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THE BOUNDARY INTEGRO-DIFFERENTIAL EQUATIONS OF A BIHARMONIC BOUNDARY VALUE PROBLEM*[1)]

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Abstract

In this paper, a new method of boundary reduction is proposed, which reduces the biharmonic boundary value problem to a system of integro-differential equations on the boundary and preserves the self-adjointness of the original problem. Moreover, a boundary finite element method based on this integro-differential equations is presented and the error estimates of the numerical approximations are given. The numerical examples show that this new method is effective.

Key words: Boundary Iintegro-differential equations, Bihamonic boundary value problem

1. Introduction

We consider a homogeneous isotropic and linear elastic Kirchhoff plate under lateral load distributed over the plate $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$. The domain $\Omega \in R^2$ is bounded with the smooth boundary Γ. In the static equilibrium, we consider the free type boundary condition on Γ. Then the deflection u satisfies the following problem:

$$\begin{cases} \Delta^2 u = \dfrac{q}{D}, & \text{in } \Omega, \\ M(x, n_x)u = 0, & \text{on } \Gamma, \\ T(x, n_x)u = 0, & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $D = \dfrac{E_0 h^3}{12(1-\nu^2)}$, is the bending stiffness of the plate with h being the plate thickness and E_0 and $\nu (0 < \nu < \frac{1}{2})$ being the modulus and Poisson's ratio respectively, q denotes the lateral loading; the boundary differential operators $M(x, n_x)$, $T(x, n_x)$ are given by:

$$\begin{aligned} M_x \equiv & M(x, n_x) = \nu \Delta_x \\ & + (1-\nu)\Big[n_1^2(x)\frac{\partial^2}{\partial x_1^2} + n_2^2(x)\frac{\partial^2}{\partial x_2^2} + 2n_1(x)n_2(x)\frac{\partial^2}{\partial x_1 \partial x_2}\Big], \end{aligned} \tag{1.2}$$

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$$T_x \equiv T(x, n_x) = -\frac{\partial \Delta_x}{\partial n_x} + (1-\nu)\frac{\partial}{\partial s_x}[n_1(x)n_2(x)\Big(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\Big) - ((n_1(x))^2 - (n_2(x))^2)\frac{\partial^2}{\partial x_1 \partial x_2}\Big], \tag{1.3}$$

where $n_x = (n_1(x), n_2(x))^T$ denotes the unit outer normal vector at $x \in \Gamma$ and $s_x = (-n_2(x), n_1(x))^T$ is the unit tangential vector at $x \in \Gamma$. For convenience, from now on we suppose that the bending stiffness D has been normalized to $D = 1$. Because the lateral loading $q(x)$ in (1.1) can always be eliminated by substracting a volume potential, hence the problem (1.1) can be reduced to the following problem:

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ M_x u = m & \text{on } \Gamma, \\ T_x u = t & \text{on } \Gamma, \end{cases} \tag{1.4}$$

for given functions $m(x), t(x)$ on the boundary Γ. Let $\Omega^c = R^2 \backslash \Omega$, then we also consider the boundary value problem on the unbounded domain Ω^c:

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega^c, \\ M_x u = m & \text{on } \Gamma, \\ T_x u = t & \text{on } \Gamma, \\ & u(x) \text{ satisfies the linear - logarithmic growth condition} \\ & \text{(see [11], p468. (8.165)), when } |x| \to \infty. \end{cases} \tag{1.5}$$

The operators M_x and T_x can be rewritten in the following form:

$$M_x = \Delta_x - (1-\nu)\frac{\partial^2}{\partial s_x^2} - (1-\nu)\omega(x, n_x)\frac{\partial}{\partial n_x}, \tag{1.6}$$

$$T_x = -\frac{\partial \Delta_x}{\partial n_x} - (1-\nu)\frac{\partial^3}{\partial s_x^2 \partial n_x} + (1-\nu)\frac{\partial}{\partial s_x}\Big[\omega(x, n_x)\frac{\partial}{\partial s_x}\Big], \tag{1.7}$$

where $\omega(x, n_x) = n_1(x)\frac{dn_2(x)}{ds_x} - n_2(x)\frac{dn_1(x)}{ds_x}$.

We will reduce the problem (1.4) to a system of boundary integro-differential equations by an indirect method.

Let

$$u(x) = \int_\Gamma M_y E(x, y) f_1(y) ds_y + \int_\Gamma T_y E(x, y) f_2(y) ds_y + p_1(x), \; x \in \Omega, \tag{1.8}$$

be the solution of problem (1.4). Here $p_1(x)$ is an arbitrary polynomial of degree one, $E(x, y) = \frac{1}{8\pi} r^2 \log r$, with $r = |x - y|$ is a fundamental solution of biharmonic equation, f_1, f_2 are two unknown density functions.

For any $x \notin \Gamma$, and an arbitrary unit vector n_x, we have

$$M_x u(x) = \int_\Gamma M_x M_y E(x, y) f_1(y) ds_y + \int_\Gamma M_x T_y E(x, y) f_2(y) ds_y, \; x \notin \Gamma, \tag{1.9}$$

$$T_x u(x) = \int_\Gamma T_x M_y E(x, y) f_1(y) ds_y + \int_\Gamma T_x T_y E(x, y) f_2(y) ds_y, \; x \notin \Gamma. \tag{1.10}$$

For $x \in \Gamma$ and n_x is the unit outward normal vector at $x \in \Gamma$ for the domain, each kernel in the integrals (1.10) has a singularity $|x-y|^{-k}$ when x and y are close for $k = 2$ (or 3, or 4). Thus the integrals in the right-hand side of (1.10) are defined as a finite part in sense of Hardmard. Before reducing the problem (1.4) to a system of boundary integro-differential equations, we will study the following two limits

$$\lim_{z\in\Omega\to x\in\Gamma}\Big\{\int_\Gamma M_z M_y E(z,y) f_1(y) ds_y + \int_\Gamma M_z T_y E(z,y) f_2(y) ds_y\Big\},$$
$$\lim_{z\in\Omega\to x\in\Gamma}\Big\{\int_\Gamma T_z M_y E(z,y) f_1(y) ds_y + \int_\Gamma T_z T_y E(z,y) f_2(y) ds_y\Big\}.$$

2. Two New Presentations of (1.9) and (1.10)

Let

$$E^*(x,y) = \Delta_x E(x,y) = \Delta_y E(x,y) = \frac{1}{2\pi}(\log r + 1), \tag{2.1}$$

then $E^*(x,y)$ is a fundamental solution of harmonic equation. Suppose that $n_x = (n_1(x), n_2(x))^T$, $n_y = (n_1(y), n_2(y))^T$ are arbitrary unit vectors and $s_x = (-n_2(x), n_1(x))^T$, $s_y = (-n_2(y), n_1(y))^T$ which are perpendicular to n_x, n_y respectively. For the fundamental solution $E^*(x,y)$ and $E(x,y)$, we have the following lemmas.

Lemma 2.1[2]**.** *For $x \neq y$, the following equality holds*

$$\frac{\partial^2 E^*(x,y)}{\partial n_x \partial n_y} = -\frac{\partial^2 E^*(x,y)}{\partial s_x s_y}.$$

Lemma 2.2. *For $x \neq y$, the following equality holds*

$$\frac{\partial^2 E^*(x,y)}{\partial n_x \partial s_y} = \frac{\partial^2 E^*(x,y)}{\partial n_y \partial s_x}.$$

Proof. We notice

$$\frac{\partial^2 E^*(x,y)}{\partial x_1 \partial y_1} = -\frac{\partial^2 E^*(x,y)}{\partial x_2 \partial y_2}, \quad \frac{\partial^2 E^*(x,y)}{\partial x_1 \partial y_2} = \frac{\partial^2 E^*(x,y)}{\partial x_2 \partial y_1}.$$

Then we obtain

$$\begin{aligned}\frac{\partial^2 E^*(x,y)}{\partial n_x \partial s_y} =& (n_1(x), n_2(x)) \begin{pmatrix} \dfrac{\partial^2 E^*(x,y)}{\partial x_1 \partial y_1} & \dfrac{\partial^2 E^*(x,y)}{\partial x_1 \partial y_2} \\ \dfrac{\partial^2 E^*(x,y)}{\partial x_2 \partial y_1} & \dfrac{\partial^2 E^*(x,y)}{\partial x_2 \partial y_2} \end{pmatrix} \begin{pmatrix} -n_2(y) \\ n_1(y) \end{pmatrix} \\ =& (-n_2(x), n_1(x)) \begin{pmatrix} \dfrac{\partial^2 E^*(x,y)}{\partial x_1 \partial y_1} & \dfrac{\partial^2 E^*(x,y)}{\partial x_1 \partial y_2} \\ \dfrac{\partial^2 E^*(x,y)}{\partial x_2 \partial y_1} & \dfrac{\partial^2 E^*(x,y)}{\partial x_2 \partial y_2} \end{pmatrix} \begin{pmatrix} n_1(y) \\ n_2(y) \end{pmatrix} = \frac{\partial^2 E^*(x,y)}{\partial n_y \partial s_x}.\end{aligned}$$

Lemma 2.3. *For $x \neq y$, the following equality holds*

$$\frac{\partial^2 E(x,y)}{\partial n_x \partial n_y} + \frac{\partial^2 E(x,y)}{\partial s_x \partial s_y} = -E^*(x,y)\cos(n_x, n_y).$$

Proof. A computation shows

$$\frac{\partial^2 E(x,y)}{\partial n_x \partial n_y} + \frac{\partial^2 E(x,y)}{\partial s_x \partial s_y} = (n_1(x)n_1(y) + n_2(x)n_2(y))\Big(\frac{\partial^2 E(x,y)}{\partial x_1 \partial y_1} + \frac{\partial^2 E(x,y)}{\partial x_2 \partial y_2}\Big).$$

Hence we have

$$\frac{\partial^2 E(x,y)}{\partial n_x \partial n_y} + \frac{\partial^2 E(x,y)}{\partial s_x \partial s_y} = -\Delta_x E(x,y)\cos(n_x, n_y) = -E^*(x,y)\cos(n_x,n_y).$$

By the lemmas 2.1–2.3 and the formulas (1.6) and (1.7), the two integrals given by (1.8) can be rewritten in the following:

$$\begin{aligned} u_1(x) \equiv & \int_\Gamma M_y E(x,y) f_1(y) ds_y = \int_\Gamma E^*(x,y) f_1(y) ds_y + (1-\nu)\int_\Gamma \frac{\partial E(x,y)}{\partial s_y}\dot{f}_1(y) ds_y \\ & - (1-\nu)\int_\Gamma \frac{\partial E(x,y)}{\partial n_y}\omega(y,n_y) f_1(y) ds_y \\ \equiv & u_{11}(x) + u_{12}(x) + u_{13}(x), \ x \in \Omega; \end{aligned} \tag{2.2}$$

$$\begin{aligned} u_2(x) \equiv & \int_\Gamma T_y E(x,y) f_2(y) ds_y = -\int_\Gamma \frac{\partial E^*(x,y)}{\partial n_y} f_2(y) ds_y - (1-\nu)\int_\Gamma \frac{\partial E(x,y)}{\partial n_y}\ddot{f}_2(y) ds_y \\ & - (1-\nu)\int_\Gamma \frac{\partial E(x,y)}{\partial s_y}\omega(y,n_y)\dot{f}_2(y) ds_y \\ \equiv & u_{21}(x) + u_{22}(x) + u_{23}(x), \ x \in \Omega, \end{aligned} \tag{2.3}$$

where $\dot{f}_i(y) = \dfrac{df_i(y)}{ds_y}, \ddot{f}_i(y) = \dfrac{d^2 f_i(y)}{ds_y^2}$ for $i = 1, 2$. Hence we obtain

$$u(x) = \sum_{i=1}^{2}\sum_{j=1}^{3} u_{ij}(x). \tag{2.4}$$

For $x \in \Gamma, z \in \Omega$, and $n_z = (n_1(x), n_2(x))^T$, $s_z = (-n_2(x), n_1(x))^T$, which are the unit outer normal vector at $x \in \Gamma$ and the unit tangential vector at $x \in \Gamma$ respectively. $M_z u(z), T_z u(z)$ can be reduced to the following forms:

$$M_z u(z) = \sum_{i=1}^{2}\sum_{j=1}^{3} M_z u_{ij}(z), \ z \in \Omega, \tag{2.5}$$

$$T_z u(z) = \sum_{i=1}^{2}\sum_{j=1}^{3} T_z u_{ij}(z), \ z \in \Omega, \tag{2.6}$$

where

$$M_z u_{11}(z) = -(1-\nu)\frac{d^2}{ds_z^2}\int_\Gamma E^*(z,y) f_1(y) ds_y - (1-\nu)\omega(z,n_z)\int_\Gamma \frac{\partial E^*(z,y)}{\partial n_z} f_1(y) ds_y, \tag{2.7}$$

$$\begin{aligned} M_z u_{12}(z) = & -(1-\nu)\int_\Gamma E^*(z,y)\ddot{f}_1(y) ds_y - (1-\nu)^2 \frac{d}{ds_z}\int_\Gamma \frac{\partial^2 E(z,y)}{\partial s_z \partial s_y}\dot{f}_1(y) ds_y \\ & - (1-\nu)^2\omega(z,n_z)\int_\Gamma \frac{\partial^2 E(z,y)}{\partial n_z \partial s_y}\dot{f}_1(y) ds_y, \end{aligned} \tag{2.8}$$

$$\begin{aligned}
M_z u_{13}(z) =& -(1-\nu)\int_\Gamma \frac{\partial E^*(z,y)}{\partial n_y}\omega(y,n_y)f_1(y)ds_y \\
&+(1-\nu)^2\frac{d}{ds_z}\int_\Gamma \frac{\partial^2 E(z,y)}{\partial s_z \partial n_y}\omega(y,n_y)f_1(y)ds_y \\
&-(1-\nu)^2\omega(z,n_z)\int_\Gamma \frac{\partial^2 E(z,y)}{\partial s_z \partial s_y}\omega(y,n_y)f_1(y)ds_y \\
&-(1-\nu)^2\omega(z,n_z)\int_\Gamma E^*(z,y)\cos(n_z,n_y)\omega(y,n_y)f_1(y)ds_y, \qquad (2.9)
\end{aligned}$$

$$\begin{aligned}
M_z u_{21}(z) =& -(1-\nu)\frac{d}{ds_z}\int_\Gamma \frac{\partial E^*(z,y)}{\partial n_z}\dot{f}_2(y)ds_y \\
&+(1-\nu)\omega(z,n_z)\frac{d}{ds_z}\int_\Gamma E^*(z,y)\dot{f}_2(y)ds_y, \qquad (2.10)
\end{aligned}$$

$$\begin{aligned}
M_z u_{22}(z) =& -(1-\nu)\int_\Gamma \frac{\partial E^*(z,y)}{\partial n_y}\ddot{f}_2(y)ds_y + (1-\nu)^2\frac{d}{ds_z}\int_\Gamma \frac{\partial^2 E(z,y)}{\partial s_z \partial n_y}\ddot{f}_2(y)ds_y \\
&-(1-\nu)^2\omega(z,n_z)\int_\Gamma \frac{\partial^2 E(z,y)}{\partial s_z \partial s_y}\ddot{f}_2(y)ds_y \\
&-(1-\nu)^2\omega(z,n_z)\int_\Gamma E^*(z,y)\cos(n_z,n_y)\ddot{f}_2(y)ds_y, \qquad (2.11)
\end{aligned}$$

$$\begin{aligned}
M_z u_{23}(z) =& (1-\nu)\int_\Gamma E^*(z,y)\frac{d}{ds_y}(\omega(y,n_y)\dot{f}_2(y))ds_y \\
&+(1-\nu)^2\frac{d}{ds_z}\int_\Gamma \frac{\partial^2 E(z,y)}{\partial s_z \partial s_y}\omega(y,n_y)\dot{f}_2(y)ds_y \\
&+(1-\nu)^2\omega(z,n_z)\int_\Gamma \frac{\partial^2 E(z,y)}{\partial n_z \partial s_y}\omega(y,n_y)\dot{f}_2(y)ds_y. \qquad (2.12)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
T_z u_{11}(z) =& -(1-\nu)\frac{d^2}{ds_z^2}\int_\Gamma \frac{\partial E^*(z,y)}{\partial n_z}f_1(y)ds_y \\
&+(1-\nu)\frac{d}{ds_z}\Big(\omega(z,n_z)\frac{d}{ds_z}\Big)\int_\Gamma E^*(z,y)f_1(y)ds_y, \qquad (2.13)
\end{aligned}$$

$$\begin{aligned}
T_z u_{12}(z) =& -(1-\nu)\frac{d}{ds_z}\int_\Gamma \frac{\partial E^*(z,y)}{\partial n_y}\dot{f}_1(y)ds_y \\
&-(1-\nu)^2\frac{d^2}{ds_z^2}\int_\Gamma \frac{\partial^2 E(z,y)}{\partial n_z \partial s_y}\dot{f}_1(y)ds_y \\
&+(1-\nu)^2\frac{d}{ds_z}\Big(\omega(z,n_z)\frac{d}{ds_z}\Big)\int_\Gamma \frac{\partial E(z,y)}{\partial s_y}\dot{f}_1(y)ds_y, \qquad (2.14)
\end{aligned}$$

$$\begin{aligned}
T_z u_{13}(z) =& (1-\nu)\frac{d}{ds_z}\int_\Gamma E^*(z,y)\frac{d}{ds_y}(\omega(y,n_y)f_1(y))ds_y \\
&-(1-\nu)^2\frac{d^2}{ds_z^2}\int_\Gamma \frac{\partial^2 E(z,y)}{\partial s_z \partial s_y}\omega(y,n_y)f_1(y)ds_y \\
&-(1-\nu)^2\frac{d^2}{ds_z^2}\int_\Gamma E^*(z,y)\cos(n_z,n_y)\omega(y,n_y)f_1(y)ds_y
\end{aligned}$$

$$- (1-\nu)^2 \frac{d}{ds_z}\Big(\omega(z,n_z)\frac{d}{ds_z}\Big)\int_\Gamma \frac{\partial E(z,y)}{\partial n_y}\omega(y,n_y)f_1(y)ds_y, \tag{2.15}$$

$$\begin{aligned} T_z u_{21}(z) =&(1-\nu)\frac{d^3}{ds_z^3}\int_\Gamma E^*(z,y)\dot{f}_2(y)ds_y \\ &- (1-\nu)\frac{d}{ds_z}\Big(\omega(z,n_z)\frac{d}{ds_z}\Big)\int_\Gamma \frac{\partial E^*(z,y)}{\partial n_y}f_2(y)ds_y, \end{aligned} \tag{2.16}$$

$$\begin{aligned} T_z u_{22}(z) =&(1-\nu)\frac{d}{ds_z}\int_\Gamma E^*(z,y)\dddot{f}_2(y)ds_y \\ &- (1-\nu)^2\frac{d^2}{ds_z^2}\int_\Gamma \frac{\partial^2 E(z,y)}{\partial s_z \partial s_y}\ddot{f}_2(y)ds_y \\ &- (1-\nu)^2\frac{d^2}{ds_z^2}\int_\Gamma E^*(z,y)\cos(n_z,n_y)\ddot{f}_2(y)ds_y \\ &- (1-\nu)^2\frac{d}{ds_z}\Big(\omega(z,n_z)\frac{d}{ds_z}\Big)\int_\Gamma \frac{\partial E(z,y)}{\partial n_y}\ddot{f}_2(y)ds_y, \end{aligned} \tag{2.17}$$

$$\begin{aligned} T_z u_{23}(z) =&(1-\nu)\frac{\partial}{\partial n_z}\int_\Gamma \frac{\partial E^*(z,y)}{\partial s_y}(\omega(y,n_y)\dot{f}_2(y))ds_y \\ &+ (1-\nu)^2\frac{d^2}{ds_z^2}\int_\Gamma \frac{\partial^2 E(z,y)}{\partial n_z \partial s_y}(\omega(y,n_y)\dot{f}_2(y))ds_y \\ &- (1-\nu)^2\frac{d}{ds_z}\Big(\omega(z,n_z)\frac{d}{ds_z}\Big)\int_\Gamma \frac{\partial E(z,y)}{\partial s_y}\omega(y,n_y)\dot{f}_2(y)ds_y. \end{aligned} \tag{2.18}$$

By the properties of the potentials in (2.7)–(2.12), we have

$$\begin{aligned} \lim_{z\in\Omega\to x\in\Gamma} M_z u(z) =& \lim_{z\in\Omega\to x\in\Gamma} M_z\Big\{\sum_{i=1}^{2}\sum_{j=1}^{3} u_{ij}(z)\Big\} \\ =&M_x\Big(u_{11}+\frac{(1-\nu)}{2}\omega(x,n_x)f_1(x)\Big)+M_x u_{12}(x) \\ &+ M_x\Big(u_{13}-\frac{(1-\nu)}{2}\omega(x,n_x)f_1(x)\Big)+M_x\Big(u_{21}+\frac{(1-\nu)}{2}\ddot{f}_2(x)\Big) \\ &+ M_x\Big(u_{22}-\frac{(1-\nu)}{2}\ddot{f}_2(x)\Big)+M_x u_{23}(x) = M_x u(x). \end{aligned}$$

Similarly, we obtain $\lim\limits_{z\in\Omega\to x\in\Gamma} T_z u(z) = T_x u(x)$. Hence if $u(x)$ given by (1.8) is a solution of (1.4), then f_1, f_2 satisfying the following system of boundary integro-differential equations

$$M_x\int_\Gamma M_y E(x,y)f_1(y)ds_y + M_x\int_\Gamma T_y E(x,y)f_2(y)ds_y = m(x),\ x\in\Gamma, \tag{2.19}$$

$$T_x\int_\Gamma M_y E(x,y)f_1(y)ds_y + T_x\int_\Gamma T_y E(x,y)f_2(y)ds_y = t(x),\ x\in\Gamma. \tag{2.20}$$

Before we discuss system (2.19) and (2.20), we define $V^*(\Gamma) = \Big\{(m,t)\in H^{-\frac{1}{2}}(\Gamma)\times H^{-\frac{3}{2}}(\Gamma)$ and $\int_\Gamma\Big(m\frac{\partial p}{\partial n}+tp\Big)ds = 0,\ \forall p\in P_1(\Gamma)\Big\}$, and recall some results of the original problems (1.4) and (1.5)[11,14]

Theorem 2.1. *Suppose that* $(m,t) \in V^*(\Gamma)$ *then the problem* (1.4) *exists a weak solution* $u(x) \in H^2(\Omega)$, *unique up to a linear function* $p \in P_1(\bar{\Omega})$.

Theorem 2.2. *Assume the given functions* $(m,t) \in V^*(\Gamma)$ *then the problem* (1.5) *has a weak solution* $u(x) \in H^2_{loc}(\Omega^c)$ *satisfying the linear - logarithmic growth condition, and* $u(x)$ *is unique up to a polynomial of degree one.*

3. The Variational Formulation of the System of Boundary Integro-Differential Equations (2.19) and (2.20)

We now discuss the system of boundary integro-differential equations (2.19) and (2.20) for given $(m,t) \in V^*(\Gamma)$. For any $(g_1, g_2) \in H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma)$, multiplying (2.19) and (2.20) by g_1 and g_2 respectively, then integrating by parts, we obtain

$$A_1(f_1, g_1) + B_0(f_2, g_1) = \langle m, g_1 \rangle, \tag{3.1}$$

$$B_0^T(f_1, g_2) + A_2(f_2, g_2) = \langle t, g_2 \rangle, \tag{3.2}$$

where

$$\begin{aligned}
A_1(f_1,g_1) =& \int_\Gamma g_1(x)\Big(\sum_{j=1}^{3} M_x u_{1j}(x)\Big)ds_x \\
=& -(1-\nu)\int_\Gamma\int_\Gamma E^*(x,y)\{g_1(x)\ddot{f}_1(y) + \ddot{g}_1(x)f_1(y)\}ds_y ds_x \\
& -(1-\nu)\int_\Gamma\int_\Gamma \Big\{\frac{\partial E^*(x,y)}{\partial n_x}(\omega(x,n_x)g_1(x))f_1(y) \\
& + \frac{\partial E^*(x,y)}{\partial n_y} g_1(x)(\omega(y,n_y)f_1(y))\Big\}ds_y ds_x \\
& + (1-\nu)^2\int_\Gamma\int_\Gamma \Big\{\frac{\partial E(x,y)}{\partial n_x}(\omega(x,n_x)g_1(x))\ddot{f}_1(y) \\
& + \frac{\partial E(x,y)}{\partial n_y}\ddot{g}_1(x)(\omega(y,n_y)f_1(y))\Big\}ds_y ds_x \\
& + (1-\nu)^2\int_\Gamma\int_\Gamma E(x,y)\ddot{g}_1(x)\ddot{f}_1(y) ds_y ds_x \\
& - (1-\nu)^2\int_\Gamma\int_\Gamma E(x,y)(\omega(x,n_x)g_1(x))^{\cdot}(\omega(y,n_y)f_1(y))^{\cdot} ds_y ds_x \\
& - (1-\nu)^2\int_\Gamma\int_\Gamma E^*(x,y)\cos(n_x,n_y) \\
& (\omega(x,n_x)g_1(x))(\omega(y,n_y)f_1(y))ds_y ds_x, \\
B_0(f_2,g_1) =& \int_\Gamma g_1(x)\Big(\sum_{j=1}^{3} M_x u_{2j}(x)\Big)ds_x \\
=& (1-\nu)\int_\Gamma\int_\Gamma \frac{\partial E^*(x,y)}{\partial n_x}\dot{g}_1(x)\dot{f}_2(y)ds_y ds_x \\
& - (1-\nu)\int_\Gamma\int_\Gamma E^*(x,y)(\omega(x,n_x)g_1(x))^{\cdot}\dot{f}_2(y) ds_y ds_x
\end{aligned}$$

$$
\begin{aligned}
&- (1-\nu)\int_\Gamma\int_\Gamma \frac{\partial E^*(x,y)}{\partial n_y} g_1(x)\ddot{f}_2(y)ds_y ds_x\\
&+ (1-\nu)^2\int_\Gamma\int_\Gamma \frac{\partial E(x,y)}{\partial n_y}\ddot{g}_1(x)\ddot{f}_2(y)ds_y ds_x\\
&- (1-\nu)^2\int_\Gamma\int_\Gamma E(x,y)(\omega(x,n_x)g_1(x))^{\cdot}\ \dddot{f}_2\ (y)ds_y ds_x\\
&- (1-\nu)^2\int_\Gamma\int_\Gamma E^*(x,y)\cos(n_x,n_y)(\omega(x,n_x)g_1(x))\ddot{f}_2(y)ds_y ds_x.\\
&+ (1-\nu)\int_\Gamma\int_\Gamma E^*(x,y)g_1(x)(\omega(y,n_y)\dot{f}_2(y))^{\cdot}ds_y ds_x\\
&- (1-\nu)^2\int_\Gamma\int_\Gamma E(x,y)\ddot{g}_1(x)(\omega(y,n_y)\dot{f}_2(y))^{\cdot}ds_y ds_x\\
&- (1-\nu)^2\int_\Gamma\int_\Gamma \frac{\partial E(x,y)}{\partial n_x}(\omega(x,n_x)g_1(x))(\omega(y,n_y)\dot{f}_2(y))^{\cdot}ds_y ds_x,
\end{aligned}
$$

$$
\begin{aligned}
B_0^T(f_1,g_2) =& B_0(g_2,f_1),\\
A_2(f_2,g_2) =& \int_\Gamma g_2(x)T_x\Big(\sum_{j=1}^{3}u_{2j}(x)\Big)ds_x\\
=& - (1-\nu)\int_\Gamma\int_\Gamma E^*(x,y)\{\dddot{g}_2\ (x)\dot{f}_2(y)+\dot{g}_2(x)\ \dddot{f}_2\ (y)\}ds_y ds_x\\
&- (1-\nu)\int_\Gamma\int_\Gamma \Big\{\frac{\partial E^*(x,y)}{\partial n_y}(\omega(x,n_x)\dot{g}_2(x))^{\cdot}f_2(y)\\
&+ \frac{\partial E^*(x,y)}{\partial n_x}g_2(x)(\omega(y,n_y)\dot{f}_2(y))^{\cdot}\Big\}ds_y ds_x\\
&- (1-\nu)^2\int_\Gamma\int_\Gamma \Big\{\frac{\partial E(x,y)}{\partial n_y}(\omega(x,n_x)\dot{g}_2(x))^{\cdot}\ddot{f}_2(y)\\
&+ \frac{\partial E(x,y)}{\partial n_x}\ddot{g}_2(x)(\omega(y,n_y)\dot{f}_2(y))^{\cdot}\}ds_y ds_x\\
&- (1-\nu)^2\int_\Gamma\int_\Gamma E(x,y)\{\dddot{g}_2\ (x)\ \dddot{f}_2\ (y)\\
&+ (\omega(x,n_x)\dot{g}_2(x))^{\cdot}(\omega(y,n_y)\dot{f}_2(y))^{\cdot}\Big\}ds_y ds_x\\
&- (1-\nu)^2\int_\Gamma\int_\Gamma E^*(x,y)\cos(n_x,n_y)\ddot{g}_2(x)\ddot{f}_2(y)ds_y ds_x.
\end{aligned}
$$

Let $I(f_1,f_2;g_1,g_2) = A_1(f_1,g_1) + B_0(f_2,g_1) + B_0^T(f_1,g_2) + A_2(f_2,g_2)$.

It is straightforward to know that $I(f_1,f_2;g_1,g_2)$ is a bounded bilinear form on $(H^{\frac{1}{2}}(\Gamma)\times H^{\frac{3}{2}}(\Gamma))^2$ and $I(f_1,f_2;g_1,g_2)$ is symmetric.

Suppose $u(x)$ given by (1.8) in domain Ω^c satisfies the regularity condition then (f_1,f_2) also satisfies the system of boundary integro-differential equations (2.19) and (2.20).

$[T_x u]_\Gamma, [M_x u]_\Gamma, \Big[\dfrac{\partial u}{\partial n}\Big]_\Gamma$, $[u]_\Gamma$ are designated for the jumps of the corresponding functions on the boundary Γ.

Let $V_*(\Gamma)=\Big\{(g_1,g_2)\in H^{1/2}(\Gamma)\times H^{\frac{3}{2}}(\Gamma)$ and $\int_\Gamma\Big(g_1\frac{\partial p}{\partial n}+g_2p\Big)ds=0,\ \forall p\in P_1(\Gamma)\Big\}$. For any $(g_1,g_2)\in V_*(\Gamma)$, let

$$u_g(x)=\int_\Gamma M_yE(x,y)g_1(y)ds_y+\int_\Gamma T_yE(x,y)g_2(y)ds_y.$$

By the properties of the triple layar and quadruple layar potentials[11], we know that u_g satisfies the linear - logarithmic growth condition when $|x|\to\infty$, then we have

$$\left.\begin{aligned}u_g=&(a_0+a_1x_1+a_2x_2)+A\log|x|+o(1)\\&\text{for some } a_0,a_1,a_2,A\in\Re,\ \text{when}x|\to\infty.\end{aligned}\right\}\tag{3.3}$$

If u_g satisfies the linear - logarithmic growth condition with $a_0=a_1=a_2=0$; then we say u_g satisfies the strongly regular condition, when $|x|\to\infty$.

Let $w_g=u_g-(a_0+a_1x_1+a_2x_2)$, where a_0,a_1,a_2 are given by (3.3).

Then w_g is the unique solution of the following problem:

$$\begin{cases}\Delta^2w_g=0, & \text{in } \Omega\cup\Omega^c,\\ [w_g]_\Gamma=g_2, & \text{on } \Gamma,\\ \Big[\dfrac{\partial w_g}{\partial n}\Big]_\Gamma=g_1, & \text{on } \Gamma,\\ [M_xw_g]_\Gamma=0, & \text{on } \Gamma,\\ [T_xw_g]_\Gamma=0, & \text{on } \Gamma,\\ \quad w_g & \text{satisfies the strongly regular condition, when } |x|\to\infty.\end{cases}\tag{3.4}$$

We now introduce the space $U=\Big\{u|\ u|_\Omega\in H^2(\Omega),\ u|_{\Omega^c}\in H^2_{loc}(\Omega^c),\ \Delta^2u=0$, in $\Omega\cup\Omega^c$, $[Mu]_\Gamma=0$, $[Tu]_\Gamma=0$, u satisfies the strongly regular condition when $|x|\to\infty$, and $\int_\Gamma\Big\{\Big[\frac{\partial u}{\partial n}\Big]_\Gamma\frac{\partial p}{\partial n}+[u]_\Gamma p\Big\}ds=0,\ \forall p\in P_1(\Gamma)\Big\}$. Let $a(u,v)=\int_{\Omega\cup\Omega^c}\Big(\Delta u\Delta v-(1-\nu)\Big(\frac{\partial^2u}{\partial x_1^2}\frac{\partial^2v}{\partial x_2^2}+\frac{\partial^2u}{\partial x_2^2}\frac{\partial^2v}{\partial x_1^2}-2\frac{\partial^2u}{\partial x_1\partial x_2}\frac{\partial^2v}{\partial x_1\partial x_2}\Big)\Big)dx$.

For any $u,w\in U, a(u,w)$ is bounded. It is straightforward to check $a(u,w)$ is an inner product of U, $\sqrt{a(u,u)}$ is a norm of U, and U is a Banach space with $|||u|||=\sqrt{a(u,u)}$. Hence we obtain a linear operator K:

$$\begin{aligned}V_*(\Gamma)\longrightarrow&U,\\ K(g_1,g_2)=&w_g,\end{aligned}$$

the operator K is one to one correspondence.

By an application of Green formula, we have $I(f_1,f_2;g_1,g_2)=a(w_f,w_g)$. Then we know the operator K is bounded. Furthermore from Banach Theorem, K^{-1} is bounded. Hence there is a constant C, such that

$$\|(g_1,g_2)\|_{V_*(\Gamma)}=\|K^{-1}w_g\|_{V_*(\Gamma)}\le C|||w_g|||,\ \forall(g_1,g_2)\in V_*(\Gamma),$$

namely

$$|||w_g||| \geq \frac{1}{C}\|(g_1, g_2)\|_{V_*(\Gamma)}.$$

Furthermore, we have

$$I(g_1, g_2; g_1, g_2) = a(w_g, w_g) \equiv |||w_g|||^2 \geq \frac{1}{C^2}\|(g_1, g_2)\|^2_{V_*(\Gamma)}.$$

Then we have the following lemma.

Lemma 3.1. *$I(f_1, f_2; g_1, g_2)$ is a bounded bilinear form on $V_*(\Gamma) \times V_*(\Gamma)$, namely there is a constant $M > 0$, such that*

$$|I(f_1, f_2; g_1, g_2)| \leq M\|(f_1, f_2)\|_{V_*(\Gamma)}\|(g_1, g_2)\|_{V_*(\Gamma)}, \ \forall (f_1, f_2), (g_1, g_2) \in V_*(\Gamma). \quad (3.5)$$

Furthermore $I(f_1, f_2; g_1, g_2)$ is a coercive in $V_*(\Gamma)$, there exists a constant $\alpha > 0$, such that

$$I(g_1, g_2; g_1, g_2) \geq \alpha\|(g_1, g_2)\|^2_{V_*(\Gamma)}, \ \forall (g_1, g_2) \in V_*(\Gamma). \quad (3.6)$$

Finally the original boundary value problem (1.4) is reduced to the following variational problem:

$$\begin{cases} \text{Find } \ (f_1, f_2) \ in V_*(\Gamma), \quad \text{such that} \\ I(f_1, f_2; g_1, g_2) = \langle (m, t), (g_1, g_2) \rangle, \ \forall (g_1, g_2) \in V_*(\Gamma), \end{cases} \quad (3.7)$$

where $\langle (m, t), (g_1, g_2) \rangle = \langle m, g_1 \rangle + \langle t, g_2 \rangle$.

By an application of Lax-Milgram theorem [10] [13], we have

Theorem 3.1. *For given $(m, t) \in V^*(\Gamma)$, the variational problem* (3.7) *has a unique solution $(f_1, f_2) \in V_*(\Gamma)$.*

Suppose that S^h is a finite dimensional subspace of $V_*(\Gamma)$, then we consider the approximation of (3.7).

$$\begin{cases} \text{Find } (f_1^h, f_2^h) \in S^h, \quad \text{such that} \\ I(f_1^h, f_2^h; g_1, g_2) = \langle (m, t), (g_1, g_2) \rangle, \ \forall (g_1, g_2) \in S^h. \end{cases} \quad (3.8)$$

By an application of Lax-Milgram theorem and Cea's lemma, we obtain:

Theorem 3.2. *The problem* (3.8) *exists a unique solution $(f_1^h, f_2^h) \in S^h$ satisfying the following estimation*

$$\|(f_1, f_2) - (f_1^h, f_2^h)\|_{V_*(\Gamma)} \leq \frac{M}{\alpha} \inf_{(g_1, g_2) \in S^h} \|(f_1, f_2) - (g_1, g_2)\|_{V_*(\Gamma)}. \quad (3.9)$$

4. Numerical Examples

Consider the following problem

$$\begin{cases} \Delta^2 u = & \text{in } \Omega, \\ M_x u = m & \text{on } \Gamma, \\ T_x u = t & \text{on } \Gamma, \end{cases} \quad (4.1)$$

where

$$\Omega = \Big\{(x_1, x_2) \in R^2 \Big| \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1, a > 0, b > 0\Big\}.$$

is an ellipse, the parametric equation of the boundary Γ is

$$\begin{cases} x_1 = a\cos t, \\ x_2 = b\sin t, \ \ (0 < t \le 2\pi). \end{cases} \tag{4.2}$$

We consider the following two groups of boundary condition $(m_1(t), t_1(t))$ and $(m_2(t), t_2(t))$, which are given in (4.3) and (4.4) respectively.

$$\begin{cases} m_1(t) = 2(1-\nu)(f_1^2(t) - f_2^2(t)), \\ t_1(t) = 4(1-\nu)f_0(t)[f_3(t)f_2(t) + f_1(t)f_4(t)], \end{cases} \tag{4.3}$$

$$\begin{cases} m_2(t) = 2\nu\cos(a\cos t)e^{b\sin t} \\ \qquad +(1-\nu)[f_1^2(t)f_5(t) + f_2^2(t)f_6(t) + 2f_1(t)f_2(t)f_7(t)], \\ t_2(t) = [2f_1(t)\sin(a\cos t) - 2f_2(t)\cos(a\cos t)]e^{b\sin t} \\ \qquad +(1-\nu)\{f_1(t)f_2(t)f_8(t) + [f_1(t)f_4(t) + f_2(t)f_3(t)]f_9(t) \\ \qquad -[f_1^2(t) - f_2^2(t)]f_{10}(t) - 2[f_1(t)f_3(t) - f_2(t)f_4(t)]f_7(t)\}f_0(t), \end{cases} \tag{4.4}$$

where

$$\begin{aligned}
f_0(t) =& \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, \\
f_1(t) =& \frac{b\cos t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, \\
f_2(t) =& \frac{a\sin t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, \\
f_3(t) =& -\frac{a^2 b\sin t}{(\sqrt{a^2\sin^2 t + b^2\cos^2 t})^3}, \\
f_4(t) =& \frac{ab^2\cos t}{(\sqrt{a^2\sin^2 t + b^2\cos^2 t})^3}, \\
f_5(t) =& (2\cos(a\cos t) - a\cos t\sin(a\cos t))e^{b\sin t}, \\
f_6(t) =& a\cos t\sin(a\cos t)e^{b\sin t}, \\
f_7(t) =& (\sin(a\cos t) + a\cos t\cos(a\cos t))e^{b\sin t}, \\
f_8(t) =& (4a\sin t\sin(a\cos t) + a^2\sin(2t)\cos(a\cos t) \\
& + 2b\cos t\cos(a\cos t) - 2ab\cos^2 t\sin(a\cos t))e^{b\sin t}, \\
f_9(t) =& 2(\cos(a\cos t) - a\cos t\sin(a\cos t))e^{b\sin t}, \\
f_{10}(t) =& \Big(-2a\sin t\cos(a\cos t) + \frac{a^2}{2}\sin(2t)\sin(a\cos t) \\
& + b\cos t\sin(a\cos t) + ab\cos^2 t\cos(a\cos t)\Big)e^{b\sin t}.
\end{aligned}$$

Then $u_1(x) = x_1^2 - x_2^2$ is a exact solution of problem (4.1) with boundary condition (4.3) and $u_2(x) = x_1\sin(x_1)e^{x_2}$ is a exact solution of problem (4.1) with boundary condition (4.4).

First of all, the boundary Γ is divided into four segmental arcs by four nodes $a_8, a_{16}, a_{24}, a_{32}$ as shown in Fig.1, the division is denoted by partition A. Then the partition is refined by dividing every segmental arcs into two parts. We obtain the partition B consisting of 8 segmental arcs corresponding to the nodes $\{a_i,\ i = 4, 8, 12, \cdots, 32\}$. Refine it again and again, then the partition C consisting of 16 segmental arcs and the partition D consisting of 32 segmental arcs are obtained as shown in Fig.1. The coordinate components of the nodes $\{a_i\}$, $(i = 1, 2, \cdots, 32)$ are $\left(a\cos\left(\frac{i2\pi}{32}\right),\ b\sin\left(\frac{i2\pi}{32}\right)\right)$, $(i = 1, 2, \cdots, 32)$.

As the solution of the equation is unique except a polynomial of degree one, we fix three nodes so that the approximation values of the solution equal the values of the solution at these three nodes. The three nodes chosen are a_8, a_{16}, a_{24}. We choose two kinds of parameters $(a, b, \nu) = (2.0, 1.0, 0.3)$, and $(a, b, \nu) = (1.5, 1.0, 0.3)$.

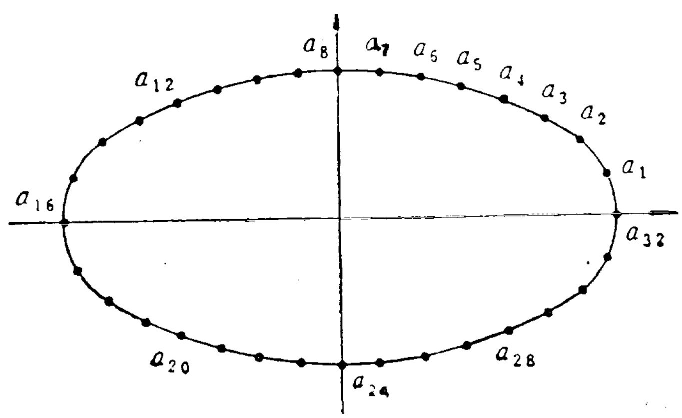
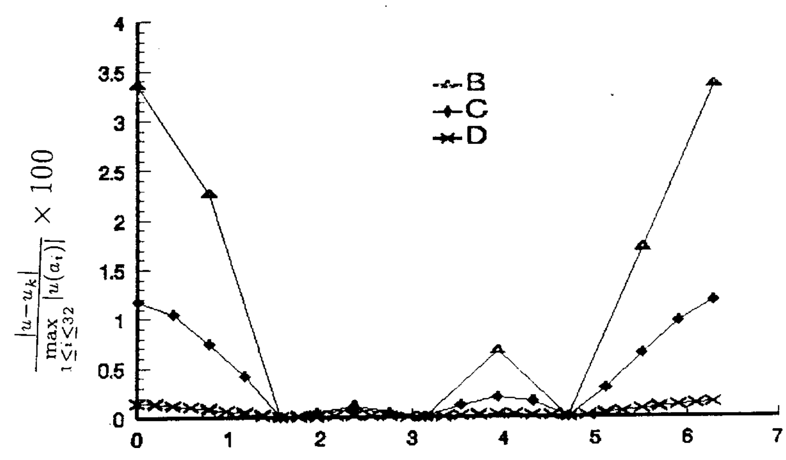


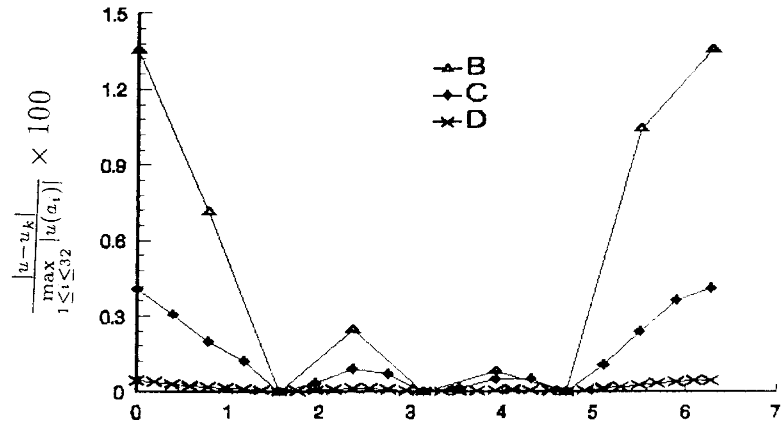
Fig. 1. Partition $A - D$.



(a=2.0, b=1.0, $\nu = 0.3$. Boundary condition (4.3))

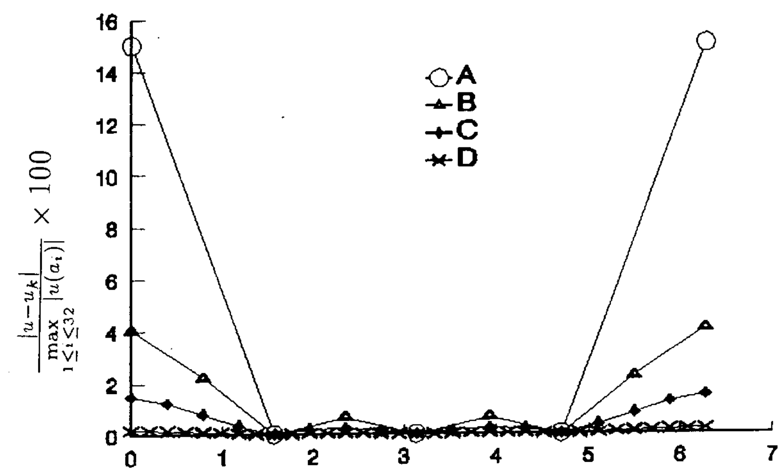
Fig. 2. Relative errors.

u_A, u_B, u_C, and u_D denote the boundary finite-element approximations of the problem (4.1) corresponding to the partition A, B, C, and D and piecewise spline boundary elements. u denotes the exact solution of the problem (4.1). We get the values of u_A, u_B, u_C, u_D and u on the nodes $\{a_i, i = 1, 2, \cdots, 32\}$. The relative errors $\frac{|u - u_k|}{\max\limits_{1\le i\le 32} |u(a_i)|}$ $(k = B, C, D)$ are given in following Fig.2–Fig.3 corresponding the boundary condition (4.3) and the relative errors $\frac{|u - u_k|}{\max\limits_{1\le i\le 32} |u(a_i)|}$ $(k = A, B, C, D)$ are given in following Fig.4–Fig.5 corresponding the boundary condition (4.4). These numerical examples show the Integro-Differential boundary finite-element method is very effective.



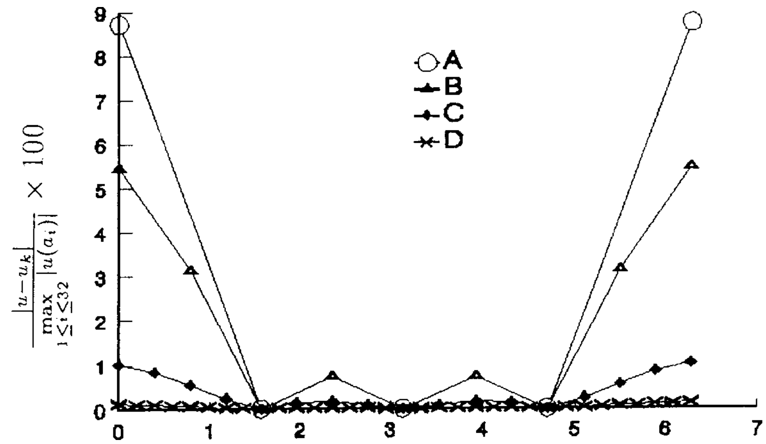
(a=1.5, b=1.0, $\nu = 0.3$. Boundary condition (4.3))

Fig. 3. Relative errors.



(a=2.0, b=1.0, $\nu = 0.3$. Boundary condition (4.4))

Fig. 4. Relative errors.



(a=1.5, b=1.0, $\nu = 0.3$. Boundary condition (4.4))

Fig. 5. Relative errors.

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