mediam. They have mainly hyperbolic nature for ε is small. This makes them difficult to solve numerically. It's well know that classical methods do not work well for (1.1)–(1.5) (see [3, 10]). The main problem is how to construct an ε-uniformly convergent scheme. Many authors have suggested various methods to solve such problems, see [2, 5, 9, 10, 13] and their references for the discussion of finite difference methods.

As to ε-uniformly convergent FE scheme, Gartland [4], Stynes and O'Rriordan [14, 16], Guo [6–8] and Sun & Stynes [17] have constructed quite a few methods. Guo 93 [8] proved that any scheme on a uniform mesh for (1.1)–(1.5) that was globally L^∞ convergent uniformly in ε, could not only have polynomial coefficients; the coefficients must depend on exponentials. But for highly nonequidistant meshes, such as Shiskin-type meshes, standard polynomial FE methods can also yield ε-uniformly convergent results (see Th 2.54 of [12]).

In the following, we'll focus on a scheme suggested by Stynes and O'Riordan 91 [16] for a steady-case of (1.1)–(1.5), which we call as "local exponentially fitted FE scheme". They used exponentially fitted splines in the boundary layer region and outside it, the normal continuous piecewise linear polynomials instead. An ε-uniform convengence order $h^{1/2}$ was obtained. Although this order is known as an optimal one for global exponentially fitted FE schemes, we can sharpen it to order $h|\ln h|^{1/2}$ in the case of local exponential fitting as a corollary of our main result for (1.1)–(1.5).

2. The Local Exponentially Fitted FE Scheme

Before describing the scheme, we need to know the behavior of the solution u of (1.1)–(1.5). Just for simplicity, we assume that $a(x,t), b(x,t), f(x,t)$ and $u_0(x)$ are sufficiently smooth and satisfy necessary compatibility assumptions on the corners of the boundary. Then we have the following lemma.

Lemma 2.1[15]. (1.1)–(1.5) *has a unique smooth solution* $u(x,t)$ *which satisfies*

$$|\partial_x^i \partial_t^j u(x,t)| \le C[1+\varepsilon^{-i} e^{-\alpha(1-x)/\varepsilon}] \quad \forall (x,t) \in [0,1]\times[0,T], \tag{2.1}$$

for $0 \le i \le 1$ *and* $0 \le i+j \le 2$.

Throughout this paper, C will denote a generic positive constant independent of ε.

We work with an arbitrary tensor product grid on $[0,1]\times[0,T]$. In the x-direction, let $0 = x_0 < x_1 < \cdots < x_N = 1$, with $h_i = x_i - x_{i-1}$ for $i = 1, \cdots, N$, and set $h = \max_i h_i$, $\bar{h}_i = (h_i + h_{i+1})/2$.

We assume that

$$\frac{h}{h_i} \le C \quad \forall i = 1, \cdots, N.$$

In the t-direction, let $0 = t_0 < t_1 < \cdots < t_M = T$, with $\tau_m = t_m - t_{m-1}$, for $m = 1, 2, \cdots, M$ and $\tau = \max_m \tau_m$.

Assuming $2\varepsilon|\ln\varepsilon|/\alpha < 1/2$ (it is not a restriction for ε is small), and set

$$K = \max\{i : 1 - x_i \ge 2\varepsilon|\ln\varepsilon|/\alpha\}. \tag{2.2}$$

From lemma 2.1, we have

Lemma 2.2. *If* $u(x,t)$ *is the solution of* (1.1)–(1.5), *then*

1) $\|u\|_{L^\infty}, \|u_x\|_{L^1} \le C, \quad \forall t \in [0,T]$

2) $|u_x|, |u_{xx}| \le C \ (x,t) \in (0, x_K) \times [0,T].$

So $[x_K, 1] \times [0,T]$ is called as the layer region.

A weak form of problem (1.1)–(1.5) is defined as: for each time t, find $u(x,t) \in H_0^1(0,1)$ such that

$$(u_t, v) + B(u,v) = (f,v) \quad \forall v \in H_0^1(0,1), \tag{2.3}$$

where $B(u,v) = \varepsilon(u_x, v_x) + (au_x, v) + (bu, v)$.

To discretize (2.3), we define a discrete L^2-inner product for each t_m, $(v^m, w^m)_h \equiv \sum_{i=1}^{N-1} \bar{h}_i v(x_i, t_m) w(x_i, t_m)$ and denote the associate norm by $\|\cdot\|_h$. Then a Petrov-Galerkin approximation of (2.3) can be formulated as follows: Set $U^0 = (u_0(x))_{\mathcal{S}_0}$ be the node point interpolant from $\mathcal{S}_0$ to $u_0(x)$. For $m = 1, 2, \cdots, M$, find $U^m \in \mathcal{S}_m \subset H_0^1(0,1)$ such that

$$\Big(\frac{U^m - U^{m-1}}{\tau_m}, v\Big)_h + \bar{B}(U^m, v) = (f, v)_h, \quad \forall v \in \mathcal{T} \subset H_0^1(0,1), \tag{2.4}$$

where $\bar{B}(v^m, w) = \varepsilon(v_x^m, w_x) + (\bar{a}_m v_x^m, w) + (bv^m, w)_h$ for $v^m, w \in H_0^1(0,1)$ and the piecewise constant $\bar{a}_m$ is an approximation of $a(x, t_m)$, which is defined by $\bar{a}_m(x) = \bar{a}_m(x)|_{(x_{i-1}, x_i)} = (a(x_{i-1}, t_m) + a(x_i, t_m))/2$. The test space $\mathcal{T}$ is composed of the normal continuous piecewise linear functions which is spaned by a basis $\{\psi_1, \psi_2, \cdots, \psi_{N-1}\}$, where each ψ_i is the hat function satisfying $\psi_i(x_j) = \delta_{ij}$ for all j. For $m = 1, 2, \cdots, M$, the trial space $\mathcal{S}m$ is constructed by local exponential fitting, which is spaned by a basis $\{\varphi_1, \varphi_2, \cdots, \varphi_{K-1}, \varphi_K^m, \cdots, \varphi_{N-1}^m\}$, where $\varphi_1, \varphi_2, \cdots, \varphi_{K-1}$ are the normal hat functions same as $\psi_1, \psi_2, \cdots, \psi_{K-1}; \varphi_{K+1}^m, \cdots, \varphi_{N-1}^m$ are so-called L-spline functions defined by (see [16])

$$L\varphi_i^m \equiv -\varepsilon(\varphi_i^m)_{xx} + \bar{a}_m(\varphi_i^m)_x = 0 \quad \text{on} \quad [x_K, 1]^\Lambda$$
$$\varphi_i^m(x_j) = \delta_{ij}, \quad \text{for} \quad j = K, \cdots, N,$$

where $[x_K, 1]^\Lambda = (x_K, x_{K+1}) \cup (x_{K+1}, x_{K+2}) \cup \cdots \cup (x_{N_1}, x_N)$; φ_K^m is a hybrid hat/L-spline defined similarly. For $m = 0$, the space $\mathcal{S}_0$ is same as $\mathcal{T}$.

Remark. Note that we still have $supp\varphi_i = (x_{i-1}, x_{i+1})$, for the L-spline functions $\varphi_i, i = K+1, \cdots, N-1$.

Define $\|\cdot\|$ to be the usual $L^2(0,1)$ norm, and then the ε-weighted energy norm is defined as

$$\|w\|_\varepsilon = (\varepsilon\|w_x\|^2 + \|w\|_h^2)^{1/2}, \quad \forall w \in H_0^1(0,1).$$

For any $v \in H_0^1(0,1)$, let $v_T \in \mathcal{T}$ interpolate to v at each node x_i. Then we have the following coercivity of $\bar{B}(\cdot,\cdot)$ (see lemma 4.3 of [16]).

Lemma 2.3.

$$\forall v^m \in \mathcal{S}_m, \bar{B}(v^m, v_T^m) \ge (\beta/2)\|v^m\|_\varepsilon^2$$

for sufficiently small h (*depending only on* a, b).

It is ready to obtain

Lemma 2.4.

$$\forall v^m \in \mathcal{S}_m, m = 1, 2, \cdots, M, \quad \bar{B}(v^m, v_T^m) + \Big(\frac{v^m - v^{m-1}}{\tau_m}, v_T^m\Big)_h$$
$$\geq (\beta/2)\|v^m\|_\varepsilon^2 + (1/(2\tau_m))[(v^m, v^m)_h - (v^{m-1}, v^{m-1})_h]$$

for sufficiently small h (independent of ε).

This lemma yields the existence and uniquenes of the solution of (2.4).

3. Error Estimates

In this section, we'll derive an ε-weighted energy norm error estimate for our discrete scheme.

First, let $u_{\mathcal{S}m}(x, t_m)$ be the interpolant from $\mathcal{S}_m$ to the exact solution $u(x, t_m)$, and set $Z^m = u_{\mathcal{S}m}(x, t_m) - U^m, \eta^m = u(x, t_m) - u_{\mathcal{S}m}(x, t_m)$. Therefore, $e^m = u(x, t_m) - U^m \equiv Z^m + \eta^m$, and $Z^0(x_i) = 0$, $\eta^m(x_i) = 0$, $m = 1, \cdots, M, i = 0, \cdots, N$

Rewrite (1.1) as $-\varepsilon u_{xx} + a u_x + bu = F(x, t) \equiv f(x, t) - u_t$. From Lemma 2.1, $|u_t|_{L^\infty} \leq C$, $\forall t \in [0, T]$. Then, similarly to [16] for the steady case, we can derive the following interpolation error estimates.

Lemma 3.1. *For $m = 1, 2, \cdots, M$,*

$$\begin{aligned} &(1) \qquad \forall x \in [x_{i-1}, x_i], \quad |\eta^m(x)| \leq C h_i^2, \quad \text{if} \quad 1 \leq i \leq K, \\ &\qquad\qquad |\eta^m(x)| \leq C h_i (1 - e^{-\rho_i}) \qquad \text{if} \quad K < i \leq N, \\ &(2) \qquad \|\eta^m\|_\varepsilon^2 \leq C h (h + (1 - e^{-\rho})\varepsilon |\ln \varepsilon|), \end{aligned}$$

where $\rho_i = \alpha h_i/\varepsilon$, $\rho = \alpha h/\varepsilon$.

It can be proved in the same way as [16] by regarding $f(x, t) - u_t$ as a right-hand-side term.

Remark. Note that $1 - e^{-\rho} \leq \rho$ for $\rho > 0$, the result (2) is $h(1 + |\ln \varepsilon|)^{1/2}$-order, which is almost optimal.

We now need to estimate $Z^m = u_{\mathcal{S}m} - U^m$. The next lemma relates the L^1 and L^2 norms of the derivative of an L-spline over each subinterval within the boundary layer region, and it plays an important role in the following anlysis.

Lemma 3.2. (see [11, 16])

For each $w \in \mathcal{S}m$, $m = 1, 2, \cdots, M$, and each $i \in \{K + 1, \cdots, N\}$

$$\int_{x_{i-1}}^{x_i} |w_x| dx \leq C (1 - e^{-\rho_i})^{1/2} \varepsilon^{1/2} \Big(\int_{x_{i-1}}^{x_i} |w_x|^2 dx \Big)^{1/2}.$$

We now come to

Theorem 3.3. *For h sufficiently small (independent of ε),*

$$\sum_{m=1}^{M} \|Z^m\|_\varepsilon^2 \tau_m + \max_m \|z^m\|_h^2 \leq C h (h + (1 - e^{-\rho})\varepsilon |\ln \varepsilon|) + C\tau^2. \tag{3.1}$$

Proof. From (2.3) and (2.4), for each $v \in \mathcal{T}$ and $m = 1, 2, \cdots, M$,

$$\Big(\frac{Z^m - Z^{m-1}}{\tau_m}, v\Big)_h + \bar{B}(Z^m, v) = R(u^m, v) + R_1(\eta^m, v), \tag{3.2}$$

where $R(u^m, v) = [(\theta^m, v) - (\theta^m, v)_h] + \Big(u_t - \dfrac{u^m - u^{m-1}}{\tau_m}, v\Big)_h + ((\bar{a}_m - a^m)u_x^m, v)$, $\theta^m = f^m - b^m u^m - u_t^m$ and $R_1(\eta^m, v) = -[\varepsilon(\eta_x^m, v_x) + (\bar{a}_m \eta_x^m, v)]$.

Taking $v = Z_{\mathcal{T}}^m \in \mathcal{T}$, and using Lemma 2.4,

$$(\beta/2)\|Z^m\|_\varepsilon^2 + (1/2/\tau_m)(\|Z^m\|_h^2 - \|Z^{m-1}\|_h^2) \le R(u^m, Z_{\mathcal{T}}^m) + R_1(\eta^m, Z_{\mathcal{T}}^m). \tag{3.3}$$

We firstly bound the second term of the righthand side. Integrating by parts and observing that $Z_{\mathcal{T}}^m \in \mathcal{T}$ is piecewise linear, we can write by Lemma 3.1 that

$$\begin{aligned}
|R_1(\eta^m, Z_{\mathcal{T}}^m)| =& |\varepsilon(\eta_x^m, (Z_{\mathcal{T}}^m)_x) + (\bar{a}_m \eta_x^m, Z_{\mathcal{T}}^m)| \\
=& \Big|\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \eta^m(-\varepsilon(Z_{\mathcal{T}}^m)_{xx} - \bar{a}_m (Z_{\mathcal{T}}^m)_x) dx| \\
\le& C \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} |\eta^m (Z_{\mathcal{T}}^m)_x| dx = C \sum_{i=0}^{K-1} \int_{x_i}^{x_{i+1}} |\eta^m| \frac{|Z^m(x_{i+1}) - Z^m(x_i)|}{h_{i+1}} dx \\
&+ C \sum_{i=K}^{N-1} \int_{x_i}^{x_{i+1}} |\eta^m| \frac{|Z^m(x_{i+1}) - Z^m(x_i)|}{h_{i+1}} dx \\
\le& C \sum_{i=1}^{K} h_i^2 |Z^m(x_i)| + Ch(1 - e^{-\rho}) \sum_{i=K}^{N-1} \int_{x_i}^{x_{i+1}} |Z_x^m| dx \\
\le& \frac{\beta}{16} \sum_{i=1}^{N-1} (Z^m(x_i))^2 \bar{h}_i + C \sum_{i=1}^{N-1} h_i^3 + Ch(1 - e^{-\rho}) \int_{x_K}^{1} |Z_x^m| dx \\
\le& \frac{\beta}{16} \|Z^m\|_h^2 + Ch^2 + Ch(1 - e^{-\rho}) \int_{x_K}^{1} |Z_x^m| dx
\end{aligned} \tag{3.4}$$

The last integration inside the boundary layer region will appear several times, and it plays a *key* role in this paper. So we treat it seperately. Using Lemma 3.2,

$$\begin{aligned}
Ch \int_{x_K}^{1} |Z_x^m| dx \le& Ch \sum_{i=K}^{N-1} (1 - e^{-\rho_{i+1}})^{1/2} \varepsilon^{1/2} \Big(\int_{x_i}^{x_{i+1}} |Z_x^m|^2 dx \Big)^{1/2} \\
\le& Ch \Big(\sum_{i=K}^{N-1} 1^2 \Big)^{1/2} \Big(\sum_{i=K}^{N-1} (1 - e^{-\rho_{i+1}}) \varepsilon \int_{x_i}^{x_{i+1}} |Z_x^m|^2 dx \Big)^{1/2} \\
\le& Ch^2 (1 - e^{-\rho}) \Big(\sum_{i=K}^{N-1} 1 \Big) + \frac{\beta}{16} \varepsilon \int_{x_K}^{1} |Z_x^m|^2 dx \\
\le& Ch^2 + Ch(1 - e^{-\rho}) \varepsilon |\ln \varepsilon| + \frac{\beta}{16} \varepsilon \int_{x_K}^{1} |Z_x^m|^2 dx_0 + ch^2
\end{aligned} \tag{3.5}$$

where we have used that $\sum_{i=K}^{N-1} 1 \le (C\varepsilon|\ln\varepsilon| + h)/h$.

We now turn to bound the term $R(u^m, Z_{\mathcal{T}}^m)$. Because of $Z_{\mathcal{T}}^m = \sum_{i=1}^{N-1} Z^m(x_i)\psi_i$ and $(1,\psi_i) = \bar{h}_i$, the first item of it can be estimated by

$$\begin{aligned}
|(\theta^m, Z_{\mathcal{T}}^m) - (\theta^m, Z_{\mathcal{T}}^m)_h| =& \Big|\sum_{i=1}^{N-1} (\theta^m - \theta(x_i, t_m), \psi_i) Z^m(x_i)\Big| \\
\le& \sum_{i=1}^{N-1} (1,\psi_i)|Z^m(x_i)| \int_{x_{i-1}}^{x_{i+1}} |\theta_x^m| dx \\
=& \sum_{i=1}^{N-1} \bar{h}_i |Z^m(x_i)| \int_{x_i}^{x_{i+1}} |\theta_x^m| dx + \sum_{i=1}^{N-1} \bar{h}_i |Z^m(x_i)| \int_{x_{i-1}}^{x_i} |\theta_x^m| dx \\
\equiv& (I) + (II).
\end{aligned}$$

These two terms can be treated in the same way. We only need to bound the first one.

$$\begin{aligned}
(I) =& \sum_{i=1}^{K-1} \bar{h}_i |Z^m(x_i)| \int_{x_i}^{x_{i+1}} |\theta_x^m| dx + \sum_{i=K}^{N-1} \bar{h}_i |Z^m(x_i)| \int_{x_i}^{x_{i+1}} |\theta_x^m| dx \\
\equiv& (I_1) + (I_2).
\end{aligned}$$

By Lemma 2.1 and 2.2, outside the boundary layer, we have $\int_0^{x_K} |\theta_x^m|^2 dx \le C$ and inside the boundary layer, $\int_{x_K}^1 |\theta_x^m| dx \le C$. Therefore,

$$\begin{aligned}
(I_1) \le& \frac{\beta}{32} \sum_{i=1}^{K-1} (Z^m(x_i))^2 \bar{h}_i + C \sum_{i=1}^{K-1} \bar{h}_i \Big(\int_{x_i}^{x_{i+1}} |\theta_x^m| dx \Big)^2 \\
\le& \frac{\beta}{32} \|Z^m\|_h^2 + Ch^2 \sum_{i=1}^{K-1} \int_{x_i}^{x_{i+1}} |\theta_x^m|^2 dx \le \frac{\beta}{32} \|Z^m\|_h^2 + Ch^2,
\end{aligned}$$

where we have used the Holder's inequality.

$$\begin{aligned}
(I_2) \le& \sum_{i=K}^{N-1} \bar{h}_i |Z^m(x_i) - Z^m(1)| \int_{x_i}^{x_{i+1}} |\theta_x^m| dx \\
\le& \sum_{i=K}^{N-1} \bar{h}_i \int_{x_i}^{x_{i+1}} |\theta_x^m| dx \int_{x_K}^1 |Z_x^m| dx \le Ch \int_{x_K}^1 |Z_x^m| dx \\
\text{(by (3.5))} \le& Ch(1 - e^{-\rho})\varepsilon|\ln\varepsilon| + \frac{\beta}{32} \varepsilon \int_{x_K}^1 |Z_x^m|^2 dx.
\end{aligned}$$

Estimating (II) in the same way, we get

$$|(\theta^m, Z_{\mathcal{T}}^m) - (\theta^m, Z_{\mathcal{T}}^m)_h| \le Ch^2 + Ch(1 - e^{-\rho})\varepsilon|\ln\varepsilon| + \frac{\beta}{16}\|Z^m\|_\varepsilon^2. \tag{3.6}$$

The second term of $R(u^m, Z_{\mathcal{T}}^m)$ can be easily bounded by using Lemma 2.1,

$$|(u_t - (u^m - u^{m-1})/\tau_m, Z_{\mathcal{T}}^m)_h| \le C\tau^2 + \frac{\beta}{16}\|Z^m\|_h^2. \tag{3.7}$$

To handle the third term of $R(u^m, Z_{\mathcal{T}}^m)$, we also seperate the integration into two parts, observing that $\|\bar{a}_m - a^m\|_{L^\infty} \le Ch$, $m = 1, 2, \cdots, M$,

$$\begin{aligned}
|(\bar{a}_m - a^m)u_x^m, Z_{\mathcal{T}}^m)| \le & \int_0^{x_K} |(\bar{a}_m - a^m)u_x^m Z_{\mathcal{T}}^m| dx + \int_{x_K}^1 |(\bar{a}_m - a^m)u_x^m Z_{\mathcal{T}}^m| dx \\
\text{(by Lemma 2.2)} \le & Ch^2 + \frac{\beta}{16}\|Z^m\|_h^2 + Ch\int_{x_K}^1 |u_x^m||Z_{\mathcal{T}}^m(x) - Z_{\mathcal{T}}^m(1)| dx \\
\le & Ch^2 + \frac{\beta}{16}\|Z^m\|_h^2 + Ch\int_{x_K}^1 |u_x^m| dx \int_{x_K}^1 |(Z_{\mathcal{T}}^m)_x| dx \\
\le & Ch^2 + \frac{\beta}{16}\|Z^m\|_h^2 + Ch\int_{x_K}^1 |(Z_{\mathcal{T}}^m)_x| dx \\
\text{(proved in (3.4))} \le & Ch^2 + \frac{\beta}{16}\|Z^m\|_h^2 + Ch\int_{x_K}^1 |Z_x^m| dx \\
\text{(by (3.5))} \le & Ch^2 + Ch(1 - e^{-\rho})\varepsilon|\ln\varepsilon| + \frac{\beta}{16}\|Z^m\|_\varepsilon^2.
\end{aligned} \tag{3.8}$$

Combining (3.3)–(3.8), we obtain for $m = 1, 2, \cdots, M$,

$$\frac{\beta}{4}\|Z^m\|_\varepsilon^2 + 1/2/\tau_m(\|Z^m\|_h^2 - \|Z^{m-1}\|_h^2) \le C(h^2 + h(1 - e^{-\rho})\varepsilon|\ln\varepsilon| + \tau^2). \tag{3.9}$$

Multiplying by τ_m, and summing form $m = 1$ to $m'(1 \le m' \le M)$,

$$\sum_{m=1}^{m'} \|Z^m\|_\varepsilon^2 \tau_m + \|Z^{m'}\|_h^2 \le C(h^2 + \tau^2 + h(1 - e^{-\rho})\varepsilon|\ln\varepsilon|).$$

Here we have used $Z^0(x_i) = 0$, $i = 0, 1, \cdots, N$.

This is the end of the proof of Theorem 3.3.

We finally come to the main error estimate. Combining Lemma 3.1 and Theorem 3.3, we get

Theorem 3.4. *If $u(x,t)$ and U^m are the solutions of* (1.1)–(1.5) *and* (2.4) *respectively. Then for h sufficiently small,*

$$\begin{aligned}
\sum_{m=1}^{M} \|u^m - U^m\|_\varepsilon^2 \tau_m + \max_m \|u^m - U^m\|_h^2 \le & C(h^2 + \tau^2 + h(1 - e^{-\rho})\varepsilon|\ln\varepsilon|) \\
\le & C(\tau^2 + h^2|\ln h|),
\end{aligned}$$

where C is only dependent on a, b, f, T.

Proof. The first inequality is directly from Lemma 3.1 and Theorem 3.3. To prove the second inequality, one needs checking two cases: (1) $\varepsilon \ge h$, and (2) $\varepsilon < h$.

(1) In the case of $\varepsilon \ge h$, since $1 - e^{-\rho} < \rho = \alpha h/\varepsilon$,

$$C(h^2 + \tau^2 + h(1 - e^{-\rho})\varepsilon|\ln\varepsilon|) \le C(\tau^2 + h^2|\ln\varepsilon|) \le C(\tau^2 + h^2|\ln h|).$$

(2) If $\varepsilon < h$, noting that the function $g(t) = t|\ln t|$ is monotonic increasing when $t \in (0, e^{-1})$, it follows that $\varepsilon|\ln\varepsilon| \leq h|\ln h|$ for h sufficiently small, so the second inequality is always true.

Remark. Our method implies that for the steady case (i.e. singularly perturbed two-point boundary value problems), the result in [16] can be improved to$\|u(x) - U(x)\|_\varepsilon^2 \leq Ch^2|\ln h|$.

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