Journal of Computational Mathematics, Vol.18, No.6, 2000, 657–672.

INFINITE ELEMENT METHOD FOR THE EXTERIOR PROBLEMS OF THE HELMHOLTZ EQUATIONS*1)

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Abstract

There are two cases of the exterior problems of the Helmholtz equation. If $\lambda \geq 0$ the bilinear form is coercive, and if $\lambda < 0$ it is the scattering problem. We give a new approach of the infinite element method, which enables us to solve these exterior problems as well as corner problems. A numerical example of the scattering problem is given.

Key words: Helmholtz equation, Exterior problem, Infinite element method.

1. Introduction

The infinite element method has been successfully applied to some boundary value problems of partial differential equations, where the solutions possess corner singular points or the domains are exterior ones. If the equations are invariant under similarity transformation the approaches have been given in [11][13] for singular solutions, and in [12][15][18][19] for the exterior problems. If the equations do not admit the above invariant property, one approach has been given in [14] to deal with the singular solutions to the Helmholtz equation, and another approach has been given in [16] to deal with the singular solutions to more general problems where the coefficients of the equations are allowed to be variable and discontinuous. For details see [17] and the references therein.

For the exterior problems of the equations which are not invariant under similarity transformation the above approaches are not valid. Because in [14] the solutions are expanded into Taylor series about the parameter λ, and in [16] the exact solution can be divided into two parts, one of which is a solution to an associated equation which is invariant under similarity transformation, and the other one of which is a regular function. Now the solutions are neither analytic nor the sum of these two parts.

We will give a new approach of the infinite element method in this paper, which enables us to deal with these problems, and it is also an efficient approach to solve the singular solutions. Firstly we study the Helmholtz equation, and the infinite element method for the exterior problems of some other equations will be given in separate papers.

The terminology of "infinite element" has been employed by many authors for different methods. For example it is employed in infinitely large elements are used on the neighborhood of the infinity and some special interpolation functions are applied

* Received November 13, 1998.
1)This work was supported by the China State Major Key Project for Basic Researches and the Science Fund of the Ministry of Education of China.

in the element to simulate the behavior of the solutions near the infinity. Our approach is different. In our approach the size of elements are finite and the number of elements is infinite. Without any truncation we solve the infinite-by-infinite algebraic systems associated with those elements. In fact for each equation we only solve one equivalent algebraic problem which is of extremely small scale. The formulation and the elements of our method are the same as the finite element method, therefore the rate of convergence is the same even if there are singularities or the domain is infinitely large. In our method no analytic expression of the true solution is needed, so there is no special requirement to the domain or to the equations. For example, this method can be applied to the equations with variable coefficients.

This paper is organized as follows: For completeness we recall the infinite element method for the Laplace equation in §2. To meet the needs of our new approach we study the critical case , where the constant of proportionality tends to one in §3. We derive the equations for the combined stiffness matrix of the infinite element method for the exterior problems of the Helmholtz equation in §4. We give some detail computation for the critical stiffness matrices and mass matrices in §5. Some further discussion on the infinite element algorithm to the Helmholtz equation is carried out in §6. We prove the convergence of the approximate solutions for the case of $\lambda > 0$ in §7. Finally we show one numerical example of the scattering problem in §8.

2. The Laplace Equation

Let $\Omega \subset \mathsf{R}^2$ be an exterior domain, the boundary of which is a closed curve Γ_0. For the sake of simplicity we assume that the origin $o \notin \Omega$ and Γ_0 is star-shape with respect to the point o, that is, all line segments connecting the points of Γ_0 with o lie outside Ω entirely. For those domains with complicated shape we can decompose them as $\Omega = \overline{\Omega_0 \bigcup \Omega'}$, where Ω_0 is a bounded domain, Ω' is an unbounded domain. Usual finite element partition is made on Ω_0 and infinite element partition is made on Ω'. Then we solve an equation which is obtained by assembly of these two. The domain decomposition technique can be applied to this decomposition if one wants. For simplicity we will assume $\Omega = \Omega'$ in the sequel.

We assume that Γ_0 is a polygonal curve. One parameter $\xi > 1$ is taken. We construct similar figures of Γ_0 with the center o and the constant of proportionality $\xi, \xi^2, \cdots, \xi^k, \cdots$, denoted by Γ_k. Let $\xi^k\Omega = \{(x, y); (x, y) \text{ is on the exterior of } \Gamma_k\}$, and $\Omega_k = \xi^{k-1}\Omega \setminus \overline{\xi^k\Omega}$. We make conventional finite element partition on each Ω_k. It is required that the meshes of all subdomains Ω_k are geometrically similar to each other and the partitions on Ω_k and Ω_{k-1} are compatible on Γ_k. For example we can construct some rays starting from the point o which divide each Ω_k into some quadrilaterals, then each quadrilateral is further divided into two triangular elements.

We define space $H^{1,*}(\Omega) = \{u \in L^2_{\rm loc}(\Omega); \|u\|_{1,*} < \infty\}$, where the norm is defined as

$$\|u\|_{1,*} = \left(\int_\Omega \left(|\nabla u(x,y)|^2 + \frac{u^2(x,y)}{r^2 \log^2 r}\right) dxdy\right)^{1/2},$$

where $r = \sqrt{x^2 + y^2}$. Then we define infinite element space

$$S(\Omega) = \{u \in H^{1,*}(\Omega); u|_{e_i} \in P_1(e_i), i = 1, 2, \cdots\},$$

where $e_i, i = 1, 2, \cdots$ are elements, and P_1 is the set of all polynomials with degree ≤ 1.

We define, moreover,

$$H_0^{1,*}(\Omega) = \{u \in H^{1,*}(\Omega); u|_{\Gamma_0} = 0\},$$

$$S_0(\Omega) = S(\Omega) \bigcap H_0^{1,*}(\Omega).$$

Let $u \in S(\Omega)$. The values of u at the notes on Γ_k are composed to a vector, which is denoted by y_k. For the Laplace equation

$$\triangle u = 0 \tag{1}$$

we denote the total stiffness matrix on Ω_1 by

$$\begin{pmatrix} K_0 & -A^T \\ -A & K_0' \end{pmatrix},$$

that is,

$$\int_{\Omega_1} \nabla u \cdot \nabla v \, dxdy = \begin{pmatrix} y_0^T & y_1^T \end{pmatrix} \begin{pmatrix} K_0 & -A^T \\ -A & K_0' \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \qquad \forall u, v \in S(\Omega),$$

where y_0 consists of the nodal values of u on Γ_0, y_1 on Γ_1, and z_0, z_1 correspond to v. We will always write $y_0 = u|_{\Gamma_0}$ and don't distinguish the difference between y_0 and the trace of u on Γ_0.

From the special feature of the equation (1) we know that all stiffness matrices on Ω_k are the same. It is easy to prove that the following boundary value problem: find $u \in S(\Omega)$, $u|_{\Gamma_0} = y_0$,

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dxdy = 0, \qquad \forall v \in S_0(\Omega), \tag{2}$$

admits a unique solution. To define the combined stiffness matrix, we prove the following lemmas:

Lemma 2.1. *Let u be the solution to the problem (2), and $v \in S(\Omega)$, then $a(u, v)$ is independent of the function v provided $v|_{\Gamma_0} = z_0$ is fixed.*

Proof. If $v_1, v_2 \in S(\Omega)$, $v_1|_{\Gamma_0} = v_2|_{\Gamma_0} = z_0$, then by (2) one has

$$a(u, v_1 - v_2) = 0.$$

that is

$$a(u, v_1) = a(u, v_2).$$

By Lemma 2.1, there exists a real matrix K_z such that $a(u, v) = y_0^T K_z z_0$. K_z is defined as the combined stiffness matrix.

Lemma 2.2. *K_z is a symmetric matrix.*

Proof. For given $y_0, z_0 \in \mathsf{R}^n$, where n is the number of nodes on Γ_0, let u, v be the solutions to (2) with boundary values y_0, z_0 respectively. By the definition of K_z one has

$$y_0^T K_z z_0 = a(u, v) = a(v, u) = z_0^T K_z y_0,$$

which yields the result since y_0, z_0 are arbitray.

Let us derive the equation for the matrix K_z. If u is the solution to the problem (2), we take an arbitrary $v \in S(\Omega)$ and let $v|_{\Gamma_0} = z_0$, then we have

$$\begin{aligned} y_0^T K_z z_0 &= \int_\Omega \nabla u \cdot \nabla v \, dxdy = \int_{\Omega_1} \nabla u \cdot \nabla v \, dxdy + \int_{\xi\Omega} \nabla u \cdot \nabla v \, dxdy \\ &= \begin{pmatrix} y_0^T & y_1^T \end{pmatrix} \begin{pmatrix} K_0 & -A^T \\ -A & K_0' \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} + y_1^T K_z z_1, \end{aligned} \tag{3}$$

where $z_1 = v|_{\Gamma_1}$. Since (3) holds for all z_1, one gets

$$-y_0^T A^T + y_1^T K_0' + y_1^T K_z = 0. \tag{4}$$

Thus

$$y_1 = (K_0' + K_z)^{-1} A y_0. \tag{5}$$

$X = (K_0' + K_z)^{-1} A$ is named as a “transfer matrix”. We substitute (5) into (3) and obtain

$$y_0^T K_z z_0 = y_0^T K_0 z_0 - y_0^T A^T (K_0' + K_z)^{-1} A z_0.$$

Since y_0, z_0 are arbitrary, it holds that

$$K_z = K_0 - A^T (K_0' + K_z)^{-1} A. \tag{6}$$

Besides

$$K_z = K_0 - A^T X,$$

therefore if one of K_z and X is known, so is the other one. From (5) we obtain y_1. By the same way we can obtain $y_2, \cdots, y_k, \cdots$, which are the complete set of the solution to the the problem (2). The readers are referred to [17] for the approach to solve K_z and X.

3. Critical Case

Let us fix the structure of the infinite element mesh and set $\xi \to 1$. We consider the limit case.

Let the nodes on Γ_0 be $(x_0^{(1)}, y_0^{(1)}), \cdots, (x_0^{(n)}, y_0^{(n)})$. We take $\kappa \in [0, \infty)$. Being analogous to Γ_k we construct Γ_κ with the constant of proportionality ξ^κ. The associated nodes on Γ_κ is denoted by $(x_\kappa^{(1)}, y_\kappa^{(1)}), \cdots, (x_\kappa^{(n)}, y_\kappa^{(n)})$. We define a space $S = \{u \in H^{1,*}(\Omega); u$ is linear on the segments connecting $(x_\kappa^{(i)}, y_\kappa^{(i)})$ and $(x_\kappa^{(i+1)}, y_\kappa^{(i+1)})$, and is continuous on $\Gamma_\kappa, \kappa \in [0, \infty), i = 1, \cdots, n\}$. Denote $S_0 = S \bigcap H_0^{1,*}(\Omega)$. Here for convenience we write $(x_\kappa^{(n+1)}, y_\kappa^{(n+1)}) = (x_\kappa^{(1)}, y_\kappa^{(1)})$.

Lemma 3.1. *Given piecewise linear boundary data y_0 on Γ_0, the infinite element solutions tend to a limit u weakly as $\xi \to 1$, which satisfies*

$$a(u, v) = 0, \qquad \forall v \in S_0. \tag{7}$$

Proof. We take a sequence $\{\xi\} \to 1$. For a fixed y_0 the infinite element solutions u_ξ are uniformly bounded in $H^{1,*}(\Omega)$ [17], then there exists a weakly convergent subsequence, still denoted by $\{u_\xi\}$. Let u be the limit.

Denote by $\Gamma^{(i)}$ the ray starting from the point o and passing through $(x_0^{(i)}, y_0^{(i)})$. By the structure of the infinite element solution, $\|\nabla u_\xi\|_{0,\Gamma^{(i)}}$ is also uniformly bounded, therefore we can extract a subsequence which converges uniformly on any compact subset of $\Gamma^{(i)}$, which yields $u \in S$ easily.

We take $v \in S_0$ and require that v is smooth on every closed subdomains bounded by $\Gamma^{(i)}$, $\Gamma^{(i+1)}$ and Γ_0. Then we take $\xi > 1$ and consider th infinite element subspace S_ξ. Let $v_I \in S_\xi$ be the interpolation function of v, then by the smoothness of v, $\|v - v_I\|_{1,*} \to 0 (\xi \to 1)$. Because $a(u_\xi, v_I) = 0$, we set $\xi \to 1$ and obtain the limit

$$a(u, v) = 0. \tag{8}$$

Since the set of v's is dense in S_0, we know that (8) holds for all $v \in S_0$, which means that u is the solution to (7). The solution to (7) is unique, consequently $u_\xi \rightharpoonup u$ as $\xi \to 1$.

Lemma 3.2. *The combined stiffness matrix K_z tends to a limit as $\xi \to 1$.*

Proof. If u_ξ is the infinite element solution with $u_\xi|_{\Gamma_0} = y_0, \xi > 1$, then

$$a(u_\xi, v) = y_0^T K_z z_0, \qquad \forall v \in S_\xi, v|_{\Gamma_0} = z_0. \tag{9}$$

Let $\xi \to 1$, then Lemma 3.1 implies that the left hand side of (9) tends to a limit, hence the right hand side does. Because y_0, z_0 are arbitrary, K_z tends to a limit.

Let us derive the equation for the critical K_z. To this end, we set $y_1 = y_0 + (\xi - 1)\eta_1$, then

$$\begin{aligned} &\begin{pmatrix} y_0^T & y_1^T \end{pmatrix} \begin{pmatrix} K_0 & -A^T \\ -A & K_0' \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \\ = &\begin{pmatrix} y_0^T & \eta_1^T \end{pmatrix} \begin{pmatrix} K_0 - A - A^T + K_0' & -(\xi-1)(A^T - K_0') \\ -(\xi-1)(A - K_0') & (\xi-1)^2 K_0' \end{pmatrix} \begin{pmatrix} y_0 \\ \eta_1 \end{pmatrix} \\ \equiv &\begin{pmatrix} y_0^T & \eta_1^T \end{pmatrix} \begin{pmatrix} \tilde{L}_0 & -\tilde{B}^T \\ -\tilde{B} & \tilde{L}_0' \end{pmatrix} \begin{pmatrix} y_0 \\ \eta_1 \end{pmatrix}. \end{aligned}$$

By the equation (6) for K_z we get

$$\begin{aligned} K_z &= \tilde{L}_0 + \frac{\tilde{L}_0'}{(\xi-1)^2} + \frac{\tilde{B}}{\xi-1} + \frac{\tilde{B}^T}{\xi-1} \\ &- \left(\frac{\tilde{L}_0'}{(\xi-1)^2} + \frac{\tilde{B}^T}{\xi-1}\right)\left(K_z + \frac{\tilde{L}_0'}{(\xi-1)^2}\right)^{-1}\left(\frac{\tilde{L}_0'}{(\xi-1)^2} + \frac{\tilde{B}}{\xi-1}\right). \end{aligned}$$

Taking into account that

$$\frac{\tilde{L}_0'}{(\xi-1)^2} + \frac{\tilde{B}}{\xi-1} = \left(K_z + \frac{\tilde{L}_0'}{(\xi-1)^2}\right) + \left(\frac{\tilde{B}}{\xi-1} - K_z\right),$$

we obtain by some deduction that

$$0 = \tilde{L}_0 - \left(\frac{\tilde{B}^T}{\xi-1} - K_z\right)\left(\frac{\tilde{L}_0'}{(\xi-1)^2} + K_z\right)^{-1}\left(\frac{\tilde{B}}{\xi-1} - K_z\right), \tag{10}$$

that is

$$0 = \frac{\tilde{L}_0}{\xi-1} - \left(\frac{\tilde{B}^T}{\xi-1} - K_z\right)\left(\frac{\tilde{L}_0'}{\xi-1} + (\xi-1)K_z\right)^{-1}\left(\frac{\tilde{B}}{\xi-1} - K_z\right).$$

Letting $\xi \to 1$ and defining

$$L_0 = \lim_{\xi\to1}\frac{\tilde{L}_0}{\xi-1}, \qquad B = \lim_{\xi\to1}\frac{\tilde{B}}{\xi-1}, \qquad L_0' = \lim_{\xi\to1}\frac{\tilde{L}_0'}{\xi-1},$$

we get the equation

$$L_0 - (B^T - K_z)(L_0')^{-1}(B - K_z) = 0, \tag{11}$$

where K_z is the stiffness matrix for the case of $\xi = 1$. The equation (11) is not uniquely solvable, so it seems difficult to get K_z directly. However the above procedure is crucial for the problems which we will discuss later on. A number of numerical examples have shown that the approximation solutions are more precise if ξ is closer to 1, so we expect the results would be better if $\xi = 1$, but the main reason to set $\xi = 1$ is that it is an efficient way to solve the equations for non-similar cases, which will be shown in the next section.

4. The Helmholtz Equation

We are concerned with

$$-\triangle u + \lambda u = 0. \tag{12}$$

where $-\infty < \lambda < +\infty$. If $\lambda > 0$ we define a bilinear form

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + \lambda uv)\, dxdy, \qquad \forall u, v \in H^1(\Omega), \tag{13}$$

where Ω is an exterior domain as the above, and we always use the usual notations of the Sobolev spaces. We fix the boundary data y_0 on Γ_0 and consider the following problam: find $u \in H^1(\Omega)$, such that $u|_{\Gamma_0} = y_0$ and

$$a(u,v) = 0, \qquad \forall v \in H_0^1(\Omega), \tag{14}$$

which admits a unique solution. We define the infinite element space $S(\Omega) \subset H^1(\Omega)$, then the infinite element solution exists uniquely. Let $K_z(\lambda)$ be the combined stiffness matrix.

We assume $\xi > 1$ first. Let u be the solution to the problem (14) and $y_1 = u|_{\Gamma_1}$. We make a similarity transformation $\xi x \to x, \xi y \to y$ and set $\tilde{u}(x,y) = u(\xi x, \xi y)$, $\tilde{v}(x,y) = v(\xi x, \xi y)$. Under the transformation the equation (12) becomes

$$-\triangle \tilde{u} + \lambda \xi^2 \tilde{u} = 0.$$

We have

$$\begin{aligned} a_{\xi\Omega}(u,v) &\equiv \int_{\xi\Omega} (\nabla u \cdot \nabla v + \lambda uv)\, dxdy \\ &= \int_\Omega (\nabla \tilde{u} \cdot \nabla \tilde{v} + \lambda \xi^2 \tilde{u}\tilde{v})\, dxdy \\ &= y_1^T K_z(\lambda\xi^2) z_1, \end{aligned}$$

where $z_1 = v|_{\Gamma_1}$. By the same way we obtain an equation like (6)

$$K_z(\lambda) = K_0 - A^T (K_0' + K_z(\lambda\xi^2))^{-1} A. \tag{15}$$

We notice that the arguments of K_z in (15) are different, so it is not a single equation. Being the same as (10) we have

$$\begin{aligned} &K_z(\lambda) = K_z(\lambda\xi^2) + \tilde{L}_0 \\ &- \left(\frac{\tilde{B}^T}{\xi - 1} - K_z(\lambda\xi^2) \right) \left(\frac{\tilde{L}_0'}{(\xi-1)^2} + K_z(\lambda\xi^2) \right)^{-1} \left(\frac{\tilde{B}}{\xi - 1} - K_z(\lambda\xi^2) \right). \end{aligned}$$

Dividing it by $\xi - 1$ and letting $\xi \to 1$ we get

$$\lambda \frac{dK_z(\lambda)}{d\lambda} = -\frac{1}{2} L_0 + \frac{1}{2} (B^T - K_z(\lambda))(L_0')^{-1}(B - K_z(\lambda)). \tag{16}$$

Generally speaking L_0, B, L_0' are dependent on λ. It follows that

Theorem 4.1. *The combined stiffness matrix $K_z(\lambda)$ satisfies the ordinary differential equation (16).*

In the equation (16) $\lambda = 0$ is a singular point, so it is difficult to solve it with the inital data $K_z(0)$. We will make a further discussion in §6 about the method to solve (16).

We turn now to consider the case of $\lambda = -\omega^2$, where $\omega > 0$ is the wave number. It is the so called scattering problem. Let us consider the most useful three dimensional case and let $x = (x_1, x_2, x_3)$ be the points. There is a particular solution

$$u = \frac{e^{i\omega r}}{r},$$

where $r=\sqrt{x_1^2+x_2^2+x_3^2}$, which satisfies the Sommerfeld radiation condition

$$\lim_{\rho\to\infty}\int_{r=\rho}|\frac{\partial u}{\partial r}-i\omega u|^2\,ds=0.$$

The function u and its gradient are not L^2 integrable on Ω which yields serious difficulties. We make a transformation

$$u=e^{i\omega r}U, \tag{17}$$

then U satisfies

$$\triangle U+2i\omega\frac{\partial U}{\partial r}+2i\omega\frac{U}{r}=0. \tag{18}$$

We introduce a weighted space

$$Y(\Omega)=\left\{U\in H^1_{\rm loc}(\Omega);|U|_{1,\Omega}<\infty,\left\|\frac{U}{r}\right\|_{0,\Omega}<\infty\right\},$$

equipped with the norm

$$\|U\|_{Y(\Omega)}=\left\{|U|^2_{1,\Omega}+\left\|\frac{U}{r}\right\|^2_{0,\Omega}\right\}^{1/2},$$

then it is a Hilbert space. The weak solution is defined as: find $U\in Y(\Omega),U|_{\Gamma_0}=y_0$, and

$$\int_\Omega\left\{\nabla U\cdot\nabla\bar v-\left(2i\omega\frac{\partial U}{\partial r}+2i\omega\frac{U}{r}\right)\bar v\right\}dxdy=0,\qquad\forall v\in H^1_0(\Omega), \tag{19}$$

where '$-$' denotes conjugate. There is another definition of weak solution in [10],

$$H^+_{1,w}(\Omega)=\{u;\|u\|^+_{1,w}<\infty\},$$

$$\|u\|^+_{1,w}=\left\{\int_\Omega\left(w|u|^2+w|\nabla u|^2+\left|\frac{\partial u}{\partial r}-i\omega u\right|^2\right)dxdy\right\}^{1/2},$$

$u\in H^+_{1,w}(\Omega)$ and satisfies

$$\int_\Omega(\nabla u\cdot\nabla\bar v-\omega^2u\bar v)\,dxdy=0,\qquad\forall v\in H^+_{1,w^*}(\Omega),v|_{\Gamma_0}=0, \tag{20}$$

where $w=r^{-2}$, $w^*=r^2$.

Lemma 4.1. *The definitions (19) and (20) are equivalent.*

Proof. If U is the solution to (19), we make the transformation (17). It is easy to see that $u\in H^+_{1,w}(\Omega)$. On the contrary if u is the solution to (20), then under the transformation (17) $U\in Y(\Omega)$. Moreover $C_0^\infty(\Omega)$ is dense in both $H_0^1(\Omega)$ and $\{u\in H^+_{1,w^*}(\Omega);u|_{\Gamma_0}=0\}$, so in (19), (20) the set of v can be replaced by $C_0^\infty(\Omega)$, therefore (19) and (20) are equivalent.

The formulation (18) is suitable to us to apply the infinite element method. Let $a(u,v)$ be the left hand side of (19),

$$\begin{pmatrix}K_0&-D\\-A&K_0'\end{pmatrix}$$

be the total stiffness matrix on Ω_1, and let $K_z(\omega)$ be the combined stiffness matrix. We make a similarity tranformation $\xi x\to x$, $\tilde U(x)=U(\xi x)$, then the equation (18) becomes

$$\triangle\tilde U+2i\omega\xi\frac{\partial\tilde U}{\partial r}+2i\omega\xi\frac{\tilde U}{r}=0. \tag{21}$$

Hence the combined stiffness matrix on $\xi\Omega$ is $\xi K_z(\omega\xi)$. We have

$$y_0^T K_z(\omega)\bar{z}_0^T = \begin{pmatrix} y_0^T & y_1^T \end{pmatrix} \begin{pmatrix} K_0 & -D \\ -A & K_0' \end{pmatrix} \begin{pmatrix} \bar{z}_0 \\ \bar{z}_1 \end{pmatrix} + y_1^T \xi K_z(\omega\xi)\bar{z}_1. \tag{22}$$

Since z_1 is arbitrary,

$$-y_0^T D + y_1^T K_0' + y_1^T \xi K_z(\omega\xi) = 0,$$

$$y_1^T = y_0^T D(K_0' + \xi K_z(\omega\xi))^{-1}. \tag{23}$$

Substituting it into (22) we get

$$K_z(\omega) = K_0 - D(K_0' + \xi K_z(\omega\xi))^{-1} A.$$

Letting $y_1 = y_0 + (\xi - 1)\eta_1$, $\tilde{L}_0 = K_0 - D - A + K_0'$, $\tilde{H} = (\xi - 1)(D - K_0')$, $\tilde{B} = (\xi - 1)(A - K_0')$, $\tilde{L}_0' = (\xi - 1)^2 K_0'$, we get the following by some computation:

$$K_z(\omega) = \xi K_z(\omega\xi) + \tilde{L}_0$$

$$- \left(\frac{\tilde{H}}{\xi - 1} - \xi K_z(\omega\xi)\right)\left(\frac{\tilde{L}_0'}{(\xi - 1)^2} + \xi K_z(\omega\xi)\right)^{-1}\left(\frac{\tilde{B}}{\xi - 1} - \xi K_z(\omega\xi)\right).$$

Let

$$L_0 = \lim_{\xi\to 1}\frac{\tilde{L}_0}{\xi - 1}, \quad B = \lim_{\xi\to 1}\frac{\tilde{B}}{\xi - 1}, \quad H = \lim_{\xi\to 1}\frac{\tilde{H}}{\xi - 1}, \quad L_0' = \lim_{\xi\to 1}\frac{\tilde{L}_0'}{\xi - 1}.$$

Dividing the equation by $\xi - 1$ and letting $\xi \to 1$ we get

$$\omega\frac{dK_z(\omega)}{d\omega} + K_z(\omega) = -L_0 + (H - K_z(\omega))(L_0')^{-1}(B - K_z(\omega)), \tag{24}$$

which is the differential equation satisfied by K_z.

5. The Element Matrices

We compute the matrices L_0, B, L_0' defined in §3 in this section. Some other matrices defined in §4 can be computed in an analogous way.

To make the program simpler, our computation is carried out on the "element level". The matrices L_0 *etc* on one layer is the assembly of the element matrices. The technique is analogous to differentiation, where all higher order terms are dropped. The symbol "=" is understood in this way in this section.

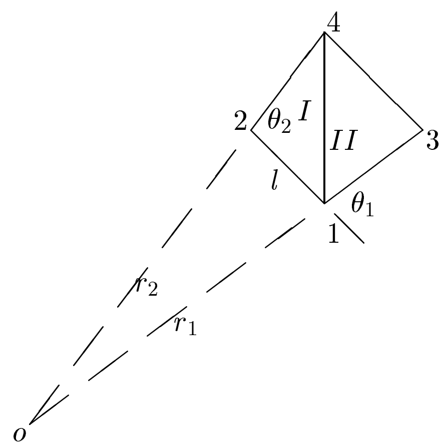

Fig. 1

Let the the triangles I, II in Fig.1 be linear elements. We study the matrices for I first. The arguments are y_0 and η_1. The basis $\{\varphi_1, \varphi_2, \varphi_3\}$ is the following:

$$\varphi_1 : f_2 = 1, f_4 = 1, f_1 = 0,$$
$$\varphi_2 : f_2 = 0, f_4 = 0, f_1 = 1,$$
$$\varphi_3 : f_2 = 0, f_1 = 0, f_4 = \xi - 1,$$

where f_j is the value of φ_i at the node j. One has

$$\varphi_1 = 1 - \frac{x}{l}, \quad \frac{\partial \varphi_1}{\partial x} = -\frac{1}{l}, \quad \frac{\partial \varphi_1}{\partial y} = \frac{\cos\theta_2}{l \sin\theta_2},$$
$$\varphi_2 = \frac{x}{l}, \quad \frac{\partial \varphi_2}{\partial x} = \frac{1}{l}, \quad \frac{\partial \varphi_2}{\partial y} = -\frac{\cos\theta_2}{l \sin\theta_2},$$
$$\varphi_3 = 0, \quad \frac{\partial \varphi_3}{\partial x} = 0, \quad \frac{\partial \varphi_3}{\partial y} = \frac{1}{r_2 \sin\theta_2},$$

and the area of I is $(\xi - 1) r_2 l \sin\theta_2 / 2$.

The element stiffness matrix is defined as

$$K = \lim_{\xi \to 1} \frac{1}{\xi - 1} \int_I \begin{pmatrix} |\nabla\varphi_1|^2 & \nabla\varphi_1 \cdot \nabla\varphi_2 & \nabla\varphi_1 \cdot \nabla\varphi_3 \\ \nabla\varphi_2 \cdot \nabla\varphi_1 & |\nabla\varphi_2|^2 & \nabla\varphi_2 \cdot \nabla\varphi_3 \\ \nabla\varphi_3 \cdot \nabla\varphi_1 & \nabla\varphi_3 \cdot \nabla\varphi_2 & |\nabla\varphi_3|^2 \end{pmatrix} dxdy,$$

and the element mass matrix is

$$M = \lim_{\xi \to 1} \frac{1}{\xi - 1} \int_I \begin{pmatrix} \varphi_1^2 & \varphi_1\varphi_2 & \varphi_1\varphi_3 \\ \varphi_2\varphi_1 & \varphi_2^2 & \varphi_2\varphi_3 \\ \varphi_3\varphi_1 & \varphi_3\varphi_2 & \varphi_3^2 \end{pmatrix} dxdy.$$

After a few computation we get

$$K = \frac{1}{2} \begin{pmatrix} \frac{r_2}{l\sin\theta_2} & -\frac{r_2}{l\sin\theta_2} & \frac{\cos\theta_2}{\sin\theta_2} \\ -\frac{r_2}{l\sin\theta_2} & \frac{r_2}{l\sin\theta_2} & -\frac{\cos\theta_2}{\sin\theta_2} \\ \frac{\cos\theta_2}{\sin\theta_2} & -\frac{\cos\theta_2}{\sin\theta_2} & \frac{l}{r_2\sin\theta_2} \end{pmatrix},$$

$$M = \frac{1}{12}\begin{pmatrix} 3lr_2\sin\theta_2 & lr_2\sin\theta_2 & 0 \\ lr_2\sin\theta_2 & lr_2\sin\theta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the same way we get for the element II that

$$K = \frac{1}{2}\begin{pmatrix} \frac{r_1}{l\sin\theta_1} & -\frac{r_1}{l\sin\theta_1} & \frac{\cos\theta_1}{\sin\theta_1} \\ -\frac{r_1}{l\sin\theta_1} & \frac{r_1}{l\sin\theta_1} & -\frac{\cos\theta_1}{\sin\theta_1} \\ \frac{\cos\theta_1}{\sin\theta_1} & -\frac{\cos\theta_1}{\sin\theta_1} & \frac{l}{r_1\sin\theta_1} \end{pmatrix},$$

$$M = \frac{1}{12}\begin{pmatrix} lr_1\sin\theta_1 & lr_1\sin\theta_1 & 0 \\ lr_1\sin\theta_1 & 3lr_1\sin\theta_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

6. The Algorithm for the Helmholtz Equation

Let us consider the way to solve the equation (16) first. To this end, we need some further discussion on the properties of K_z. We will always denote by C a generic constant.

Lemma 6.1. *$K_z(\lambda_1) \le K_z(\lambda_2)$ for $\lambda_1 < \lambda_2$, that is, $K_z(\lambda_2) - K_z(\lambda_1)$ is a nonnegative matrix.*

Proof. Given y_0, let u_{λ_1} and u_{λ_2} be the infinite element solutions associated with λ_1 and λ_2, then

$$\int_\Omega (|\nabla u_{\lambda_2}|^2 + \lambda_1 u_{\lambda_2}^2)\,dxdy \le \int_\Omega (|\nabla u_{\lambda_2}|^2 + \lambda_2 u_{\lambda_2}^2)\,dxdy.$$

Besides, since u_{λ_1} is a solution, one has

$$\int_\Omega (|\nabla u_{\lambda_2}|^2 + \lambda_1 u_{\lambda_2}^2)\,dxdy \ge \int_\Omega (|\nabla u_{\lambda_1}|^2 + \lambda_1 u_{\lambda_1}^2)\,dxdy.$$

Thus we get $K_z(\lambda_1) \le K_z(\lambda_2)$.

Lemma 6.2. *$K_z(\lambda) \le C\lambda^{1/2} I$, where I is the unit matrix.*

Proof. Given y_0, letting $\xi = \lambda^{-1/2} + 1$, we define

$$v = \begin{cases} 0 & (x,y) \in \xi\Omega, \\ \text{linear about } r, & (x,y) \in \Omega\backslash\overline{\xi\Omega}. \end{cases}$$

By virtue of the computation in §5 we can see that

$$a(v,v) \le C(\xi-1)^{-1}|y_0|^2 + C(\xi-1)\lambda|y_0|^2 = C\lambda^{1/2}|y_0|^2,$$

which gives the upper bound of the infinite element solution.

According to Lemmas 6.1, 6.2, we can make a formal asymptotic expansion,

$$K_z(\lambda) = k_1\lambda^{1/2} + k_0 + k_{-1}\lambda^{-1/2} + \cdots. \tag{25}$$

The computation in §5 shows that for linear elements the matrices B and L_0' in the equation (16) are in fact independent of λ, and L_0 depends linearly on λ. Letting $L_0 = l_1 + \lambda l_2$, we substitute (25) into (16) and obtain

$$-\frac{1}{2}l_2 + \frac{1}{2}k_1(L_0')^{-1}k_1 = 0,$$

$$\frac{1}{2}k_1 = -k_1(L_0')^{-1}B + k_1(L_0')^{-1}k_0,$$

$$-\frac{1}{2}l_1 + k_1(L_0')^{-1}k_{-1} - \frac{1}{2}(B^T - k_0)(L_0')^{-1}(B - k_0) = 0,$$

$$\cdots\cdots$$

then get the solution

$$k_1 = (L_0')^{1/2}((L_0')^{-1/2}l_2(L_0')^{-1/2})^{1/2}(L_0')^{1/2},$$

$$k_0 = \frac{1}{2}L_0' + B,$$

$$k_{-1} = \frac{1}{2}L_0'k_1^{-1}(l_1 + (B^T - k_0)(L_0')^{-1}(B - k_0)),$$

$$\cdots\cdots$$

Letting λ_0 be large enough, we use the above expression to get an approximation to $K_z(\lambda_0)$, then solve (16) to get $K_z(\lambda)$ for$\lambda \le \lambda_0$.

We turn now to the equation (24). Let

$$b(u,v) = \int_\Omega \left\{ \frac{1}{\omega}\nabla u \cdot \nabla \bar{v} - 2i\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\bar{v}\right\} dxdy,$$

and denote by $K(\omega)$ the corresponding combined stiffness matrix, then $K(\omega) = K_z(\omega)/\omega$.

Let the equation of Γ_0 be $r = \gamma(\theta,\varphi)$, where (r,θ,φ) are the spheroidal coordinates. The infinite element solution coincides with the true solution as $\omega = +\infty$,

$$u = \frac{\gamma(\theta,\varphi)}{r}y_0,$$

then $b(u,v) = 0$, hence $K(+\infty) = 0$. From the equation (24) we have the equation for $K(\omega)$,

$$\frac{dK(\omega)}{d\omega} = -2\frac{K(\omega)}{\omega} - \frac{L_0}{\omega^2} + \left(\frac{H}{\omega} - K(\omega)\right)(L_0')^{-1}\left(\frac{B}{\omega} - K(\omega)\right). \tag{26}$$

We take a large ω_0 and set $K(\omega_0) = 0$, then (26) gives $K(\omega)$ for $\omega \le \omega_0$.

Once K_z is obtained, it is routine to get the solution of the original boundary value problem. In §2 we have given the formula to get X from K_z, and to get $y_1, y_2, \cdots, y_k, \cdots$. To deal with the case of $\xi = 1$, we introduce the concept of "infinitesimal transfer matrix".

For example for the scattering problem, by (23) we write

$$y_1 = Xy_0, \qquad X = (K_0'^T + \xi K_z^T(\omega\xi))^{-1}D^T.$$

The infinitesimal transfer matrix is defined as

$$Y(\omega) = \lim_{\xi\to 1}\frac{X - I}{\xi - 1}.$$

After a few computation we have

$$Y(\omega) = (L_0')^{-1}(H^T - K_z^T(\omega)).$$

In view of the equation (21) we obtain

$$\begin{aligned} y_\xi &= \lim_{\Delta\xi\to 0}(I + Y(\omega\xi)\Delta\xi)\cdots(I + Y(\omega + \omega\Delta\xi)\Delta\xi)(I + Y(\omega)\Delta\xi)y_0 \\ &= e^{\int_1^\xi Y(\omega\xi)\,d\xi}y_0, \end{aligned} \tag{27}$$

where $\xi = r/\gamma(\theta,\varphi)$ and $y_\xi = u(r,\theta,\varphi)$, which is the explicit expression of the infinite element solution.

7. Convergence

We prove the convergence of the infinite element method for the case of $\lambda > 0$. To get the optimal error bounds, we assume that the boundary $\partial\Omega$ is sufficiently smooth, Ω

is decomposed into a bounded domain Ω_0 and an exterior polygon Ω', and the boundary value u_0 on $\partial\Omega$ is sufficiently smooth, then the solution u is regular enough.

Finite element partition is made on Ω_0, while infinite element partition is made on Ω'. We assume that the partition on Ω_0 is regular, that is, all interior angles of all elements admit a common lower bound $\theta_0 > 0$. Let the domain composed by all elements by Ω_h. We take $\tilde{\Omega} \supset (\bar{\Omega} \bigcup \bar{\Omega}_h)$ and extend the solution u smoothly on $\tilde{\Omega}$, then u is well defined on Ω_h.

Let Π be the interpolation operator, then by [7] one has

$$\|u - \Pi u\|_{1,\Omega_h\setminus\Omega'} \le Ch|u|_{2,\tilde{\Omega}\setminus\Omega'}, \tag{28}$$

where h is the greatest length of element sides. The following lemma is about the property of Π on Ω'.

Lemma 7.1. *If there is a constant $\theta_0 > 0$ such that the angle between each ray starting from the point o and the interior normal direction of Ω' on Γ_0 is not greater than $\frac{\pi}{2} - \theta_0$, then*

$$\|\Pi u - u\|_{0,\Omega'} \le Ch|u|_{1,w,\Omega'}, \tag{29}$$

$$|\Pi u - u|_{1,\Omega'} \le Ch|u|_{2,w,\Omega'}, \tag{30}$$

where $w = r^2$ is a weight, r is the distance to the point o,

$$|u|^2_{s,w,\Omega'} = \int_{\Omega'} r^2 |\partial^s u|^2 \, dxdy,$$

and h is the greatest length of element sides on Γ_0.

Proof. We take two neighboring nodes on Γ_0 and consider the domain Ω^* bounded by Γ_0 and two rays passing through these two nodes (Fig.2). Let (o, x, y) be local coordinates where the line segment linking these two nodes is perpendicular to the x-axis. Without loss of generality we assume that $u \in C^\infty(\overline{\Omega^*})$.

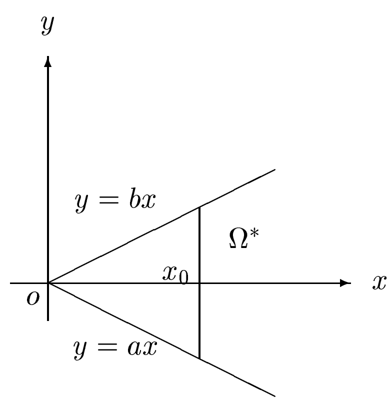

Fig. 2

We consider the line segment $y \in [ax, bx]$ for any $x \ge x_0$. We have

$$\Pi u - u = (u(x, bx) - u(x,y))\frac{y - ax}{bx - ax} + (u(x, ax) - u(x,y))\frac{bx - y}{bx - ax}, \tag{31}$$

thus

$$|\Pi u - u| \le \int_{ax}^{bx} |\frac{\partial u}{\partial y}| \, dy. \tag{32}$$

By the Hölder inequality

$$\begin{aligned}\int_{ax}^{bx} (\Pi u - u)^2 \, dy &\leq \int_{ax}^{bx} dy \left(\int_{ax}^{bx} |\frac{\partial u}{\partial y}| \, dy \right)^2 \\ &\leq (bx - ax)^2 \int_{ax}^{bx} (\frac{\partial u}{\partial y})^2 \, dy \\ &= \frac{(bx_0 - ax_0)^2}{x_0^2} \int_{ax}^{bx} x^2 (\frac{\partial u}{\partial y})^2 \, dy,\end{aligned}$$

hence

$$\int_{x_0}^{\infty} \int_{ax}^{bx} (\Pi u - u)^2 \, dy dx \leq \frac{h^2}{x_0^2} \int_{\Omega^*} r^2 |\partial u|^2 \, dx dy,$$

which gives (29) by summing up with respect to all subdomains.

For the derivatives we have

$$\frac{\partial \Pi u}{\partial y} = \frac{\partial u}{\partial y}(x, y^*),$$

where y^* is a mean value, which yields

$$\left| \frac{\partial \Pi u}{\partial y} - \frac{\partial u}{\partial y} \right| = \left| \int_{y}^{y^*} \frac{\partial^2 u}{\partial y^2} \, dy \right| \leq \int_{ax}^{bx} |\frac{\partial^2 u}{\partial y^2}| \, dy.$$

Comparing it with (32) we get the desired estimate immediately. Differentiating (31) gives

$$\begin{aligned}\frac{\partial \Pi u}{\partial x} - \frac{\partial u}{\partial x} &= \left(\frac{\partial u}{\partial x}(x, bx) - \frac{\partial u}{\partial x}(x, y) \right) \frac{y - ax}{bx - ax} + \left(\frac{\partial u}{\partial x}(x, ax) - \frac{\partial u}{\partial x}(x, y) \right) \frac{bx - y}{bx - ax} \\ &+ \left(\frac{\partial u}{\partial y}(x, bx) - \frac{u(x, bx) - u(x, ax)}{bx - ax} \right) \frac{b(y - ax)}{bx - ax} \\ &+ \left(\frac{\partial u}{\partial y}(x, ax) - \frac{u(x, bx) - u(x, ax)}{bx - ax} \right) \frac{a(bx - y)}{bx - ax},\end{aligned}$$

which implies

$$\left| \frac{\partial \Pi u}{\partial x} - \frac{\partial u}{\partial x} \right| \leq \int_{ax}^{bx} \left| \frac{\partial^2 u}{\partial x \partial y} \right| \, dy + \max(|a|, |b|) \int_{ax}^{bx} \left| \frac{\partial^2 u}{\partial y^2} \right| \, dy.$$

Comparing it with (32) we get the desired estimate.

Under the above assumtions we have the theorem.

Theorem 7.1. *If u_h is a solution of the infinite element method, then*

$$\|u - u_h\|_{1,\Omega_h} \leq Ch(\|u\|_{2,\infty,\Omega_0} + |u|_{1,w,\Omega'} + |u|_{2,w,\Omega'}). \tag{33}$$

Proof. u satisfies the equation

$$- \triangle u + \lambda u = f \tag{34}$$

on Ω_h. By the equation (12), $f = 0$ on Ω, hence supp $f \subset \overline{\Omega_h \backslash \Omega}$ and

$$|f| \leq Ch\|u\|_{2,\infty,\Omega_0}.$$

Since u_h and Πu coincide at the nodes on the boundary, $(\Pi u - u_h)|_{\partial \Omega_h} = 0$, that is $\Pi u - u_h \in S_0(\Omega_h)$. By (34) we get

$$\int_{\Omega_h} (\nabla u \cdot \nabla (\Pi u - u_h) + \lambda u (\Pi u - u_h)) \, dx dy = \int_{\Omega_h} f (\Pi u - u_h) \, dx dy.$$

Because u_h is the infinite element solution,

$$\int_{\Omega_h} (\nabla u_h \cdot \nabla (\Pi u - u_h) + \lambda u_h (\Pi u - u_h)) \, dx dy = 0.$$

By subtraction we have

$$\int_{\Omega_h} (\nabla(u-u_h)\cdot\nabla(\Pi u-u_h)+\lambda(u-u_h)(\Pi u-u_h))\,dxdy = \int_{\Omega_h} f(\Pi u-u_h)\,dxdy,$$

thus

$$\begin{aligned}
&\int_{\Omega_h} (|\nabla(\Pi u-u_h)|^2+\lambda(\Pi u-u_h)^2)\,dxdy \\
= &\int_{\Omega_h} (\nabla(\Pi u-u)\cdot\nabla(\Pi u-u_h)+\lambda(\Pi u-u)(\Pi u-u_h))\,dxdy \\
&+\int_{\Omega_h} f(\Pi u-u_h)\,dxdy \\
\le &|\Pi u-u|_{1,\Omega_h}\cdot|\Pi u-u_h|_{1,\Omega_h}+\lambda\|\Pi u-u\|_{0,\Omega_h}\cdot\|\Pi u-u_h\|_{0,\Omega_h} \\
&+\|f\|_{0,\Omega_h\backslash\Omega}\|\Pi u-u_h\|_{0,\Omega_h\backslash\Omega} \\
\le &(|\Pi u-u|^2_{1,\Omega_h}+\lambda\|\Pi u-u\|^2_{0,\Omega_h})^{1/2}\cdot(|\Pi u-u_h|^2_{1,\Omega_h}+\lambda\|\Pi u-u_h\|^2_{0,\Omega_h})^{1/2} \\
&+Ch\|u\|_{2,\infty,\Omega_0}|\Pi u-u_h|_{1,\Omega_h\backslash\Omega},
\end{aligned}$$

where we have noticed that the area of $\Omega\backslash\Omega_h$ is boundeed by Ch^2. We cancel one factor and obtain

$$\begin{aligned}
&(|\Pi u-u_h|^2_{1,\Omega_h}+\lambda\|\Pi u-u_h\|^2_{0,\Omega_h})^{1/2} \\
&\le (|\Pi u-u|^2_{1,\Omega_h}+\lambda\|\Pi u-u\|^2_{0,\Omega_h})^{1/2}+Ch\|u\|_{2,\infty,\Omega_0}.
\end{aligned}$$

By (28) and Lemma 7.1

$$\begin{aligned}
&|\Pi u-u_h|^2_{1,\Omega_h}+\lambda\|\Pi u-u_h\|^2_{0,\Omega_h} \\
&\le Ch^2(|u|^2_{2,w,\tilde\Omega}+|u|^2_{1,w,\tilde\Omega}+\|u\|_{2,\infty,\Omega_0}),
\end{aligned}$$

where we have affirmed that $|u|_{2,\tilde\Omega\backslash\Omega'}\le C\|u\|_{2,\infty,\Omega_0}$. Applying the estimate of $\Pi u-u$ and the triangle inequality, we get (33).

The convergence problem for the scattering problems is still open.

8. Numerical Example

We take the scattering problem as an example. Let Ω be the exterior domain of the sphere $\Gamma=\{x\in \mathsf{R}^3; r=\sqrt{3}\}$. The boundary condition is given as $U|_\Gamma=1$, then the solution to the equation (18) is $U=\frac{\sqrt{3}}{r}$. We take only 8 nodes $(\pm1,\pm1,\pm1)$ on Γ and 6 elements $\{(x_1,x_2,x_3); 1\le|x_k|\le\xi, |x_i|\le|x_k|, |x_j|\le|x_k|\}$, $1\le i,j,k\le3$, $i\ne j, j\ne k, k\ne i$, which are trilinear isoparametric hexahedron elements, then solve it by the infinite element method. The approximate boundary is no longer a sphere but a box, $\{(x_1,x_2,x_3); x_k=\pm1, -1\le x_i\le1, -1\le x_j\le1, 1\le i,j,k\le3, i\ne j, j\ne k, k\ne i\}$, indeed. Therefore our approximation is very coarse. However the numerical result is still quite close to the exact one.

We take $\omega_0=100$ and set $K(\omega_0)=0$. Using the Runge-Kutta method we solve (26) with time step 0.1. The infinite element solutions for the case of $\omega\le\omega_0$ are thus obtained. We show some of them as examples.

$\omega = 90$

ξ	1.011	1.022	1.033	1.044	1.056
IEM	0.982 $+0.007i$	0.972 $+0.017i$	0.969 $+0.023i$	0.968 $+0.02i$	0.962 $+0.012i$
exact	0.989	0.978	0.968	0.958	0.947
ξ	1.067	1.078	1.089	1.1	1.11
IEM	0.949 $+0.002i$	0.93 $-0.0005i$	0.912 $+0.004i$	0.901 $+0.013i$	0.898 $+0.02i$
exact	0.937	0.927	0.918	0.909	0.900

$\omega = 50$

ξ	1.02	1.04	1.06	1.08	1.1
IEM	0.987 $-0.007i$	0.966 $-0.016i$	0.94 $-0.021i$	0.913 $-0.019i$	0.892 $-0.01i$
exact	0.98	0.962	0.943	0.926	0.909
ξ	1.12	1.14	1.16	1.18	1.2
IEM	0.88 $-0.002i$	0.872 $+0.0006i$	0.865 $-0.004i$	0.851 $-0.012i$	0.831 $-0.018i$
exact	0.892	0.877	0.862	0.847	0.833

$\omega = 10$

ξ	1.1	1.2	1.3	1.4	1.5
IEM	0.893 $+0.007i$	0.813 $+0.015i$	0.752 $+0.019i$	0.701 $+0.015i$	0.654 $+0.008i$
exact	0.909	0.833	0.769	0.714	0.667
ξ	1.6	1.7	1.8	1.9	2.0
IEM	0.608 $+0.001i$	$0.564i$ -0.0001	0.526 $+0.003i$	0.496 $+0.008i$	0.473 $+0.011i$
exact	0.625	0.588	0.556	0.526	0.500

$\omega = 5$

ξ	1.2	1.4	1.6	1.8	2.0
IEM	0.81 $+0.003i$	0.682 $-0.001i$	0.585 $-0.007i$	0.508 $-0.011i$	0.447 $-0.01i$
exact	0.833	0.714	0.625	0.556	0.500
ξ	2.2	2.4	2.6	2.8	3.0
IEM	0.399 $-0.005i$	0.364 $-0.003i$	0.336 $+0.0008i$	0.314 $-0.0001i$	0.293 $-0.003i$
exact	0.455	0.417	0.385	0.357	0.333

We see that the precision is quite high for big wave numbers ω, and for smaller ω the infinite element solutions differ more to the exact solution. We think it is because for smaller ω the dissipation term has more influence to the solutions, so the solutions exterior to a box differ more to that exterior to a sphere. The round-off error after about one thousand steps is also a remarkable amount. However the trend of all of them are correct and the results for such a coarse mesh are better than our expectation.

We have also compared the limit of the combined stiffness matrix as $\omega \to 0$ to that of the Laplace equation with $\xi = 1.01$. They are in a good agreement.

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