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BOUNDING PYRAMIDS AND BOUNDING CONES FOR TRIANGULAR BÉZIER SURFACES*1)

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Abstract

This paper describes practical approaches on how to construct bounding pyramids and bounding cones for triangular Bézier surfaces. Examples are provided to illustrate the process of construction and comparison is made between various surface bounding volumes. Furthermore, as a starting point for the construction, we provide a way to compute hodographs of triangular Bézier surfaces and improve the algorithm for computing the bounding cone of a set of vectors.

Key words: Triangular Bézier surface patch, Hodograph, Bounding pyramid, Bounding cone.

1. Introduction

A *bounding pyramid/cone* of a rational Bézier surface patch is a pyramid/cone which has the property that if its vertex is translated to any point on the Bézier patch, the patch will lie completely outside the pyramid/cone. These kinds of pyramids/cones are useful tools in detecting closed loops in surface/surface intersections[2, 3] and determining directions for which a surface is single valued[5]. While methods of finding bounding pyramids and bounding cones for rectangular Bézier surfaces are widely addressed[2, 3, 4, 5, 7], no analogous results have ever been obtained for triangular surface patches. The purpose of this paper is to discuss the problem of computing bounding pyramids and bounding cones for triangular Bézier surfaces. Although the construction process presented in this paper shares some similarities with that for rectangular Bézier surfaces, it is still very valuable to fully describe the detailed process of the constructions due to the specialties of triangular Bézier surfaces.

The organization of this paper is as follows. We first provide an algorithm to compute the hodograph of a triangular Bézier surface and derive an upper bound for any partial derivative direction of the Bézier surface in Section 2. Then in Section 3, we present methods to compute bounding pyramid/cone of a set of vectors which are used to obtain the tangent bounding pyramids/cones of a Bézier surface in the

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next section. Section 4 describes different approaches to construct surface bounding pyramids/cones based on the tangent bounding pyramids/cones, and comparison is made between these different bounding volumes. Finally in Section 5, an example is illustrated to demonstrate the whole construction process.

2. Hodographs of Triangular Bézier Surfaces

A triangular rational Bézier surfaces in homogeneous form is defined by

$$\begin{aligned}\mathbf{B}(\mathbf{P}) &:= \mathbf{B}(u,v,w) := (X(u,v,w), Y(u,v,w), Z(u,v,w), W(u,v,w)) \\ &:= \sum_{i+j+k=n} \mathbf{P}_{ijk} B_{ijk}^n(u,v,w)\end{aligned} \tag{1}$$

where $\mathbf{P}_{ijk} = (X_{ijk}, Y_{ijk}, Z_{ijk}, W_{ijk})$ are homogenous control points, and $B_{ijk}^n(u,v,w)$ are Bernstein bases with (u,v,w) being the barycentric coordinates of points $\mathbf{P}$ with respect to triangle domain $\mathbf{T} = \triangle \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3$.

A direction in the domain plane can be expressed using barycentric coordinates as $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. For example, $\overrightarrow{\mathbf{T}_1\mathbf{T}_2} = (-1,1,0)$ and $\overrightarrow{\mathbf{T}_2\mathbf{T}_3} = (0,-1,1)$.

In this section a single character in bold typeface signifies a homogenous point, while one with tilde denotes the corresponding Cartesian point or vector. For any two points $\mathbf{P}_i = (X_i, Y_i, Z_i, W_i), i = 1,2$, we define[6]

$$Dir(\mathbf{P}_1, \mathbf{P}_2) = (W_1X_2 - W_2X_1, W_1Y_2 - W_2Y_1, W_1Z_2 - W_2Z_1) \tag{2}$$

'*Dir*' function indicates the direction of the Cartesian vector between two points, since

$$Dir(\mathbf{P}_1, \mathbf{P}_2) = W_1W_2(\tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_2). \tag{3}$$

2.1 Derivative Directions of Triangular Rational Bézier Surfaces

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be any direction in domain plane. The derivative of $\mathbf{B}(\mathbf{P})$ along direction α is

$$\frac{\partial \mathbf{B}(\mathbf{P})}{\partial \alpha} = n \sum_{i+j+k=n-1} (\alpha_1 \mathbf{P}_{i+1,j,k} + \alpha_2 \mathbf{P}_{i,j+1,k} + \alpha_3 \mathbf{P}_{i,j,k+1}) B_{ijk}^{n-1}(u,v,w). \tag{4}$$

In Cartesian coordinates,

$$\frac{\partial}{\partial \alpha}\tilde{\mathbf{B}}(\mathbf{P}) = \frac{Dir(\mathbf{B}(\mathbf{P}), \frac{\partial}{\partial \alpha}\mathbf{B}(\mathbf{P}))}{W(u,v,w)^2}. \tag{5}$$

Thus the scaled hodograph $Dir(\mathbf{B}(\mathbf{P}), \frac{\partial}{\partial \alpha}\mathbf{B}(\mathbf{P}))$ gives the direction of $\frac{\partial}{\partial \alpha}\tilde{\mathbf{B}}(\mathbf{P})$.

Direct computation yields

$$\begin{aligned}
&Dir(\mathbf{B}(\mathbf{P}), \frac{\partial}{\partial\alpha}\mathbf{B}(\mathbf{P})) \\
&= n \sum_{i+j+k=n} \sum_{r+s+t=n-1} Dir(\mathbf{P}_{ijk}, \alpha_1\mathbf{P}_{r+1,s,t} + \alpha_2\mathbf{P}_{r,s+1,t} + \alpha_3\mathbf{P}_{r,s,t+1}) \\
&\qquad B^n_{ijk}(u,v,w)B^{n-1}_{rst}(u,v,w) \\
&= \frac{n}{\binom{2n-1}{n}} \sum_{i+j+k=2n-1} \sum_{r+s+t=n-1} \\
&Dir(\mathbf{P}_{i-r,j-s,k-t}, \alpha_1\mathbf{P}_{r+1,s,t} + \alpha_2\mathbf{P}_{r,s+1,t} + \alpha_3\mathbf{P}_{r,s,t+1}) \times \\
&\binom{i}{r}\binom{j}{s}\binom{k}{t} B^{2n-1}_{ijk}(u,v,w) \\
&= \frac{n}{\binom{2n-1}{n}} \sum_{i+j+k=2n-1} \tilde{\mathbf{H}}_{ijk} B^{2n-1}_{ijk}(u,v,w),
\end{aligned} \tag{6}$$

where

$$\tilde{\mathbf{H}}_{ijk} = \sum_{r+s+t=n-1} \binom{i}{r}\binom{j}{s}\binom{k}{t} Dir(\mathbf{P}_{i-r,j-s,k-t}, \alpha_1\mathbf{P}_{r+1,s,t} + \alpha_2 P_{r,s+1,t} + \alpha_3\mathbf{P}_{r,s,t+1}) \tag{7}$$

Thus the derivative direction of rational Bézier surface $\mathbf{B}(\mathbf{P})$ is totally determined by the vectors $\tilde{\mathbf{H}}_{ijk}$, $i+j+k=2n-1$.

The scaled hodograph generally is degree $2n-1$. For a quadratic rational Bézier surfaces, the scaled hodograph along direction $\alpha^1 = (1,-1,0)$ is a degree three Bézier surface with control points

$$\begin{aligned}
\tilde{\mathbf{H}}_{300} &= 3Dir(\mathbf{P}_{110}, \mathbf{P}_{200}), \\
\tilde{\mathbf{H}}_{210} &= Dir(\mathbf{P}_{110}, \mathbf{P}_{200}) + Dir(\mathbf{P}_{020}, \mathbf{P}_{200}), \\
\tilde{\mathbf{H}}_{120} &= Dir(\mathbf{P}_{020}, \mathbf{P}_{110}) + Dir(\mathbf{P}_{020}, \mathbf{P}_{200}), \\
\tilde{\mathbf{H}}_{030} &= 3Dir(\mathbf{P}_{110}, \mathbf{P}_{020}), \\
\tilde{\mathbf{H}}_{201} &= Dir(\mathbf{P}_{101}, \mathbf{P}_{200}) + 2Dir(\mathbf{P}_{110}, \mathbf{P}_{101}) + Dir(\mathbf{P}_{011}, \mathbf{P}_{200}), \\
\tilde{\mathbf{H}}_{111} &= Dir(\mathbf{P}_{011}, \mathbf{P}_{200}) + Dir(\mathbf{P}_{020}, \mathbf{P}_{101}), \\
\tilde{\mathbf{H}}_{021} &= Dir(\mathbf{P}_{020}, \mathbf{P}_{011}) + 2Dir(\mathbf{P}_{011}, \mathbf{P}_{110}) + Dir(\mathbf{P}_{020}, \mathbf{P}_{101}), \\
\tilde{\mathbf{H}}_{102} &= Dir(\mathbf{P}_{002}, \mathbf{P}_{200}) + 2Dir(\mathbf{P}_{011}, \mathbf{P}_{101}) + Dir(\mathbf{P}_{110}, \mathbf{P}_{200}), \\
\tilde{\mathbf{H}}_{012} &= Dir(\mathbf{P}_{002}, \mathbf{P}_{110}) + 2Dir(\mathbf{P}_{011}, \mathbf{P}_{101}) + Dir(\mathbf{P}_{020}, \mathbf{P}_{002}), \\
\tilde{\mathbf{H}}_{003} &= 3(Dir(\mathbf{P}_{002}, \mathbf{P}_{101}) + Dir(\mathbf{P}_{011}, \mathbf{P}_{002})).
\end{aligned} \tag{8}$$

We especially note that, for polynomial Bézier surface (1), its derivative is simply given by

$$\frac{\partial}{\partial\alpha}\tilde{\mathbf{B}}(\mathbf{P}) = n \sum_{r+s+t=n-1} (\alpha_1\tilde{\mathbf{P}}_{i+1,j,k} + \alpha_2\tilde{\mathbf{P}}_{i,j+1,k} + \alpha_3\tilde{\mathbf{P}}_{i,j,k+1}) B^{n-1}_{ijk}(u,v,w), \tag{9}$$

so its derivative direction is completely determined by vectors $\alpha_1\tilde{\mathbf{P}}_{i+1,j,k} + \alpha_2\tilde{\mathbf{P}}_{i,j+1,k} + \alpha_3\tilde{\mathbf{P}}_{i,j,k+1}$, $i+j+k=n-1$.

2.2 Bounds of Derivative Directions By convex hull property, the derivative direction of triangular Bézier surface is bounded by the convex of $n(2n+1)$ vectors

$\tilde{\mathbf{H}}_{ijk}$, $i+j+k=2n-1$ defined by (7). Especially if the triangular Bézier surface is a polynomial surface, the derivative direction can be bounded by the convex hull of $n(n-1)/2$ vectors $\alpha_1\tilde{\mathbf{P}}_{i+1,j,k}+\alpha_2\tilde{\mathbf{P}}_{i,j+1,k}+\alpha_3\tilde{\mathbf{P}}_{i,j,k+1}$, $i+j+k=n-1$.

It is possible to bound the derivative directions using smaller number of vectors by playing some tricks as was done in [6], but it generally gives looser bound.

3. Vectors Bounding Pyramids and Cones

Given a closed planar polygon Pg and a point P not in the plane of the polygon, the surface generated by all the lines through P and the points on the polygon Pg is called a *pyramidal surface*, the infinite volume enclosed by the surface is called a *pyramid* generated by Pg and P. We denote it as $Pyrd(Pg, P)$. The polygon Pg is called the *directrix* of pyramid $Pyrd(Pg, P)$ while point P the pyramid's *vertex*. The lines from P to points on the polygon Pg are called the *generators* of the pyramid.

A *conical surface* is identical to a pyramidal surface except that its directrix is a smooth curve, such as a circle or an ellipse. A *right circular* conical surface has a circular directrix, and the line from the vertex to the center of the circle, which is called the *axis*, is perpendicular to the plane of the circle. The volume enclosed by the conical surface is called a *cone*. In this paper, we mostly deal with the cones generated by right circular conical surfaces since they are easy to control in practice, and when we say a cone, it refers to a right circular cone. Any cone can be uniquely represented by its axis and half vertex angle, and if we always set its vertex at the origin, then the cone can be represented by a point P which is the intersection of the axis and the unit sphere and its half vertex angle θ. We denote this cone as $Cone(P, \theta)$. The complement of a cone is the locus of all lines through a common vertex, none of which is perpendicular to any line inside the given cone. It is easy to see the complement of $Cone(P, \theta)$ is $Cone(-P, \pi/2-\theta)$.

Both pyramids and cones are comprised of two symmetric nappes which meet at the vertex. According to different situations, we sometimes make use of just one nappe of a pyramid/cone. This will be made clear in the context.

An essential step to construct a surface bounding pyramid/cone is to construct a pyramid/cone(called *vectors bounding pyramid/cone*) which bounds a given set of vectors whose tails are at the origin. In this section, we provide methods to compute such a vectors bounding pyramid/cone (We assume it contains just one nappe).

Firstly, we describe a method to compute vectors bounding pyramid.

Given n vectors $\overrightarrow{OP}_1, \overrightarrow{OP}_2, \ldots, \overrightarrow{OP}_n$, where O is the origin, the pyramid which bounds this set of vectors is denoted by $Pyrd(P_1, P_2, \ldots, P_n)$. It turns out that $Pyrd(P_1, P_2, \ldots, P_n)$ can be obtained by applying the algorithm of constructing convex hull for the set of points $\{O, P_1, \ldots, P_n\}$[1]. Suppose $OP_{i_1}, OP_{i_2}, \ldots, OP_{i_m}$ are edges on the convex hull of the set of points $\{O, P_1, \ldots, P_n\}$, then the generators of the pyramid $Pyrd(P_1, P_2, \ldots, P_n)$ are $OP_{i_1}, OP_{i_2}, \ldots, OP_{i_m}$.

Next we consider constructing bounding cones for a set of vectors. In [2], Sederberg etal. presented an algorithm to compute such a bounding cone. In the following, we give two improved algorithms, one's time complexity is linear in the number of vectors but not necessarily gives best bounding cone, while the other is quartic in time but produces best solution.

Algorithm 1.

Input n vectors $\overrightarrow{OP_1}, \overrightarrow{OP_2}, \ldots, \overrightarrow{OP_n}$. Without loss of generality, we assume points $P_1, P_2, \ldots, P_n$ are on the unit sphere.

Output $Cone(Q, \theta)$ which bounds vectors $\overrightarrow{OP_1}, \overrightarrow{OP_2}, \cdots, \overrightarrow{OP_n}$.

Procedure

Step 1. Let $m \leftarrow n, Q_j \leftarrow P_j, \theta_j \leftarrow 0, j = 1, 2, \cdots, n$;

Step 2. If m is odd, $m \leftarrow m + 1, Q_m \leftarrow P_{m-1}, \theta_m \leftarrow 0$;

Step 3. For i vary from 1 to $m/2$, there are three cases:

- **Case 1** If cone $Cone(Q_{2i-1}, \theta_{2i-1})$ encloses cone $Cone(Q_{2i}, \theta_{2i})$, let $Q_i \leftarrow Q_{2i-1}$, $\theta_i \leftarrow \theta_{2i-1}$;
- **Case 2** If cone $Cone(Q_{2i}, \theta_{2i})$ encloses cone $Cone(Q_{2i-1}, \theta_{2i-1})$, let $Q_i \leftarrow Q_{2i}$, $\theta_i \leftarrow \theta_{2i}$;
- **Case 3** Otherwise, find a cone $Cone(Q_i, \theta_i)$ which exactly bounds two cones $Cone(Q_{2i-1}, \theta_{2i-1})$ and $Cone(Q_{2i}, \theta_{2i})$. This can be done as follows.

  Assume the angle between vectors $\overrightarrow{OQ_{2i-1}}$ and $\overrightarrow{OQ_{2i}}$ be α_i. Let $\theta_i \leftarrow (\theta_{2i-1} + \alpha + \theta_{2i})/2$, and choose point Q_i on the unit sphere such that it lies in the plane $OQ_{2i-1}Q_{2i}$, and the angles $\angle Q_{2i-1}OQ_i = (\alpha + \theta_{2i} - \theta_{2i-1})/2$, $\angle Q_iOQ_{2i} = (\theta_{2i-1} + \alpha - \theta_{2i})/2$. It is easy to see $Cone(Q_i, \theta_i)$ bounds $Cone(Q_{2i-1}, \theta_{2i-1})$ and $Cone(Q_{2i}, \theta_{2i})$ exactly.

Step 4. Let $m \leftarrow [m/2]$, if $m = 0$, go to **Step 5**, else go to **Step** 2.

Step 5. Let $Q \leftarrow Q_1$ and $\theta \leftarrow \theta_1$.

We test Algorithm 1 through tens of examples which are constructed according to random numbers, and the result shows that, in almost all cases, the vertex angles of cones obtained by Algorithm 1 are 8-10% smaller than those obtained from the algorithm in [2].

We can go even further. In fact, we can get the best vectors bounding cone. Before describing the algorithm, we point out a simple fact.

Lemma 1. *If Q is the tightest cone which bounds vectors $\overrightarrow{OP_1}, \overrightarrow{OP_2}, \ldots, \overrightarrow{OP_n}$, then there are at least two points among P_i lying on Q, and if there are only two points lying on Q, then the directrix center of cone Q must be in the plane through these two points and the origin.*

Proof. The proof is a little tedious though the result is obvious. We omit the details.

Remark. The smallest cone which bounds a set of vectors is called the *cone spanned by this set of vectors.*

Algorithm 2.

The input and output are the same as in Algroithm 1.

Procedure

Step 1. Find two points P_{i_1} and P_{i_2} from the given set of points such that the cone spanned by vectors $\overrightarrow{OP_{i_1}}$ and $\overrightarrow{OP_{i_2}}$ is the smallest cone which bounds all the given set of vectors. Denote this cone by $Cone(Q_1, \theta_1)$. Note that $Cone(Q_1, \theta_1)$ may not exist.

Step 2. Find three points P_{i_1}, P_{i_2} and P_{i_3} from the given set of points such that the cone spanned by vectors $\overrightarrow{OP}_{i_1}, \overrightarrow{OP}_{i_2}, \overrightarrow{OP}_{i_3}$ contains all the other vectors and it is the smallest. Denote this cone by $Cone(Q_2, \theta_2)$, where Q_2 can be found by computing the interior center of triangle $\triangle P_{i_1}P_{i_2}P_{i_3}$.

Step 3. If $\theta_1 < \theta_2$, set $Q \leftarrow Q_1$, $\theta \leftarrow \theta_1$, else let $Q \leftarrow Q_2$, $\theta \leftarrow \theta_2$.

The computation complexity of Algorithm 2 is $O(n^4)$, but it produces best bounding cone for a given set of vectors by Lemma 1.

4. Surface Bounding Pyramids and Cones

When we have computed the hodograph of a given surface patch along any direction, we can get the tangent bounding pyramids/cones by applying the vectors bounding pyramid/cone algorithm presented in the last section. It turns out that the surface bounding pyramid or cone can be constructed from these tangent bounding pyramids or cones through some kinds of geometrical operations.

4.1 Tangent Bounding Pyramids and Cones

Given a rational triangular Bézier surface patch $\mathbf{B}(\mathbf{P})$, and a direction α in the base triangle $\mathbf{T}$, the *tangent bounding pyramid* $Pyrd_\alpha(\mathbf{B})$ of the surface patch $\mathbf{B}(\mathbf{P})$ along direction α is the pyramid which contains all the tangent directions $\frac{\partial \tilde{\mathbf{B}}}{\partial \alpha}(\mathbf{P})$ for all $\mathbf{P} \in \mathbf{T}$. By the convex hull property and (6), the pyramid which bounds the $n(2n+1)$ vectors $\tilde{\mathbf{H}}_{ijk}, i+j+k = 2n-1$ defined by (7), also bounds all derivative directions of surface $\mathbf{B}(\mathbf{P})$ along α, and thus can be served as the tangent bounding pyramid(Similar to the vectors bounding pyramid, we assume tangent bounding pyramid is composed of just one nappe). Three special tangent bounding pyramids which are along three special directions $\alpha^1 = (1,-1,0)$, $\alpha^2 = (0,1,-1)$ and $\alpha^3 = (-1,0,1)$ are essential to the construction of the surface bounding pyramid/cone. We denote them by $Pyrd_1$, $Pyrd_2$ and $Pyrd_3$ respectively. Similarly, we can construct three *tangent bounding cones* $Cone_i$ along directions α^i, $i = 1,2,3$.

A useful property of tangent bounding pyramids/cones is expressed in the following theorem.

Theorem 1. *If pyramid $Pyrd_1$(or cone $Cone_1$) is translated so that its vertex lies at $\mathbf{B}(u_0, v_0, w_0)$, then the isoparametric curve segment $\mathbf{B}(u, v, w_0)$, $u+v = 1-w_0$, $u \geq u_0$ and $v \geq 0$, lies within $Pyrd_1$(or $Cone_1$). In fact, similar result holds also for any tangent bounding pyramid(or cone).*

Proof. Similar to the proof of theorem 2 in [7].

4.2 Bounding Pyramid of a Surface Patch

We first construct a surface bounding pyramid based on the tangent bounding cones. Before proceeding, we give a simple lemma.

Lemma 2. *The partial derivative of a triangular surface patch along any direction can be represented by the convex combination of the derivatives of the patch along two directions among α^1, α^2 and α^3.*

Proof. For any direction α in the base triangle, there exist two directions α^{i_1} and α^{i_2} among α^1, α^2 and α^3 such that

$$\alpha = \lambda_1 \alpha^{i_1} + \lambda_2 \alpha^{i_2},$$

where $\lambda_i \geq 0$, $i = 1,2$. Thus

$$\frac{\partial \tilde{\mathbf{B}}}{\partial \alpha}(\mathbf{P}) = \lambda_1 \frac{\partial \tilde{\mathbf{B}}}{\partial \alpha^{i_1}}(\mathbf{P}) + \lambda_2 \frac{\partial \tilde{\mathbf{B}}}{\partial \alpha^{i_2}}(\mathbf{P}),$$

which completes the proof.

Now we are ready to construct a surface bounding pyramid based on the tangent bounding cones as follows.

Let

$$\begin{aligned} C_{ij} = & \quad \{ P \in \mathbb{R}^3 |\ \overrightarrow{OP} = \lambda_1 \vec{v^i} + \lambda_2 \vec{v^j},\ \lambda_1 + \lambda_2 = 1, \\ & \quad \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \vec{v^k} \in Cone_k,\ k = i, j\}, \\ (i,j) = & \quad \{(1,2),(2,3),(3,1)\} \end{aligned} \tag{10}$$

and

$$C_{123} = \overline{C_{12} \cup C_{23} \cup C_{31}} \tag{11}$$

(Here $\bar{A}$ means the complement set of A). Then we have

Theorem 2. (1) *If C_{12}, C_{23} and C_{31} are translated so that their vertices lie at any point $\mathbf{B}(u_0, v_0, w_0)$ on the surface patch $\mathbf{B}(\mathbf{P})$, then the entire patch $\mathbf{B}(\mathbf{P})$ will lie completely within $C_{12} \cup C_{23} \cup C_{31}$.*

(2) *C_{123} is a bounding volume of surface patch $\mathbf{B}(\mathbf{P})$, and it is composed of two nappes of two (possibly) different pyramids.*

Proof. (1) Consider an arbitrary curve segment $\mathbf{B}(u_0+(u_1-u_0)t, v_0+(v_1-v_0)t, w_0+(w_1-w_0)t)$, $t \in [0,1]$ on the surface patch $\mathbf{B}(\mathbf{P})$, where (u_1, v_1, w_1) is a point in the base triangle $\mathbf{T}$. By Lemma 2, there exist constants $\lambda_i \geq 0$, $i = 1,2$ and two directions α^{i_1} and $\alpha^{i_2} (i_1, i_2 = 1,2,3)$ such that

$$\begin{aligned} & \tilde{\mathbf{B}}_t(u_0 + (u_1 - u_0)t, v_0 + (v_1 - v_0)t, w_0 + (w_1 - w_0)t) \\ = & \quad \lambda_1 \frac{\partial \tilde{\mathbf{B}}}{\partial \alpha^{i_1}}(u_0 + (u_1 - u_0)t, v_0 + (v_1 - v_0)t, w_0 + (w_1 - w_0)t) \\ + & \quad \lambda_2 \frac{\partial \tilde{\mathbf{B}}}{\partial \alpha^{i_2}}(u_0 + (u_1 - u_0)t, v_0 + (v_1 - v_0)t, w_0 + (w_1 - w_0)t). \end{aligned}$$

From the above equation and Theorem 1, curve segment $\mathbf{B}(u_0 + (u_1 - u_0)t, v_0 + (v_1 - v_0)t, w_0 + (w_1 - w_0)t)$ must lie in $C_{i_1 i_2}$. Thus the whole patch $\mathbf{B}(\mathbf{P})$ lies within $C_{12} \cup C_{23} \cup C_{31}$.

(2) Obviously, C_{123} bounds surface patch $\mathbf{B}(\mathbf{P})$ in the sense that if the vertex of C_{123} is translated to any point on $\mathbf{B}(\mathbf{P})$, the patch will lie entirely outside of C_{123}. Notice that C_{123} is generally not a pyramid, but is comprised of two nappes of two (possibly) different pyramids $P_1(\mathbf{B})$ and $P_2(\mathbf{B})$, and these two nappes do not intersect except meeting at a common vertex. However, this fact doesn't influence the convenience of C_{123} as a bounding volume.

If we intersect $P_1(\mathbf{B})$ and $P_2(\mathbf{B})$, we will get a surface bounding pyramid $P(\mathbf{B})$ which generally has a hexagonal directrix since both $P_1(\mathbf{B})$ and $P_2(\mathbf{B})$ have triangular directrices.

Remark. In the following, we will call C_{123} a *bi-pyramid* and C_{ij} the *convex set* of $Cone_i$ and $Cone_j$.

To actually compute C_{123}, we should notice the fact that the planar faces of the two nappes of C_{123} are in fact those planes which are tangent to two of the three cones $Cone_i$, $i = 1,2,3$. Thus to get the explicit representation of the two nappes of C_{123}, it is enough to determine the plane which is tangent to two arbitrary cones.

Suppose we are given two cones $Cone(P_1, \theta_1)$ and $Cone(P_2, \theta_2)$ whose vertices are at the origin. The normal of the plane which is tangent to both of $Cone(P_1, \theta_1)$ and $Cone(P_2, \theta_2)$ must be the normals of these cones along the tangent lines. Hence the intersect of the complement cones of $Cone(P_1, \theta_1)$ and $Cone(P_2, \theta_2)$ gives the normal of the tangent plane. Recall the complement cones of $Cone(P_1, \theta_1)$ and $Cone(P_2, \theta_2)$ are $Cone(-P_1, \pi/2 - \theta_1)$ and $Cone(-P_2, \pi/2 - \theta_2)$, the normal of the tangent plane can be easily computed as

$$\overrightarrow{OP} = \lambda_0 \vec{v^1} + \lambda_1(\vec{v^1} \times \vec{v^2}) + \lambda_2[(\vec{v^1} \times \vec{v^2}) \times \vec{v^1}], \tag{12}$$

where

$$\vec{v^1} = -\overrightarrow{OP_1}, \qquad \vec{v^2} = -\overrightarrow{OP_2}, \tag{13}$$

and

$$\lambda_0 = \sin\theta_1, \lambda_2 = [\sin\theta_2 - \sin\theta_1 \cos\theta_0]/\sin^2\theta_0, \lambda_1 = \pm\sqrt{1 - \lambda_0^2 - \lambda_2^2 \sin^2\theta_0}/\sin\theta_0. \tag{14}$$

Here θ_0 is the angle between $\vec{v^1}$ and $\vec{v^2}$. Notice that there exist two normals which correspond to two tangent planes of cones $Cone(P_1, \theta_1)$ and $Cone(P_2, \theta_2)$.

For each nappe of C_{123}, there are three planar faces whose normals $\mathcal{N}_i$, $i = 1, 2, 3$ can be computed as above. The three generators of the nappe are simply $\mathcal{N}_1 \times \mathcal{N}_2$, $\mathcal{N}_2 \times \mathcal{N}_3$, $\mathcal{N}_3 \times \mathcal{N}_1$.

If we use tangent bounding pyramids instead of tangent bounding cones, we can construct another surface bounding bi-pyramid by taking the similar process as before.

Let

$$\begin{aligned} P_{ij} = \quad & \{ P \in \mathbb{R}^3 |\ \overrightarrow{OP} = \lambda_1 \vec{v^i} + \lambda_2 \vec{v^j}, \\ & \lambda_1 + \lambda_2 = 1,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \vec{v^k} \in Pyrd_k,\ k = i, j\}, \\ (i, j) = \quad & \{(1,2), (2,3), (3,1)\} \end{aligned} \tag{15}$$

and

$$P_{123} = \overline{P_{12} \cup P_{23} \cup P_{31}} \tag{16}$$

Then we have

Theorem 3. *P_{123} is a bi-pyramid which bounds surface patch $\mathbf{B}(\mathbf{P})$, and each of the two nappes of P_{123} has a triangular directirx.*

Proof. Essentially the same as the proof of Theorem 2.

Remark. The intersection of the two pyramids corresponding to the two nappes of P_{123} is a surface bounding pyramid whose directrix is generally hexagonal.

To compute the two nappes of P_{123}, we also need find the three pairs of planes, each of which are tangent to two of the three pyramids $Pyrd_i$, $i = 1, 2, 3$. The details of the algorithm are omitted.

4.3 Bounding Cone of a Surface Patch

Because cones are more easily controlled than pyramids, one may wish to construct bounding cones instead of bounding pyramids sometimes. One way to obtain a surface bounding cone is to first compute a surface bounding pyramid according to the algorithm presented in the last subsection, and then find the largest cone which is enclosed in the pyramid.

Suppose we are given a pyramid $\mathcal{P}$ with a triangular directrix whose vertex is at the origin, and the generators of $\mathcal{P}$ are unit vectors $\overrightarrow{OP_i} = v_i$, $i = 1, 2, 3$. Denote the angle between v_i and v_j by β_{ij}, $i, j = 1, 2, 3$.

Theorem 4. *Let*

$$\overrightarrow{OP} = \vec{v} = \vec{v^1} \sin\beta_{23} + \vec{v^2} \sin\beta_{31} + \vec{v^3} \sin\beta_{12}, \tag{17}$$

and

$$\gamma = \arcsin \frac{(\vec{v^1} \times \vec{v^2}) \cdot \vec{v^3}}{\|\vec{v}\|}. \tag{18}$$

Then $Cone(P, \gamma)$ is the largest cone bounded in the pyramid $Pyrd(P_1, P_2, P_3)$.

Proof. Let π_{ij} be the plane which is through OP_i and OP_j, $(i, j) = \{(1, 2), (2, 3), (3, 1)\}$. The distance from P to π_{ij} is

$$d_{ij} = (\vec{v^i} \times \vec{v^j} \cdot \vec{v}) / \|\vec{v^i} \times \vec{v^j}\| = \vec{v^1} \times \vec{v^2} \cdot \vec{v^3},$$

that is, the distances from P to three planes π_{ij}, $(i, j) = \{(1, 2), (2, 3), (3, 1)\}$ are the same. Thus $\vec{v}$ is the axis of the cone which is tangent to the three planes π_{ij}. Obviously, such a cone is the largest cone enclosed in the pyramid $Pyrd(P_1, P_2, P_3)$, and its half vertex angle γ is determined by

$$d_{ij} = \|\vec{v}\| \cdot \sin \gamma = \vec{v^1} \times \vec{v^2} \cdot \vec{v^3}.$$

The theorem is confirmed.

Based on the surface bounding bi-pyramid obtained in the last subsection and Theorem 4, we can easily get a surface bounding volume which is composed of two nappes of two different cones $Cone(P_1, \theta_1)$ and $Cone(P_2, \theta_2)$, and each of the nappes is enclosed in the corresponding nappe of the surface bounding bi-pyramid. We call such a bounding volume a *bi-cone*. However, if we want to get a surface bounding cone, we have to intersect $Cone(P_1, \theta_1)$ and $Cone(P_2, \theta_2)$ and find a cone enclosed in the intersection. Such a cone $Cone(P, \theta)$ can be simply computed as

$$\overrightarrow{OP} = \lambda_1 \overrightarrow{OP_1} + \lambda_2 \overrightarrow{OP_2}, \qquad \theta = (\theta_1 + \theta_2 - \theta_0)/2, \tag{19}$$

where

$$\lambda_1 = [\cos(\theta_1 - \theta_0) - \cos\theta_0 \cos(\theta_2 - \theta)] / \sin^2 \theta_0, \tag{20}$$

$$\lambda_2 = [\cos(\theta_2 - \theta_0) - \cos\theta_0 \cos(\theta_1 - \theta)] / \sin^2 \theta_0, \tag{21}$$

and θ_0 is the angle between $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. Note that the above solution doesn't account for the trivial cases where $Cone(P_1, \theta_1) \subset Cone(P_2, \theta_2)$ or $Cone(P_2, \theta_2) \subset Cone(P_1, \theta_1)$. For these two cases, the solution is simply $Cone(P, \theta) = Cone(P_1, \theta_1)$ or $Cone(P, \theta) = Cone(P_2, \theta_2)$.

Of course, we can also find a surface bounding cone by finding the largest cone enclosed in a surface bounding pyramid(which generally has hexagonal directrix).

4.4 Comparisons

We have provided two kinds of surface bounding volumes-surface bounding pyramids (bi-pyramids) and surface bounding cones(bi-cones). While the surface bounding cones have the merit that they are easier to handle in practice, they suffer from the disadvantage that they have smaller volumes than surface bounding pyramids. In fact, it can be shown that the volume of a surface bounding pyramid is at least $1 - \pi/3\sqrt{3} \approx 40\%$ larger than the volume of the corresponding surface bounding cone. Thus, in practice the users have to determine which bounding volume is to be used.

We also derived two different surface bounding bi-pyramids C_{123} and P_{123} depending on whether using tangent bounding cones or tangent bounding pyramids as the bases. Since tangent bounding pyramids are generally smaller than corresponding tangent bounding cones, the surface bounding bi-pyramid P_{123} are generally larger than C_{123}. Of course, it is a little more convinent to construct C_{123} than to construct P_{123}.

5. Examples

In this section, we give an example to illustrate the process of constructing bounding pyramids(bi-pyramids) and bounding cones(bi-cones) for a triangular Bézier surface patch.

We consider a degree three triangular Bézier surface patch(Fig. 1) whose control points are as follows

$$P_{3,0,0} = (0.0, 0.0, 0.0),\quad P_{2,1,0} = (1.0, 0.0, 0.4),\quad P_{1,2,0} = (2.0, 0.0, 0.5),$$
$$P_{0,3,0} = (3.0, 0.0, 0.0),\quad P_{2,0,1} = (1.0, 1.0, 0.3),\quad P_{1,1,1} = (1.8, 1.2, 0.6),$$
$$P_{0,2,1} = (3.0, 1.0, 0.2),\quad P_{1,0,2} = (2.0, 2.0, 0.5),\quad P_{0,1,2} = (3.0, 2.0, 0.5),$$
$$P_{0,0,3} = (3.0, 3.0, 0.0).$$

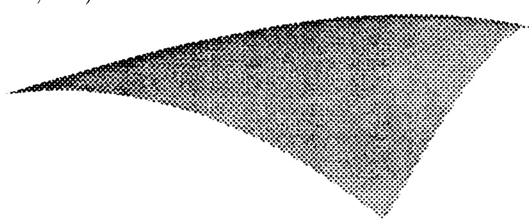

Figure 1. A cubic triangular Bézier surface patch

By computing the directional derivatives along $\alpha^1 = (1, -1, 0)$, $\alpha^2 = (0, 1, -1)$ and $\alpha^3 = (-1, 0, 1)$, we get three tangent bounding pyramids

$$\begin{aligned} Pyrd_1 &= ((-1.0, 0.0, -0.4), (-1.0, 0.0, -0.1), (-1.0, 0.0, 0.5), \\ &\quad (-0.8, -0.2, -0.3), (-1.2, 0.2, 0.4), (-1.0, 0.0, 0.0)) \\ Pyrd_2 &= ((0.0, -1.0, -0.2), (0.0, -1.0, -0.3), (0.0, -1.0, 0.5), \\ &\quad (0.2, -1.2, -0.1), (-0.2, -0.8, 0.1), (0.0, -1.0, 0.1)) \\ Pyrd_3 &= (1.0, 1.0, -0.5), (1.0, 1.0, 0.2), (1.0, 1.0, 0.3), \\ &\quad (1.2, 0.8, -0.1), (0.8, 1.2, 0.2), (1.0, 1.0, -0.3)) \end{aligned}$$

The convex set of $Pyrd_1$ and $Pyrd_2$ is computed by using the convex hull algorithm[1] as

$$\begin{aligned} P_{12} &= Pyrd((-1.0, 0.0, -0.4), (-1.2, 0.2, 0.4), (-1.0, 0.0, 0.5), \\ &\quad (0.0, -1.0, 0.5), (0.2, -1.2, -0.1), (0.0, -1.0, -0.3)), \end{aligned}$$

and the normal vectors of the two planes which are tangent to both of $Pyrd_1$ and $Pyrd_2$ are thus $v_{12}^1 = (-1.0, 0.0, 0.5) \times (0.0, -1.0, 0.5) = (0.5, 0.5, 1.0)$ and $v_{12}^2 = (0.0, -1.0, -0.3) \times (-1.0, 0.0, -0.4) = (0.4, 0.3, -1.0)$.

Similarily, we can obtain the other two convex sets of the tangent bounding pyramids and their corresponding normal vectors of the tangent planes

$$\begin{aligned} P_{23} &= Pyrd((1.0, 1.0, 0.2), (0.8, 1.2, 0.2), (1.0, 1.0, 0.3), \\ &\quad (-0.2, -0.8, 0.1), (0.0, -1.0, -0.2)), \\ &\quad v_{23}^1 = (-0.8, 0.4, 0.8), \qquad v_{23}^2 = (-0.8, 0.3, -1.0) \\ P_{31} &= Pyrd((1.0, 1.0, -0.5), (1.2, 0.8, -0.1), (1.0, 1.0, 0.3), \\ &\quad (-1.0, 0.0, 0.5), (-0.8, -0.2, -0.3)), \\ &\quad v_{31}^1 = (0.5, -0.8, 1.0), \qquad v_{31}^2 = (0.4, -0.7, -0.6) \end{aligned}$$

From vectors v_{ij}^k, $i, j = 1, 2, 3$, $k = 1, 2$, we found the two nappes of a surface bounding bi-pyramid are(Fig. 2)

$$Pyrd((0.00, -1.20, 0.60), (1.04, 1.20, 0.44), (-1.30, 0.00, 0.65))$$

and

$$Pyrd((0.88, 0.88, -0.44), (0.00, -1.20, -0.36), (-0.88, -0.16, -0.40)).$$

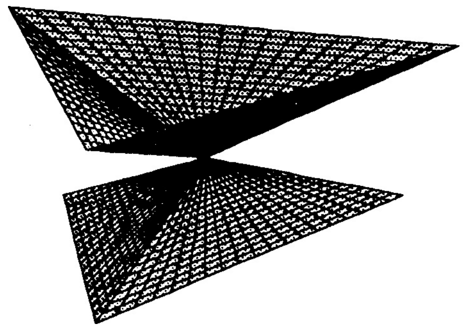

Figure 2. A surface bounding bi-pyramid

The intersection of the two pyramids corresponding to the two nappes is also a surface bounding pyramid which has a hexagonal directrix(Fig. 3)

$$Pyrd((0.80, -0.30, 0.75), (0.72, 0.54, 0.45), (0.44, 0.80, 0.42),$$
$$(-0.66, 0.39, 0.64), (-0.79, -0.30, 0.54), (-0.41, -0.72, 0.56)).$$

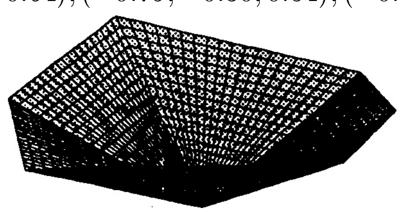

Figure 3. One nappe of a surface bounding pyramid

By theorem 4, we can also find a surface bounding bi-cone, the two nappes of which are(Fig. 4)

$Cone((-0.13, -0.08, 0.99), 0.80)$ and $Cone((-0.12, -0.30, -0.95), 0.81)$.

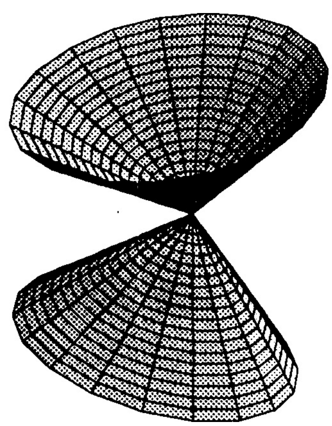

Figure 4. A surface bounding bi-cone

The cone enclosed in the intersection of the two cones corresponding to the above two nappes is $Cone((0.01, -0.07, -0.68), 0.58)$, which is a surface bounding cone(Fig. 5).

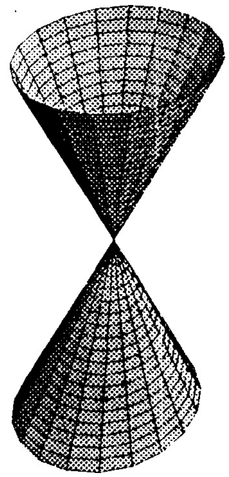

Figure 5. A surface bounding cone

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