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CONVERGENCE OF CHORIN-MARSDEN FORMULA FOR THE NAVIER-STOKES EQUATIONS ON CONVEX DOMAINS*

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Abstract

We prove the convergence of the Chorin-Marsden product formula for solving the initial-boundary value problems of the Navier-Stokes equations on convex domains. As a particular case we consider the case of the half plane.

Key words: Navier Stokes equation, Vortex method, Fractional step method, Convergence

1. Introduction

Different kinds of fractional step methods have been applied to solve the initial-boundary value problems of the Navier-Stokes equations for viscous incompressible flow. The vortex method developed in [5] by Chorin is a scheme with three intermediate steps where the effects of convection and viscosity are separated, and vortex sheets are created along the boundary. A set of vortex blobs is introduced to approximate the vorticity field. These vortex blobs move along the particle trajectories in the convection step, and they move randomly in the diffusion step. The convergence of the scheme is an interesting problem which has called the attention of many authors.

Related to this scheme, the splitting of the initial-boundary value problems of the Navier-Stokes equations to the corresponding problems of the Euler equations and the Stokes equations has been extensively studied, see [2] [3] [7] [9] [10] [11] [12] [13] [14] and the references therein. By the results a simple splitting converges in $L^p, p < \infty$, and in $H^s, s < \frac{5}{2}$, and if the vortex sheets are smeared out such that the vorticity is smooth, then the scheme with some modification still converges.

Marsden gave one mathematical formulation of Chorin's scheme which is a product of three operators,

$$u_k(ik) = (H_k \circ \phi \circ E_k)^i u_0,$$

where u_0 is the initial data, E_k is the local flow defined by the Euler equations with temporal step k, ϕ is the "vorticity creation operator", and H_k is the solver of the heat equation with step k. This formula is known as the Chorin-Marsden formula[6]. It involves a further approximation beyond the splitting. In [6] the velocity field is extended oddly to the exterior of the domain and the Cauchy problem of the heat

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equation for the velocity is solved in the diffusion step rather than the initial-boundary value problem of the Stokes equation . This approximation is consistent to the random walk procedure. Convergence of the linear problems was proved in [6]. Benfatto and Pulvirenti studied the Chorin-Marsden formula in the case of the half plane for the Navier-Stokes equations and proved the convergence[4]. The scheme in [4] is different from that in [6] by two respects: The tangential component of the velocity is also extended oddly but the normal component is extended evenly, and an explicit Euler scheme is applied in the convection step rather than using the particle method. The first modification bears the advantage that the velocity field keeps incompressible after the extension.

The purpose of this paper is to prove the convergence of the Chorin-Marsden formula for arbitrary two dimensional convex domains. In the convection step we use the velocity of the previous step to slove the particle trajectories, making the step in fact linear. In the diffusion step we use a modified approach to extend the velocity. Particularly if the domain is the half plane then the extension here is the same as that in [6].

In section 2 we state the scheme in details and introduce some notations. In section 3 we prove the convergence of the scheme for convex domains, where for simplicity we assume that the domains are bounded. In section 4 we apply our approach to the case of the half plane, and we will show that both approaches of extension, by Chorin-Marsden and by Benfatto and Pulvirenti, yield the results of convergence.

2. Scheme

Let $\Omega \subset R^2$ be a domain with sufficiently smooth boundary $\partial\Omega$ and $x = (x_1, x_2)$ be the points in R^2. We consider the following initial-boundary value problems,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = \nu \triangle u + f, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u|_{\partial\Omega} = 0, \tag{3}$$

$$u|_{t=0} = u_0, \tag{4}$$

where $u = (u_1, u_2)$ is the velocity, p is the pressure, f is the external force, ρ is the constant density, ν is the constant kinematic viscosity, and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$. We introduce the vorticity $\omega = -\nabla \wedge u$ and the stream function ψ such that $u = \nabla \wedge \psi$, where $\nabla\wedge = (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$, then the vorticity-stream function formulation of the problems is

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla\omega = \nu \triangle \omega + F, \tag{5}$$

$$-\triangle\psi = \omega, \quad \psi|_{\partial\Omega} = 0, \quad \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = 0, \tag{6}$$

$$u = \nabla \wedge \psi, \tag{7}$$

$$\omega|_{t=0} = \omega_0, \tag{8}$$

where $F = -\nabla \wedge f$, $\omega_0 = -\nabla \wedge u_0$, and n is the unit exterior normal vector on the boundary.

Following [6] we define a mapping Φ from the interior of Ω to the exterior. We take a positive constant d, then consider the set of all straight line segments through $\partial\Omega$ and normal to it, the length of each segment is d inside and d outside. The union of line segments is a tubular neighborhood of $\partial\Omega$, denoted by S. $\Phi : S \to S$ is the map which reflects across the boundary relative to these line segments, then Φ is a smooth mapping. Let $J(x)$ be the Jacobian of Φ at point x.

The three steps scheme is the following: Let $k > 0$ be the length of temporal steps, $t = 0, k, \cdots, ik, \cdots$. If the approximate solutions ω_k, u_k, $\tilde{\omega}_k$, $\tilde{u}_k$ are already known for $t \in [0, ik)$, then on $[ik, (i+1)k)$ the approximate solutions are solved by:

Step 1. Extension. Define an extension operator E as

$$u_k^*(x, ik) = E\tilde{u}_k(x, ik-0) = \begin{cases} \tilde{u}_k(x, ik-0), & x \in \bar{\Omega}, \\ -|J(x)|\tilde{u}_k(\Phi(x), ik-0), & x \in S \setminus \bar{\Omega}, \\ 0, & x \notin S \bigcup \Omega. \end{cases}$$

Then we set $\omega_k(x, ik) = -\nabla \wedge u_k^*(x, ik)$. In general $\tilde{u}_k$ does not vanish on the boundary, so $\omega_k^*(x, ik)$ is a distribution composed by a piecewise smooth function and a vortex sheet.

Step 2. Diffusion. Solve the heat equation

$$\frac{\partial \omega_k}{\partial t} = \nu \triangle \omega_k$$

on $R^2 \times [ik, (i+1)k)$ with the initial data $\omega_k(x, ik)$. The velocity u_k is obtained by

$$\begin{aligned} & -\triangle \psi_k = \omega_k, \quad x \in \Omega, \\ & \psi_k|_{\partial\Omega} = 0, \\ & u_k = \nabla \wedge \psi_k. \end{aligned}$$

Step 3. Convection. Solve the following problem

$$\begin{aligned} & \frac{\partial \tilde{\omega}_k}{\partial t} + u_k \cdot \nabla \tilde{\omega}_k = F_k, \\ & -\triangle \tilde{\psi}_k = \tilde{\omega}_k, \\ & \tilde{\psi}_k|_{x \in \partial\Omega} = 0, \\ & \tilde{u}_k = \nabla \wedge \tilde{\psi}_k, \\ & \tilde{\omega}_k(x, ik) = \omega_k(x, (i+1)k - 0) \end{aligned}$$

on $\Omega \times [ik, (i+1)k)$, where F_k is the approximate right hand side to be defined later. Then repeat the procedure. $\tilde{u}_k(-0)$ is understood as u_0.

In fact the above scheme is a semi-discretization scheme. The vorticities ω_k and $\tilde{\omega}_k$ can be further approximated by linear combinations of vortex blobs, then the particle method and the random walk procedure can be applied.

To meet the need of our convergence proof, we derive an equivalent form of the above scheme. As usual we define a subspace of $(L^2(\Omega))^2$ as $X = \{u \in (L^2(\Omega))^2; \nabla \cdot u = 0, u \cdot n|_{\partial\Omega} = 0\}$. The Helmholtz operator P is an orthogonal projection $P : (L^2(\Omega))^2 \to X$. At the second step we solve the heat equation for the velocity,

$$\frac{\partial u_k^*}{\partial t} = \nu \triangle u_k^*, \tag{9}$$

with the initial data $u_k^*(x, ik)$. Applying the operator $-\nabla\wedge$ to (9) we find that $\omega_k = -\nabla \wedge u_k^*$. Since u_k^* and u_k correspond to the same vorticity ω_k, we have $u_k = Pu_k^*$. In the velocity-pressure form, Step 3 can be written as

$$\frac{\partial \tilde{u}_k}{\partial t} + (Pu_k^* \cdot \nabla)\tilde{u}_k + \frac{1}{\rho}\nabla \tilde{p}_k = f, \tag{10}$$

$$\nabla \cdot \tilde{u}_k = 0, \tag{11}$$

$$\tilde{u}_k \cdot n|_{\partial\Omega} = 0, \tag{12}$$

$$\tilde{u}_k|_{t=ik} = Pu_k^*(x, (i+1)k - 0). \tag{13}$$

Then we set

$$u_k^*|_{t=(i+1)k} = E\tilde{u}_k(x, (i+1)k - 0) \tag{14}$$

and repeat the fractional step procedure at the next temporal step. From (10) it is deduced that

$$F_k = F - \begin{vmatrix} \frac{\partial (u_k)_1}{\partial x_1} & \frac{\partial (\tilde{u}_k)_1}{\partial x_1} \\ \gamma_k & \tilde{\gamma}_k \end{vmatrix},$$

where

$$\gamma_k = \frac{\partial (u_k)_1}{\partial x_2} + \frac{\partial (u_k)_2}{\partial x_1}, \quad \tilde{\gamma}_k = \frac{\partial (\tilde{u}_k)_1}{\partial x_2} + \frac{\partial (\tilde{u}_k)_2}{\partial x_1}$$

are the velocities of shear strain and $(\ )_1$ and $(\ )_2$ are the components in the x_1 and x_2 direction. We will prove the convergence of the scheme (9)–(14).

If (10) is replaced by the Euler equation

$$\frac{\partial \tilde{u}_k}{\partial t} + (\tilde{u}_k \cdot \nabla)\tilde{u}_k + \frac{1}{\rho}\nabla \tilde{p}_k = f, \tag{15}$$

then accordingly we have

$$\frac{\partial \tilde{\omega}_k}{\partial t} + \tilde{u}_k \cdot \nabla \tilde{\omega}_k = F \tag{16}$$

in the convection step. We will prove that for the case of the half plane the scheme (9) (15) (11)–(14) also converges.

3. Convergence for Bounded Domains

In this section we first prove some estimates for the approximate solutions, then prove the convergence. The usual notations of the Sobolev spaces $H^s(\Omega)$ are applied throughout the paper, and the norms and seminorms are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ respectively. We will always denote by C a generic constant. For simplicity we assume that Ω is bounded. For notational convenience sometimes we omit the spatial variable and simply write the solutions as $u_k^*(t), \tilde{u}_k(t), \cdots$. To study the property of the functions u_k^*, we consider an auxiliary problem

$$\frac{\partial h}{\partial t} = \nu \triangle h, \quad x \in R^2, t > 0, \tag{17}$$

$$h|_{t=0} = Eh_0, \tag{18}$$

where E is the extension operator defined in the previous section.

Lemma 1. *If $h_0 \in L^2(\Omega)$ and $h_0 \geq 0$, then there exists $T^* > 0$, such that*

$$h \geq 0, \quad \frac{\partial h}{\partial n} \leq 0$$

on $\partial\Omega \times (0, T^*]$, where h is the solution to the problem (17) (18).

Proof. By the change of variables we have

$$\begin{aligned} h(x,t) &= \int \frac{1}{4\pi t\nu} e^{-\frac{|x-\xi|^2}{4t\nu}} Eh_0(\xi)\, d\xi \\ &= \int_{\Omega \setminus S} \frac{1}{4\pi t\nu} e^{-\frac{|x-\xi|^2}{4t\nu}} h_0(\xi)\, d\xi \\ &\quad + \int_{\Omega \cap S} \frac{1}{4\pi t\nu} \left(e^{-\frac{|x-\xi|^2}{4t\nu}} - e^{-\frac{|x-\Phi(\xi)|^2}{4t\nu}} \right) h_0(\xi)\, d\xi. \end{aligned} \tag{19}$$

Since Ω is convex, $|x - \Phi(\xi)| \geq |x - \xi|$ for $x \in \partial\Omega$ and $\xi \in \Omega$. The integrants are positive, hence $h|_{x\in\partial\Omega} \geq 0$.

It is easy to see that the normal derivative of the first term of (19) is non-positive. Let us consider the second term and set

$$\varphi(r) = e^{-\frac{r^2}{4t\nu}},$$

then

$$\varphi'(r) = -\frac{r}{2t\nu}\varphi(r), \tag{20}$$

We intend to prove

$$\left(\frac{x-\xi}{|x-\xi|}\varphi'(|x-\xi|) - \frac{x-\Phi(\xi)}{|x-\Phi(\xi)|}\varphi'(|x-\Phi(\xi)|) \right) \cdot n \leq 0, \quad x \in \partial\Omega.$$

By (20) it is

$$((x-\xi)\varphi(|x-\xi|) - (x-\Phi(\xi))\varphi(|x-\Phi(\xi)|)) \cdot n \geq 0, \quad x \in \partial\Omega. \tag{21}$$

We take an arbitrary $o \in \partial\Omega$ and construct local coordinates (o, ξ_1, ξ_2) with the origin o and the ξ_2 coordinate axis pointing to the interior normal direction. Locally the boundary $\partial\Omega$ can be expressed in terms of a function $\xi_2 = f(\xi_1)$, $a < \xi_1 < b$, $a < 0$, $b > 0$. Since the domain is convex, $f'' \geq 0$. We assume that there is a constant $\delta > 0$ such that

$$|f'(a)| > \delta, \qquad |f'(b)| > \delta, \tag{22}$$

otherwise we can expand the interval (a,b) to achieve it. Denote by $\partial\Omega_1$ the subset of $\partial\Omega$ lying in (a,b), then we set $\partial\Omega_2=\partial\Omega\setminus\partial\Omega_1$. Let ξ be on the ξ_2-axis, then $\xi=(0,\xi_2)$ and $\Phi(\xi)=(0,-\xi_2)$.

We consider $\partial\Omega_2$ first. Let $x\in\partial\Omega_2$, then we have

$$\begin{aligned}\frac{\partial}{\partial\xi_2}\left(\frac{\varphi(|x-\xi|)}{\varphi(|x-\Phi(\xi)|)}\right)=&\frac{\partial}{\partial\xi_2}\left(e^{-\frac{|x-\xi|^2}{4t\nu}+\frac{|x-\Phi(\xi)|^2}{4t\nu}}\right)\\ =&\frac{x_2}{t\nu}e^{-\frac{|x-\xi|^2}{4t\nu}+\frac{|x-\Phi(\xi)|^2}{4t\nu}},\end{aligned}$$

hence

$$\frac{\varphi(|x-\xi|)}{\varphi(|x-\Phi(\xi)|)}\geq 1+\frac{x_2}{t\nu}\xi_2. \tag{23}$$

We may assume that $(x-\Phi(\xi))\cdot n\geq 0$, otherwise (21) is obvious, then we have

$$\frac{(x-\Phi(\xi))\cdot n}{(x-\xi)\cdot n}=1+\frac{(\xi-\Phi(\xi))\cdot n}{(x-\xi)\cdot n}=1+\frac{(0,2\xi_2)\cdot n}{(x-\xi)\cdot n}.$$

The inequality (22) implies

$$\frac{1}{(x-\xi)\cdot n}\leq C,$$

consequently

$$\frac{(x-\Phi(\xi))\cdot n}{(x-\xi)\cdot n}\leq 1+C\xi_2. \tag{24}$$

We take T^* small enough such that $\frac{x_2}{t\nu}\geq C$ for all $x\in\partial\Omega_2$ and $t\in(0,T^*]$, then (23) and (24) imply (21).

Next let us consider $\partial\Omega_1$. Let $n_1=(f'(x_1),-1)$ for $x\in\partial\Omega_1$, then $n=\frac{n_1}{|n_1|}$. We have

$$\begin{aligned}(x-\xi)\cdot n&=(x_1f'(x_1)-(x_2-\xi_2))/|n_1|\\ &=(x_1f'(x_1)-f(x_1)+\xi_2)/|n_1|,\end{aligned}$$

and

$$(x-\Phi(\xi))\cdot n=(x_1f'(x_1)-f(x_1)-\xi_2)/|n_1|.$$

Therefore

$$(x-\xi)\cdot n\geq(x-\Phi(\xi))\cdot n.$$

Besides

$$\varphi(|x-\xi|)\geq\varphi(|x-\Phi(\xi)|).$$

So (21) holds. ■

Lemma 2. *If $h_0\in L^2(\Omega)$, and h is the solution to the problem* (17) (18), *then*

$$\|h(t)\|_{0,\Omega}^2+\nu\int_0^t|h(\tau)|_{1,\Omega}^2\,d\tau\leq\|h(0)\|_{0,\Omega}^2,\quad t\in(0,T^*],$$

where T^* is given in Lemma 1.

Proof. Let $h_0^+ = \max(0, h_0)$, $h_0^- = \max(0, -h_0)$, then $h_0 = h_0^+ - h_0^-$. The solutions with initial data Eh_0^+ and Eh_0^- are denoted by h^+ and h^- respectively, then $h = h^+ - h^-$. We have

$$h^+ \frac{\partial h^+}{\partial t} = h^+ \nu \triangle h^+.$$

Taking integration we obtain

$$\frac{1}{2}\|h^+(t)\|_0^2 - \frac{1}{2}\|h^+(0)\|_0^2 + \nu \int_0^t |h^+(\tau)|_1^2 \, d\tau = \nu \int_0^t \int_{\partial\Omega} h^+ \frac{\partial h^+}{\partial n} \, ds d\tau.$$

By Lemma 1 we have

$$\frac{1}{2}\|h^+(t)\|_0^2 - \frac{1}{2}\|h^+(0)\|_0^2 + \nu \int_0^t |h^+(\tau)|_1^2 \, d\tau \leq 0.$$

Analogously

$$\frac{1}{2}\|h^-(t)\|_0^2 - \frac{1}{2}\|h^-(0)\|_0^2 + \nu \int_0^t |h^-(\tau)|_1^2 \, d\tau \leq 0.$$

Thus we obtain the estimate for h,

$$\begin{aligned}
\|h(t)\|_0^2 + \nu \int_0^t |h(\tau)|_1^2 \, d\tau &= \|h^+(t) - h^-(t)\|_0^2 + \nu \int_0^t |h^+(\tau) - h^-(\tau)|_1^2 \, d\tau \\
&= \int_\Omega (h^+(t) - h^-(t))^2 \, dx + \nu \int_0^t \int_\Omega |\nabla h^+(\tau) - \nabla h^-(\tau)|^2 \, dx d\tau \\
&\leq \int_\Omega (h^+(t))^2 \, dx + \int_\Omega (h^-(t))^2 \, dx + 2\nu \int_0^t \int_\Omega (|\nabla h^+(\tau)|^2 + |\nabla h^-(\tau)|^2) \, dx d\tau \\
&\leq \int_\Omega (h^+(0))^2 \, dx + \int_\Omega (h^-(0))^2 \, dx = \|h(0)\|_0^2.
\end{aligned}$$

We turn now to estimate the solutions of the scheme (9)–(14). ■

Lemma 3. *If $u_0 \in X$, $f \in L^1(0,T;X)$, and $k \leq T^*$, then the following estimates hold for $t \in [0,T]$:*

$$\|\tilde{u}_k(t)\|_{0,\Omega} \leq C, \tag{25}$$

$$\|u_k^*(t)\|_{0,\Omega} \leq C, \tag{26}$$

$$\int_0^t |u_k^*(\tau)|_{1,\Omega}^2 \, d\tau \leq C, \tag{27}$$

$$\int_0^t |Pu_k^*(\tau)|_{1,\Omega}^2 \, d\tau \leq C, \tag{28}$$

where the constant C is independent of k.

Proof. Multiplying (10) by $\tilde{u}_k$ and taking integration we get

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}_k\|_{0.\Omega}^2 = \int_\Omega \tilde{u}_k \cdot f \, dx \leq \|\tilde{u}_k\|_{0,\Omega} \cdot \|f\|_{0,\Omega},$$

which gives

$$\frac{d}{dt}\|\tilde{u}_k\|_{0.\Omega} \leq \|f\|_{0,\Omega},$$

hence

$$\|\tilde{u}_k(t)\|_{0,\Omega} \le \|\tilde{u}_k(ik)\|_{0,\Omega} + \int_{ik}^{t} \|f(\tau)\|_{0,\Omega}\, d\tau, \quad t \in [ik,(i+1)k). \tag{29}$$

Since P is an orthogonal operator, we have

$$\|\tilde{u}_k(ik)\|_{0,\Omega} \le \|u_k^*((i+1)k-0)\|_{0,\Omega} \tag{30}$$

by (13). We apply Lemma 2 to the components of u_k^*, and get

$$\|u_k^*(t)\|_{0,\Omega}^2 + \nu \int_{ik}^{t} |u_k^*(\tau)|_{1,\Omega}^2\, d\tau \le \|u_k^*(ik)\|_{0,\Omega}^2, \quad t \in [ik,(i+1)k) \tag{31}$$

by (9) (14), which implies

$$\|u_k^*(t)\|_{0,\Omega} \le \|\tilde{u}_k(ik-0)\|_{0,\Omega}, \quad t \in [ik,(i+1)k). \tag{32}$$

The combination of (29) (30) (32) gives

$$\|\tilde{u}_k(t)\|_{0,\Omega} \le \|\tilde{u}_k(ik-0)\|_{0,\Omega} + \int_{ik}^{t} \|f(\tau)\|_{0,\Omega}\, d\tau, \quad t \in [ik,(i+1)k).$$

By induction we obtain

$$\|\tilde{u}_k(t)\|_{0,\Omega} \le \|u_0\|_{0,\Omega} + \int_{0}^{t} \|f(\tau)\|_{0,\Omega}\, d\tau, \tag{33}$$

thus (25) is proved. Then (32) implies (26). By (29) we have

$$\begin{aligned}\|\tilde{u}_k(t)\|_{0,\Omega}^2 - \|\tilde{u}_k(ik)\|_{0,\Omega}^2 &\le (\|\tilde{u}_k(t)\|_{0,\Omega} + \|\tilde{u}_k(ik)\|_{0,\Omega}) \int_{ik}^{t} \|f(\tau)\|_{0,\Omega}\, d\tau \\ &\le C \int_{ik}^{t} \|f(\tau)\|_{0,\Omega}\, d\tau,\end{aligned}$$

together with (31) (13) (14) which gives

$$\nu \int_{ik}^{(i+1)k} |u_k^*(\tau)|_{1,\Omega}^2\, d\tau \le \|\tilde{u}_k((i-1)k)\|_{0,\Omega}^2 + C \int_{(i-1)k}^{ik} \|f(\tau)\|_{0,\Omega}\, d\tau - \|\tilde{u}_k(ik)\|_{0,\Omega}^2.$$

Summing them up with respect to i, we obtain

$$\nu \int_{0}^{(i+1)k} |u_k^*(\tau)|_{1,\Omega}^2\, d\tau \le 2\|u_0\|_{0,\Omega}^2 - \|\tilde{u}_k(ik)\|_{0,\Omega}^2 + C \int_{0}^{ik} \|f(\tau)\|_{0,\Omega}\, d\tau \le C,$$

which gives (27). (27) implies (28) since P is bounded in $H^1(\Omega)$.[8] ■

Applying the above estimates we obtain the following results of convergence.

Lemma 4. *If $u_0 \in X$, $f \in L^2(0,T;X)$, then for a sequence of approximate solutions with $k \to 0$, there exixts a subsequence, such that $\tilde{u}_k$, u_k^* and u_k converge in $L^\infty(0,T;(L^2(\Omega))^2)$ weak *, $\tilde{u}_k$ converges in $L^2(0,T;(H^\gamma(\Omega))^2), \gamma < 0$, strongly, u_k^* and u_k converge in $L^2(0,T;(H^s(\Omega))^2), s < 1$, strongly and in $L^2(0,T;(H^1(\Omega))^2)$ weakly. The limits of them are equal, and are the weak solution to the equations* (1) (2).

Proof. Since u_k are uniformly bounded in $L^2(0,T;(H^1(\Omega))^2)$, u_k and $\tilde{u}_k$ are uniformly bounded in $L^\infty(0,T;(L^2(\Omega))^2)$, we can extract subsequences such that they converge weakly and weak * respectively. Let u and $\tilde{u}$ be the limits. We define

$$v_k(t) = \begin{cases} u_k(t-ik), & t \in [2ik,(2i+1)k), \\ \tilde{u}_k(t-(i+1)k), & t \in [(2i+1)k, 2(i+1)k), \end{cases} \quad i = 0,1,\cdots,$$

then $v_k \in C([0,2T];(L^2(\Omega))^2)$. By Lemma 3 v_k are uniformly bounded in $L^\infty(0,2T;(L^2(\Omega))^2)$. Let us estimate $v_k' = \frac{\partial v_k}{\partial t}$. The Stokes operator A is defined as $A = -P\triangle$ with domain $\{u \in (H^2(\Omega))^2 \cap (H_0^1(\Omega))^2;\ \nabla\cdot u = 0\}$. Let $\beta \in (1,\frac{3}{2})$, we set $W = D(A^{\frac{\beta}{2}})$, then $\mathcal{D} = \{u \in (C_0^\infty(\Omega))^2;\ \nabla\cdot u = 0\}$ is dense in W. Let $\chi \in \mathcal{D}$ be an arbitrary test function, then by (10) we have

$$\int_\Omega \frac{\partial \tilde{u}_k}{\partial t}\cdot\chi\,dx + \int_\Omega (u_k\cdot\nabla)\tilde{u}_k\cdot\chi\,dx = \int_\Omega f\cdot\chi\,dx. \tag{34}$$

Let us estimate the terms of (34) as the following:

$$\begin{aligned} \left|\int_\Omega f\cdot\chi\,dx\right| &\le \|f\|_0\|\chi\|_0 \le \|f\|_0\|\chi\|_\beta, \\ \left|\int_\Omega (u_k\cdot\nabla)\tilde{u}_k\cdot\chi\,dx\right| &= \left|\int_\Omega (u_k\cdot\nabla)\chi\cdot\tilde{u}_k\,dx\right| \\ &\le C\|\tilde{u}_k\|_0\|u_k\|_{0,\frac{2}{\beta-1}}\|\nabla\chi\|_{0,\frac{2}{2-\beta}} \le C\|\tilde{u}_k\|_0\|u_k\|_1\|\chi\|_\beta \le C\|u_k\|_1\|\chi\|_\beta, \end{aligned}$$

where we have applied the imbedding theorem[1] and Lemma 3. It follows from (34) that

$$\left\|\frac{\partial\tilde{u}_k}{\partial t}\right\|_{W'} \le C(\|f\|_0 + \|u_k\|_1). \tag{35}$$

We apply the operator P to the equation (9) and get

$$\frac{\partial u_k}{\partial t} = \nu P\triangle u_k^*,$$

hence

$$\left\|\frac{\partial u_k}{\partial t}\right\|_{W'} \le C\left\|\frac{\partial u_k}{\partial t}\right\|_{-1} \le C\|u_k^*\|_1. \tag{36}$$

By Lemma 3 we get the upper bound of the right hand sides of (35) and (36), which yields

$$\int_0^{2T}\left\|\frac{\partial v_k}{\partial t}\right\|_{W'}^2 dt \le C.$$

Thus v_k' is uniformly bounded in $L^2(0,2T;W')$. We define the Hölder space $C^{\frac{1}{2}}([0,2T];W')$ equipped with the norm

$$\|v\| = \max_{t\in[0,2T]}\|v(t)\|_{W'} + \sup_{t,\tau\in[0,2T]}|t-\tau|^{-\frac{1}{2}}\|v(t)-v(\tau)\|_{W'},$$

then by the Schwarz inequality

$$\begin{aligned}\|v_k(t)-v_k(\tau)\|_{W'} &= \left\|\int_\tau^t \frac{\partial v_k}{\partial t}\,dt\right\|_{W'} \le \int_\tau^t \left\|\frac{\partial v_k}{\partial t}\right\|_{W'} dt \\ &\le \left(\int_\tau^t \left\|\frac{\partial v_k}{\partial t}\right\|_{W'}^2 dt\right)^{\frac{1}{2}} (t-\tau)^{\frac{1}{2}}, \quad t>\tau, \end{aligned} \tag{37}$$

so v_k is uniformly bounded in $C^{\frac{1}{2}}([0,2T];W')$. We extract a convergent subsequence in $C([0,2T];W')$, still denoted by $\{v_k\}$. Let us consider the functions u_k and $\tilde{u}_k$ on $[0,T]$. For two different temporal steps k and k' we have

$$\begin{aligned}\|u_k(t)-u_{k'}(t)\|_{W'} &= \left\|v_k\left(\left[\frac{t}{k}\right]k+t\right)-v_{k'}\left(\left[\frac{t}{k'}\right]k'+t\right)\right\|_{W'} \\ &\le \left\|v_k\left(\left[\frac{t}{k}\right]k+t\right)-v_{k'}\left(\left[\frac{t}{k}\right]k+t\right)\right\|_{W'} + C\max(k^{\frac{1}{2}},k'^{\frac{1}{2}}).\end{aligned}$$

Therefore $u_k(t)$ converges in the norm $\|\cdot\|_{W'}$ uniformly with respect to $t\in[0,T]$. In general u_k is not continuous in t, so u_k converges in $L^\infty(0,T;W')$. The same is true for $\tilde{u}_k$. By the interpolation inequality we have for $\gamma\in(-1,0)$ that

$$\begin{aligned}\int_0^T \|\tilde{u}-\tilde{u}_k\|_\gamma^2\,dt \le& C\int_0^T \|\tilde{u}-\tilde{u}_k\|_0^{\frac{2(\beta+\gamma)}{\beta}} \|\tilde{u}-\tilde{u}_k\|_{W'}^{-\frac{2\gamma}{\beta}}\,dt \\ \le& C\left(\int_0^T \|\tilde{u}-\tilde{u}_k\|_0^2\,dt\right)^{\frac{\beta+\gamma}{\beta}} \left(\int_0^T \|\tilde{u}-\tilde{u}_k\|_{W'}^2\,dt\right)^{-\frac{\gamma}{\beta}},\end{aligned}$$

consequently $\tilde{u}_k$ converges in $L^2(0,T;(H^\gamma(\Omega))^2)$. Because u_k is bounded in $L^2(0,T;(H^1(\Omega))^2)$, by the same reason u_k converges in $L^2(0,T;(H^s(\Omega))^2)$ for $s<1$.

We notice that (37) implies $\|u_k(t)-\tilde{u}_k(t)\|_{W'}\le Ck^{\frac{1}{2}}$, so $u=\tilde{u}$.

To study the convergence of u_k^*, we derive an analogue of (37) from the equation (9) that

$$\|u_k^*(t)-u_k^*(ik)\|_{W'} \le C|t-ik|^{\frac{1}{2}}, \qquad t\in[ik,(i+1)k).$$

Applying the interpolation inequality we get

$$\begin{aligned}\|u_k^*(t)-u_k^*(ik)\|_\gamma &\le C\|u_k^*(t)-u_k^*(ik)\|_{W'}^{-\frac{\gamma}{\beta}} \|u_k^*(t)-u_k^*(ik)\|_0^{\frac{\beta+\gamma}{\beta}} \\ &\le C(t-ik)^{-\frac{\gamma}{\beta}}.\end{aligned}$$

We notice that $u_k^*(ik)=u_k(ik)$, hence

$$\|u_k^*(t)-u_k(t)\|_\gamma \le Ck^{-\frac{\gamma}{\beta}}.$$

Therefore u_k^* also converges to u in $L^2(0,T;(H^\gamma(\Omega))^2)$. Following the same lines we can prove that u_k^* possesses the same convergent property as u_k.

Finally let us prove that u is a solution to the equations (1) (2). We take $\chi\in\mathcal{D}$, then we have

$$\frac{d}{dt}\int_\Omega u_k^*\cdot\chi\,dx = \nu\int_\Omega u_k^*\cdot\triangle\chi\,dx,$$

which gives

$$\int_\Omega (u_k^*(t) - u_k^*(\tau)) \cdot \chi \, dx = \nu \int_\tau^t \int_\Omega u_k^* \cdot \triangle \chi \, dxdt, \quad t, \tau \in [ik, (i+1)k).$$

Analogously

$$\int_\Omega (\tilde{u}_k(t) - \tilde{u}_k(\tau)) \cdot \chi \, dx - \int_\tau^t \int_\Omega (u_k \cdot \nabla)\chi \cdot \tilde{u}_k \, dxdt = \int_\tau^t \int_\Omega \tilde{u}_k \cdot f \, dxdt.$$

Taking the initial conditions (13) (14) into account, we get for $t > \tau, t, \tau \in [0, T]$, that

$$\begin{aligned} &\int_\Omega (u_k^*(t) - u_k^*(\tau)) \cdot \chi \, dx - \int_{[\frac{\tau}{k}]k}^{[\frac{t}{k}]k} \int_\Omega (u_k \cdot \nabla)\chi \cdot \tilde{u}_k \, dxdt \\ &\quad = \nu \int_\tau^t \int_\Omega u_k^* \cdot \triangle \chi \, dxdt + \int_{[\frac{\tau}{k}]k}^{[\frac{t}{k}]k} \int_\Omega \tilde{u}_k \cdot f \, dxdt. \end{aligned}$$

Letting $k \to 0$, we obtain

$$\begin{aligned} &\int_\Omega (u(t) - u(\tau)) \cdot \chi \, dx - \int_\tau^t \int_\Omega (u \cdot \nabla)\chi \cdot u \, dxdt \\ &\quad = \nu \int_\tau^t \int_\Omega u \cdot \triangle \chi \, dxdt + \int_\tau^t \int_\Omega u \cdot f \, dxdt, \end{aligned}$$

which is the weak form of (1) in the sense of Leray. It is clear that (2) holds, because both u_k and $\tilde{u}_k$ satisfy this equation. ■

We turn now to prove that u satisfies the boundary condition (3).

Lemma 5. *Under the conditions of Lemma 4 the limit function u belongs to $L^2(0, T; (H_0^1(\Omega))^2)$.*

Proof. We use the formula (19) for u_k^*,

$$\begin{aligned} u_k^*(x,t) = &\int_{\Omega \setminus S} \frac{1}{4\pi\tau\nu} e^{-\frac{|x-\xi|^2}{4\tau\nu}} u_k^*(\xi, ik) \, d\xi \\ &+ \int_{\Omega \cap S} \frac{1}{4\pi\tau\nu} \left(e^{-\frac{|x-\xi|^2}{4\tau\nu}} - e^{-\frac{|x-\Phi(\xi)|^2}{4\tau\nu}} \right) u_k^*(\xi, ik) \, d\xi, \end{aligned} \tag{38}$$

where $\tau = t - ik$, $t \in (ik, (i+1)k)$. Let $x \in \partial\Omega$ be an arbitrary point and $\varepsilon > 0$ be a small positive constant. We construct a disc Ω_1 with center x and radius ε, then

$$\int_{\Omega \cap S} = \int_{(\Omega \cap S) \setminus \Omega_1} + \int_{\Omega \cap \Omega_1}.$$

If $\xi \notin \Omega_1$, then $|x - \xi| \geq \varepsilon$, $|x - \Phi(\xi)| \geq \varepsilon$. Therefore if k is small enough, then we have

$$\left| \int_{\Omega \setminus S} \right| + \left| \int_{(\Omega \cap S) \setminus \Omega_1} \right| < \varepsilon$$

for $\tau < k$. It remains to consider the integral on $\Omega \cap \Omega_1$. Let $\xi \in \Omega \cap \Omega_1$. We construct the same local coordinates as in the proof of Lemma 1, then

$$|x - \Phi(\xi)|^2 - |x - \xi|^2 = (x_1^2 + (x_2 + \xi_2)^2) - (x_1^2 + (x_2 - \xi_2)^2)$$

$$= 4x_2\xi_2 = 4f(x_1)\xi_2 \le C|x-\xi|^2\varepsilon,$$

hence

$$0 \le e^{-\frac{|x-\xi|^2}{4\tau\nu}} - e^{-\frac{|x-\Phi(\xi)|^2}{4\tau\nu}} \le \frac{C|x-\xi|^2\varepsilon}{4\tau\nu} e^{-\frac{|x-\xi|^2}{4\tau\nu}}.$$

We take an arbitrary $\zeta \in C^\infty(\partial\Omega)$, then we have

$$\begin{aligned} I =& \left| \int_{\partial\Omega} \zeta(x)\,dx \int_{\Omega\cap\Omega_1} \frac{1}{4\pi\tau\nu} \left(e^{-\frac{|x-\xi|^2}{4\tau\nu}} - e^{-\frac{|x-\Phi(\xi)|^2}{4\tau\nu}} \right) u_k^*(\xi, ik)\,d\xi \right| \\ \le& \int_{\partial\Omega} |\zeta(x)|\,dx \int_{\Omega\cap\Omega_1} \frac{C|x-\xi|^2\varepsilon}{16\pi\tau^2\nu^2} e^{-\frac{|x-\xi|^2}{4\tau\nu}} |u_k^*(\xi, ik)|\,d\xi. \end{aligned}$$

We extend $u_k^*(\xi, ik)$ by zero to the exterior of Ω, still denoted by $u_k^*(\xi, ik)$. We notice that $\nabla \cdot u_k^*(\xi, ik) = 0$. By the change of variables $\xi = x + \sqrt{\tau}\eta$ we have

$$\begin{aligned} I \le& \int_{\partial\Omega} |\zeta(x)|\,dx \int_{|\eta|\le\frac{1}{\sqrt{\tau}}} C\varepsilon|\eta|^2 e^{-\frac{|\eta|^2}{4\nu}} |u_k^*(x+\sqrt{\tau}\eta, ik)|\,d\eta \\ =& \int_{|\eta|\le\frac{1}{\sqrt{\tau}}} C\varepsilon|\eta|^2 e^{-\frac{|\eta|^2}{4\nu}}\,d\eta \int_{\partial\Omega} |\zeta(x)| \cdot |u_k^*(x+\sqrt{\tau}\eta, ik)|\,dx \\ \le& \int_{|\eta|\le\frac{1}{\sqrt{\tau}}} C\varepsilon|\eta|^2 e^{-\frac{|\eta|^2}{4\nu}} \|\zeta\|_{\frac{1}{2},\partial\Omega} \|u_k^*(\cdot+\sqrt{\tau}\eta, ik)\|_{-\frac{1}{2},\partial\Omega}\,d\eta \\ \le& \|\zeta\|_{\frac{1}{2},\partial\Omega} \|u_k^*(\cdot, ik)\|_{0,\Omega} \int_{|\eta|\le\frac{1}{\sqrt{\tau}}} C\varepsilon|\eta|^2 e^{-\frac{|\eta|^2}{4\nu}}\,d\eta \le C\varepsilon \|\zeta\|_{\frac{1}{2},\partial\Omega} \|u_k^*(\cdot, ik)\|_{0,\Omega}. \end{aligned}$$

Due to Lemma 3 we have

$$\left| \int_{\partial\Omega} \zeta(x) u_k^*(x,t)\,dx \right| < C\varepsilon, \quad \forall\zeta \in C^\infty(\partial\Omega),$$

where the constant C depends on ζ. Consequently

$$\overline{\lim_{k\to 0}} \left| \int_{\partial\Omega} \zeta(x) u_k^*(x,t)\,dx \right| < C\varepsilon.$$

But ε is arbitrary, therefore

$$\lim_{k\to 0} \int_{\partial\Omega} \zeta(x) u_k^*(x,t)\,dx = 0, \quad \forall\zeta \in C^\infty(\partial\Omega), \tag{39}$$

which implies u_k^* tends to zero on the boundary $\partial\Omega$ in the sense of distributions. By Lemma 4 u_k^* converges to u in $L^2(0,T;(H^s(\Omega))^2)$ strongly, $s < 1$. We take $s > \frac{1}{2}$ and extract a subsequence such that $u_k^*(t)$ converges to $u(t)$ in $(H^s(\Omega))^2$ for almost all $t \in [0,T]$. By the trace theorem $u_k^*(t)|_{\partial\Omega}$ converges to $u(t)|_{\partial\Omega}$ in $(H^{s-\frac{1}{2}}(\partial\Omega))^2$. (37) implies the limit is zero. Therefore $u(t) \in (H_0^s(\Omega))^2$ for almost all $t \in [0,T]$. But $u \in L^2(o,T;(H^1(\Omega))^2)$, so $u \in L^2(o,T;(H_0^1(\Omega))^2)$. ■

The conclusions of the above lemmas lead to the following theorem.

Theorem 1. *If* $u_0 \in X$*,* $f \in L^2(0,T;X)$*, then the weak solution* $u \in L^2(0,T;(H_0^1(\Omega))^2) \bigcap L^\infty\ (0,T;X)$ *to the problem* (1)–(4) *in the sense of Leray is the limit of* $\tilde{u}_k$*,* u_k*,* u_k^* *in the sense of*

$$\begin{aligned}
&u_k, \tilde{u}_k \overset{*}{\rightharpoonup} u, (L^\infty(0,T;X)),\\
&u_k^* \overset{*}{\rightharpoonup} u, (L^\infty(0,T;(L^2(\Omega))^2)),\\
&u_k, u_k^* \rightharpoonup u, (L^2(0,T;(H^1(\Omega))^2)),\\
&\tilde{u}_k \to u, (L^2(0,T;(H^\gamma(\Omega))^2), \gamma < 0),\\
&u_k, u_k^* \to u, (L^2(0,T;(H^s(\Omega))^2), s < 1).
\end{aligned}$$

Proof. We have already proved that there exists a subsequence which converges to u in the above sense. The weak solution to (1)–(4) is unique, therefore the above results of convergence hold for the whole $k \to 0$. ■

4. Convergence for the Half Plane

If Ω is the half plane $\{x; x_2 > 0\}$, then the absolute value of the Jacobian $|J(x)| = 1$. The extension operator E is the same as that in [6]. Since u_k^* is an odd function with respect to x_2, it is not only the solution to the Cauchy problem of (9), but also the solution to the initial-boundary value problem with the Dirichlet boundary condition

$$u_k^*|_{\partial\Omega} = 0. \tag{40}$$

We notice that (40) does not imply $u_k = u_k^*$, because generally speaking $\nabla \cdot u_k^* \neq 0$. However, Lemma 1 and Lemma 2 become trivial at this case. Lemma 3 holds without any change. Since the domain is unbounded, the results of strong convergence in Lemma 4 should be changed to be locally. By (40) Lemma 5 is also trivial. We have the following theorem by analogy to Theorem 1.

Theorem 2. *If* $\Omega = \{x; x_2 > 0\}$*,* $u_0 \in X$*,* $f \in L^2(0,T;X)$*, then the weak solution* $u \in L^2(0,T;(H_0^1(\Omega))^2) \bigcap L^\infty(0,T;X)$ *to the problem (1)-(4) in the sense of Leray is the limit of* $\tilde{u}_k$*,* u_k*,* u_k^* *in the sense of*

$$\begin{aligned}
&u_k, \tilde{u}_k \overset{*}{\rightharpoonup} u, (L^\infty(0,T;X)),\\
&u_k^* \overset{*}{\rightharpoonup} u, (L^\infty(0,T;(L^2(\Omega))^2)),\\
&u_k, u_k^* \rightharpoonup u, (L^2(0,T;(H^1(\Omega))^2)),\\
&\tilde{u}_k \to u, (L^2(0,T;(H^\gamma_{loc}(\Omega))^2), \gamma < 0),\\
&u_k, u_k^* \to u, (L^2(0,T;(H^s_{loc}(\Omega))^2), s < 1).
\end{aligned}$$

We turn now to study the extension given in [4],

$$(u_k^*)_1(x, ik) = \begin{cases} (\tilde{u}_k)_1(x, ik-0), & x \in \bar{\Omega},\\ -(\tilde{u}_k)_1(-x, ik-0), & x \notin \Omega, \end{cases} \tag{41}$$

$$(u_k^*)_2(x, ik) = \begin{cases} (\tilde{u}_k)_2(x, ik-0), & x \in \bar{\Omega},\\ (\tilde{u}_k)_2(-x, ik-0), & x \notin \Omega. \end{cases} \tag{42}$$

The corresponding boundary conditions are

$$(u_k^*)_1|_{\partial\Omega} = 0, \quad \left.\frac{\partial (u_k^*)_2}{\partial n}\right|_{\partial\Omega} = 0. \tag{43}$$

Since $\nabla\cdot u_k^* = 0$, u_k^* are not only the solutions to (9) but also the solutions to the Stokes equation

$$\begin{aligned}&\frac{\partial u_k^*}{\partial t} + \frac{1}{\rho}\nabla p_k^* = \nu \triangle u_k^*,\\ &\nabla\cdot u_k^* = 0.\end{aligned}$$

The results of Lemma 2 still hold, because we can use the boundary condition (43) to get the energy estimates for $(u_k^*)_1$ and $(u_k^*)_2$ respectively. The remarks for Lemma 3 and Lemma 4 are the same as the previous case. Lemma 5 is also trivial because by (43) we get $u_1|_{\partial\Omega} = 0$ and by $(u_k)_2|_{\partial\Omega} = 0$ we get $u_2|_{\partial\Omega} = 0$. Therefore for the extension (41) (42) the result of comvergence also holds.

Theorem 3. *Under the extension* (41) (42) *the conclusion of Theorem* 2 *holds.*

Finally let us study the scheme (9) (15) (11)-(14). From the proof in section 3 we see that we need an estimate of $\tilde{u}_k$ in the form of (27).

Lemma 6. *If $u_0 \in X$, $f \in L^2(0,T;H^1(\Omega))$. $k \le T^*$, and if Ω' is a compact subdomain of $\bar{\Omega}$, then for the scheme* (9) (15) (11)-(14) *the following estimate holds for $(i+1)k \le T$:*

$$\int_{ik}^{(i+1)k} |\tilde{u}_k(\tau)|_{1,\Omega'}^2 \, d\tau \le C \int_{ik}^{(i+1)k} |u_k^*(\tau)|_{1,\Omega}^2 \, d\tau + Ck, \tag{44}$$

where the constant C depends on Ω'.

Proof. Taking the inner product of (9) with $\triangle u_k^*$ we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_k^*|^2 \, dx = \nu \int_\Omega |\triangle u_k^*|^2 \, dx \ge 0,$$

which gives

$$\int_\Omega |\nabla u_k^*(x,(i+1)k-0)|^2 \, dx \le \frac{1}{k}\int_{ik}^{(i+1)k} d\tau \int_\Omega |\nabla u_k^*(x,\tau)|^2 \, dx. \tag{45}$$

Multiplying (16) by $\tilde{\omega}_k$ and taking integration we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\tilde{\omega}_k|^2 \, dx = \int_\Omega \tilde{\omega}_k F \, dx,$$

which gives

$$\frac{d}{dt}\|\tilde{\omega}_k(t)\|_{0,\Omega} \le \|F(t)\|_{0,\Omega},$$

therefore

$$\|\tilde{\omega}_k(t)\|_{0,\Omega} \le \|\tilde{\omega}_k(ik)\|_{0,\Omega} + \int_{ik}^t \|F(\tau)\|_{0,\Omega} \, d\tau, \quad t \in [ik,(i+1)k),$$

consequently

$$\begin{aligned}\int_{ik}^{(i+1)k} \|\tilde{\omega}_k(\tau)\|_{0,\Omega}^2\, d\tau &\le 2k\|\tilde{\omega}_k(ik)\|_{0,\Omega}^2 + 2k\int_{ik}^{(i+1)k} \|F(\tau)\|_{0,\Omega}^2\, d\tau \\ &= 2k\|\omega_k((i+1)k-0)\|_{0,\Omega}^2 + 2k\int_{ik}^{(i+1)k} \|F(\tau)\|_{0,\Omega}^2\, d\tau \\ &\le 2k\|\nabla u_k^*((i+1)k-0)\|_{0,\Omega}^2 + 2k\int_{ik}^{(i+1)k} \|F(\tau)\|_{0,\Omega}^2\, d\tau.\end{aligned}$$

By (45)

$$\int_{ik}^{(i+1)k} \|\tilde{\omega}_k(\tau)\|_{0,\Omega}^2\, d\tau \le 2\int_{ik}^{(i+1)k} |u_k^*(\tau)|_{1,\Omega}^2\, d\tau + 2k\int_{ik}^{(i+1)k} \|F(\tau)\|_{0,\Omega}^2\, d\tau. \tag{46}$$

We take Ω' such that $\Omega' \subset\subset \Omega'' \subset\subset \bar{\Omega}$. Noting the definition of the stream function $\tilde{\psi}_k$ and the estimate (25), by virtue of the Friedrichs inequality we get

$$\|\tilde{\psi}_k\|_{0,\Omega''} \le C. \tag{47}$$

Then applying the interior estimate of elliptic equations we have

$$|\tilde{u}_k|_{1,\Omega'} \le C\{\|\tilde{\psi}_k\|_{0,\Omega''} + \|\tilde{\omega}_k\|_{0,\Omega}\},$$

which together with (46) (47) yields (44). ■

Following the same lines as section 3 we get the convergence theorem as follows.

Theorem 4. *If* $\Omega = \{x; x_2 > 0\}$, $u_0 \in X$, $f \in L^2(0,T;H^1(\Omega))$, *then the solutions to the scheme* (9) (15) (11)–(14) *converge to u in the sense of*

$$\begin{aligned}&u_k, \tilde{u}_k \overset{*}{\rightharpoonup} u, (L^\infty(0,T;X)),\\ &u_k^* \overset{*}{\rightharpoonup} u, (L^\infty(0,T;(L^2(\Omega))^2)),\\ &u_k, u_k^* \rightharpoonup u, (L^2(0,T;(H^1(\Omega))^2)),\\ &u_k, u_k^*, \tilde{u}_k \to u, (L^2(0,T;(H^s_{loc}(\Omega))^2), s<1).\end{aligned}$$

Remark For the extension (41) (42) we can obtain the same result. Since the argument is just the same, it is omitted here.

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