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SEMIDISCRETIZATION IN SPACE OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH BLOW-UP OF THE SOLUTIONS*

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Abstract

Semidiscretization in space of nonlinear degenerate parabolic equations of non-divergent form is presented, under zero Dirichlet boundary condition. It is shown that semidiscrete solutions blow up in finite time. In particular, the asymptotic behavior of blowing-up solutions, is discussed precisely.

Key words: Semi-discrete problem, Blow-up of solutions, Blow-up rate, Blow-up set, Limiting profile.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider the following initial boundary value problem :

$$\text{(P1)} \qquad \begin{cases} u_t = u^\delta(\triangle u + \mu u), & x \in \Omega,\, t > 0, \qquad (1)\\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \qquad (2)\\ u(x,0) = u_0(x), & x \in \Omega, \qquad (3) \end{cases}$$

where δ, μ are positive constants and $u_0(x)$ is a nonnegative bounded continuous function on $\bar{\Omega}$.

When $N = 1$ and $\delta = 2$, the problem arises in a model for the resistive diffusion of a force-free magnetic field in a plasma confined between two walls in one dimension (see [5], [8], [9], [10] and [14]). Equation (1) also describes the evolution of the curvature of a locally convex plane curve, and it has been studied in [2] and [6] under periodic boundary condition.

A. Friedman and B. McLeod [5] considered (P1) in the case $\delta = 2$ and $\mu = 1$. They showed that the behavior of solutions depends on the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet problem for the Laplacian on the domain Ω. If $\lambda_1(\Omega) > 1$, then there exists a unique global solution which tends to zero as $t \to \infty$. If $\lambda_1(\Omega) < 1$, then

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there exists a positive constant T such that we have a unique solution in $0 < t < T$, which blows up as $t \uparrow T$. They also showed that the blow-up set has positive Lebesgue measure. In particular, when $N = 1$ and initial data u_0 satisfies $u_0(-x) = u_0(x)$ and $u_{0xx} + u_0 \geq 0$, they showed that the blow-up set S is exactly $S = \{-\pi/2 \leq x \leq \pi/2\}$. Qi [12] discussed the Cauchy problem for (1) and (3) with $0 < \delta < 2$. For the case $\delta > 1$, M. Wiegner [15] studied the existence and uniqueness of smooth positive solutions and gave an upper bound of the blow-up time for the positive initial data. When $N = 1$ and $\delta > 0$, K. Anada, I. Fukuda and M. Tsutsumi [1] got precise information on the blow-up set and asymptotic behavior near the blow-up time. When $N \geq 2$, in [13] we have obtained the detailed results on the blow-up sets and asymptotic behavior of solutions of the problem (P1) with radially symmetric positive initial data. In [7], we solved this problem numerically by using a finite difference scheme with a variable time increment with suitable control and showed numerical results for symmetric and non-symmetric blowing-up solutions.

We consider the following two different levels in discretization of the problem:

Step 1. First, the problem (P1) is discretized in space. We use finite difference method as this discretization and get an ordinary differential system in a finite dimension. We call it "semidiscrete problem."

Step 2. Next, we discrete the semidiscrete problem in time by finite difference method. In order to compute a blowing-up solution suitably, we have to apply some control to time increment. (See [7].) We call this "variable time increment method". This idea is seen in [11], [3] and [4], in which they use variable time increment method for semilinear parabolic equations.

In this paper we consider the semidiscrete problem of (P1) for rectangle domain $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$ and analyze properties of solutions of semidiscrete problem. Of course, we can get the same results in higher dimension. We prove the blow-up of solutions of the semidiscrete problem for $\delta > 0$ and obtain lower and upper rates of blowing-up solution. We also get the lower and upper estimates of blow-up time and discuss the asymptotic behavior of solutions near the blow-up time.

2. Semidiscretization in Space of the Problem

First of all, in order to analyze a problem on a grid point set, we define a grid point set R_h with mesh size h (> 0) by $R_h = \{x_m \in \mathbb{R} | x_m = hm, m \in \mathbb{Z}\}$. Let M and N be positive integers and take $h_x = a/M$ and $h_y = b/N$. Then we now define Ω_h, which is the discretization of Ω, as the following:

$$\Omega_h = \{(x_m, y_n) \in R_{h_x} \times R_{h_y} | (x_m, y_n) \in \Omega\}.$$

Next, we introduce the following terms and notion to express the statements precisely.

Definition 2.1. (*neighboring grid points, neighboring set*) (x_{m+1}, y_n),(x_{m-1}, y_n), (x_m, y_{n+1}) *and* (x_m, y_{n-1}) *are called neighboring grid points of* (x_m, y_n). *The neighbor-*

ing set of (x_m, y_n)*, denoted by* $N_2(x_m, y_n)$*, is defined by*

$$N_2(x_m, y_n) = \{(x_{m+1}, y_n), (x_{m-1}, y_n), (x_m, y_{n+1}), (x_m, y_{n-1})\}.$$

Remark 2.2. We can define neighboring grid points and neighboring set in higher dimensional cases.

Let D be a grid point set in $R_{h_x} \times R_{h_y}$. We define the boundary of D.

Definition 2.3. (x_m, y_n) *is a boundary point of* D *if* $(x_m, y_n) \notin D$ *and* $\exists (x_{m'}, y_{n'}) \in N_2(x_m, y_n)$ *such that* $(x_{m'}, y_{n'}) \in D$.

Definition 2.4. *The boundary of* D*, denoted by* ∂D*, consists of all boundary point of* D.

Now, using the above definition, we can define the boundary of Ω_h, which is denoted by $\partial\Omega_h$.

Therefore, we may introduce the semidiscrete problem of (P1) as the following:

$$\text{(SP1)} \qquad \begin{cases} \dfrac{d}{dt}u_{m,n} = u_{m,n}^{\delta}(\triangle_d u_{m,n} + \mu u_{m,n}), & (x_m, y_n) \in \Omega_h, t > 0, \quad (4) \\ u_{m,n}(t) = 0, & (x_m, y_n) \in \partial\Omega_h, t > 0, \quad (5) \\ u_{m,n}(0) = u_{0_{m,n}} \geq 0, & (x_m, y_n) \in \Omega_h, \quad (6) \end{cases}$$

where $u_{m,n}(t)$ is a real-valued function defined on $R_{h_x} \times R_{h_y} \times \mathbb{R}$ and denotes a value at (x_m, y_n, t) and $\triangle_d$ denotes the discrete Laplacian, that is,

$$\triangle_d u_{m,n} = (D_x D_{\bar{x}} + D_y D_{\bar{y}})u_{m,n},$$

where D_x,$D_{\bar{x}}$,D_y,and $D_{\bar{y}}$ are finite difference operators, that is,

$$\begin{aligned} D_x u_{m,n} &= \frac{1}{h_x}(u_{m+1,n} - u_{m,n}), \\ D_{\bar{x}} u_{m,n} &= \frac{1}{h_x}(u_{m,n} - u_{m-1,n}), \\ D_y u_{m,n} &= \frac{1}{h_y}(u_{m,n+1} - u_{m,n}), \\ D_{\bar{y}} u_{m,n} &= \frac{1}{h_y}(u_{m,n} - u_{m,n-1}). \end{aligned}$$

We define sets $P(t)$ and $Z(t)$ by

$$P(t) = \{(x_m, y_n) \in \Omega_h | u_{m,n}(t) > 0\},$$
$$Z(t) = \{(x_m, y_n) \in \Omega_h | u_{m,n}(t) = 0\}.$$

We have the following propositions.

Proposition 2.5. *Let* $\delta \geq 1$. *If* $u_{0_{m,n}}$ *is nonnegative, then the solutions is nonnegative. Moreover,* $P(t) = P(0)$ *and* $Z(t) = Z(0)$ *for* $t \in [0, T_{max})$. *Here* T_{max} *is a maximal existence time.*

Proof. We only consider the case $\delta > 1$. We may prove the case $\delta = 1$ in the same manner.

Clearly, $Z(0) \subseteq Z(t)$ for $t > 0$ because zero is the solution of ordinary equation

$$\frac{d}{dt}u_{m,n} = u_{m,n}^{\delta}(\triangle_d u_{m,n} + \mu u_{m,n})$$

for some m and n and uniqueness of the solution may be hold since $\delta \geq 1$.

From the continuity of solution, for $(x_m, y_n) \in P(0)$, $u_{m,n}(t)$ remains positive near $t = 0$. Assume that there exist $t_1 > 0$ and $Z_1 \subset P(0)$ such that Z_1 is not empty and

$$\begin{cases} u_{m,n}(t) > 0, & \text{for } 0 \leq t < t_1 \text{ and } (x_m, y_n) \in P(0), \\ u_{m,n}(t_1) = 0, & \text{for } (x_m, y_n) \in Z_1, \\ u_{m,n}(t_1) > 0, & \text{for } (x_m, y_n) \in P(0) \setminus Z_1. \end{cases}$$

Note that from the above assumption, $u_{m,n}(t_1)$ is nonnegative for $(x_m, y_n) \in P(0)$. From eq.(4) we get

$$\frac{1}{1-\delta} \frac{d}{dt} u_{m,n}^{1-\delta} = \triangle_d u_{m,n} + \mu u_{m,n} \ , \ (x_m, y_n) \in P(0), \ 0 \leq t < t_1.$$

Taking a summation on $(x_m, y_n) \in P(0)$ and integrating it from 0 to t, for any $t < t_1$, we have

$$\begin{aligned} \sum_{(x_m,y_n)\in P(0)} u_{m,n}^{1-\delta}(t) &= -(\delta-1) \int_0^t \sum_{(x_m,y_n)\in P(0)} (\triangle_d u_{m,n}(s) + \mu u_{m,n}(s)) ds \\ &+ \sum_{(x_m,y_n)\in P(0)} u_{m,n}^{1-\delta}(0). \end{aligned} \tag{7}$$

Since $u_{m,n}(t_1)$ $((x_m, y_n) \in P(0))$ is nonnegative and $u_{m,n}(t_1) = 0$ for $(x_m, y_n) \in Z_1 \subset P(0)$, the l.h.s of (7) tends to infinity and the r.h.s of (7) tends to a finite value as $t \to t_1$. This is a contradiction and we have $P(0) \subseteq P(t)$ for $t > 0$. Since the initial data is nonnegative, that is, $\Omega_h = P(0) \cup Z(0)$ and we already get $Z(0) \subseteq Z(t)$ for $t > 0$, then we obtain

$$Z(t) = Z(0), \quad t > 0,$$

$$P(t) = P(0), \quad t > 0,$$

and the solution is nonnegative. Thus we complete the proof.

Proposition 2.6. *Let $\delta > 0$. If $u_{0_{m,n}}$ is nonnegative, then $u_{m,n}(t)$ is nonnegative for $t \in [0, T_{max})$.*

Proof. The non-negativity of the solution in the case $\delta \geq 1$ is obvious from the above proposition. Thus we treat only the case $0 < \delta < 1$. Consider the following problem

$$\begin{aligned} \frac{d}{dt} u_{m,n} = |u_{m,n}|^{\delta} (\triangle_d u_{m,n} + \mu u_{m,n}), \quad (x_m, y_n) \in \Omega_h, t > 0, \\ u_{m,n}(t) = 0, \quad (x_m, y_n) \in \partial\Omega_h, t > 0, \\ u_{m,n}(0) = u_{0_{m,n}} \geq 0, \quad (x_m, y_n) \in \Omega_h. \end{aligned}$$

Take $T_0 < T_{max}$, fixed. Putting $v_{m,n}(t) = u_{m,n}(t)e^{-\lambda t}$ where λ is a positive constant satisfying

$$\lambda > \mu \max_{\substack{m,n \\ 0 \leq t \leq T_0}} |u_{m,n}|^{\delta},$$

we have

$$\frac{d}{dt} v_{m,n}(t) = e^{-\lambda t} \{-\lambda u_{m,n}(t) + |u_{m,n}(t)|^{\delta} (\triangle_d u_{m,n}(t) + \mu u_{m,n}(t))\}. \tag{8}$$

Assume that there exist $t_0 > 0$ and (x_{m_0}, y_{n_0}) such that $v_{m_0,n_0}(t_0)$ is a negative minimum over $\{t|\ 0 \le t \le t_0\} \times \Omega_h$. Note that $\frac{d}{dt} v_{m_0,n_0}(t_0) \le 0$, $u_{m_0,n_0}(t_0) < 0$ and $\triangle_d u_{m_0,n_0}(t_0) \ge 0$.

From eq. (8), we obtain

$$\frac{d}{dt} v_{m_0,n_0}(t)|_{t=t_0} \ge e^{-\lambda t_0} u_{m_0,n_0}(t_0)(-\lambda + \mu |u_{m_0,n_0}(t_0)|^{\delta}). \tag{9}$$

The l.h.s. of (9) is non-positive and the r.h.s. of (9) is positive. This leads a contradiction and we have the assertion.

3. Blow-up of Semidiscrete Solutions

Let λ_d and $\Phi_{m,n}$ be the first eigenvalue and eigenfunction of the discrete eigenvalue problem

$$\begin{cases} -\triangle_d \Phi_{m,n} = \lambda_d \Phi_{m,n} & (x_m, y_n) \in \Omega_h, \\ \Phi_{m,n} = 0, & (x_m, y_n) \in \partial\Omega_h. \end{cases}$$

In the same manner as the continuous case we can assume that $\Phi_{m,n}$ are positive for $m = 1, 2, \cdots, M-1$ and $n = 1, 2, \cdots, N-1$ and $\langle \Phi_{m,n}, 1 \rangle = 1$, where $\langle\ ,\ \rangle$ denotes

$$\langle a_{m,n}, b_{m,n} \rangle = h_x \cdot h_y \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} a_{m,n} b_{m,n}.$$

We also denote $\|\ \|_2$ by

$$\|a_{m,n}\|_2 = \langle a_{m,n}, a_{m,n} \rangle^{1/2}.$$

Theorem 3.1. *Let $\delta > 0$ and $u_{0_{m,n}}$ be a bounded positive function in Ω_h. If $\lambda_d < \mu$, then there exists a finite time $T_d > 0$ such that the solution $u_{m,n}$ of* (SP1) *blows up to infinity as $t \uparrow T_d$ where*

$$xT_d \le \begin{cases} \dfrac{1}{\delta(\mu - \lambda_d)} \langle u_{0_{m,n}}^{1-\delta}, \Phi_{m,n} \rangle^{\frac{\delta}{\delta-1}} & \text{if } \delta \ne 1, \\ \dfrac{1}{\mu - \lambda_d} e^{-\langle \log u_{0_{m,n}}, \Phi_{m,n} \rangle} & \text{if } \delta = 1. \end{cases}$$

Proof. We first consider the case $\delta > 1$. (4) may be written as

$$-\frac{1}{\delta - 1} \frac{d}{dt} u_{m,n}^{1-\delta} = \triangle_d u_{m,n} + \mu u_{m,n}\ . \tag{10}$$

Taking the inner product of the both sides of (10) with $\Phi_{m,n}$, we have

$$\frac{d}{dt} \langle u_{m,n}^{1-\delta}, \Phi_{m,n} \rangle = -(\delta - 1)\ (\mu - \lambda_d)\ \langle u_{m,n}, \Phi_{m,n} \rangle. \tag{11}$$

Here we make use of the following equality

$$\sum_{m=1}^{M-1} h \cdot (D_x D_{\bar{x}} v_m) w_m = \sum_{m=1}^{M-1} h \cdot v_m (D_x D_{\bar{x}} w_m),$$

for any v_m and w_m with $v_0 = v_M = w_0 = w_M = 0$. Since $f(x) = x^{1-\delta} (\delta > 1)$ is a convex function on $(0, \infty)$, Jensen's inequality yields that

$$\langle u_{m,n}^{1-\delta}, \Phi_{m,n} \rangle^{-\frac{1}{\delta-1}} \le \langle u_{m,n}, \Phi_{m,n} \rangle$$

for $\delta > 1$. Since $\lambda_d < \mu$, we have

$$\frac{d}{dt}\langle u_{m,n}^{1-\delta}, \Phi_{m,n}\rangle \le -(\delta-1)\ (\mu-\lambda_d)\ \langle u_{m,n}^{1-\delta}, \Phi_{m,n}\rangle^{-\frac{1}{\delta-1}}. \tag{12}$$

From eq.(12), we get

$$\langle u_{m,n}^{1-\delta}, \Phi_{m,n}\rangle \le \left\{\langle u_{0_{m,n}}^{1-\delta}, \Phi_{m,n}\rangle^{\frac{\delta}{\delta-1}} - \delta(\mu-\lambda_d)\ t\right\}^{\frac{\delta-1}{\delta}} ,$$

which gives that

$$\sup_{m,n} u_{m,n} \ge \left\{\langle u_{0_{m,n}}^{1-\delta}, \Phi_{m,n}\rangle^{\frac{\delta}{\delta-1}} - \delta(\mu-\lambda_d)\ t\right\}^{-\frac{1}{\delta}} . \tag{13}$$

This completes the proof of the case $\delta > 1$.

Next we consider the case $0 < \delta < 1$. We have

$$\frac{d}{dt}\langle u_{m,n}^{1-\delta}, \Phi_{m,n}\rangle = (1-\delta)\ (\mu-\lambda_d)\ \langle u_{m,n}, \Phi_{m,n}\rangle.$$

Since $f(x) = -x^{1-\delta}(0 < \delta < 1)$ is a convex function on $(0,\infty)$, Jensen's inequality yields that

$$\langle u_{m,n}^{1-\delta}, \Phi_{m,n}\rangle^{\frac{1}{1-\delta}} \le \langle u_{m,n}, \Phi_{m,n}\rangle$$

for $0 < \delta < 1$. Since $\lambda_d < \mu$, we have

$$\frac{d}{dt}\langle u_{m,n}^{1-\delta}, \Phi_{m,n}\rangle \ge (1-\delta)\ (\mu-\lambda_d)\ \langle u_{m,n}^{1-\delta}, \Phi_{m,n}\rangle^{\frac{1}{1-\delta}}. \tag{14}$$

From eq.(14),

$$\langle u_{m,n}^{1-\delta}, \Phi_{m,n}\rangle \ge \left\{\langle u_{0_{m,n}}^{1-\delta}, \Phi_{m,n}\rangle^{\frac{\delta}{\delta-1}} - \delta(\mu-\lambda_d)\ t\right\}^{\frac{\delta-1}{\delta}} . \tag{15}$$

Since $0 < \delta < 1$ and eq.(15), we get (13). This completes the proof of the case $0 < \delta < 1$.

Finally when $\delta = 1$, we have

$$\frac{d}{dt}\langle \log u_{m,n}, \Phi_{m,n}\rangle = (\mu-\lambda_d)\ \langle u_{m,n}, \Phi_{m,n}\rangle.$$

Since $f(x) = \log x^{-1}$ is a convex function on $(0,\infty)$, Jensen's inequality yields that

$$e^{\langle \log u_{m,n}, \Phi_{m,n}\rangle} \le \langle u_{m,n}, \Phi_{m,n}\rangle .$$

Since $\lambda_d < \mu$, we have

$$\frac{d}{dt}\langle \log u_{m,n}, \Phi_{m,n}\rangle \ge (\mu-\lambda_d)\ e^{\langle \log u_{m,n}, \Phi_{m,n}\rangle} \tag{16}$$

From eq.(16),

$$\langle \log u_{m,n}, \Phi_{m,n}\rangle \ge -\log\left\{e^{-\langle \log u_{0_{m,n}}, \Phi_{m,n}\rangle} - (\mu-\lambda_d)\ t\right\}. \tag{17}$$

We conclude from (17) that

$$\sup_{m,n} u_{m,n} \ge \left\{e^{-\langle \log u_{0_{m,n}}, \Phi_{m,n}\rangle} - (\mu-\lambda_d)\ t\right\}^{-1}.$$

This completes the proof of Theorem 3.1.

To clear the statement in the future argument, we again define some terms and notations. Let A and B be a grid point sets in $R_{h_x} \times R_{h_y}$.

Definition 3.2. *(x_m, y_n) is directly connected to $(x_{m'}, y_{n'})$ if $(x_m, y_n) \in N_2(x_{m'}, y_{n'})$, or equivalently if $(x_{m'}, y_{n'}) \in N_2(x_m, y_n)$.*

Definition 3.3. *$(x_m, y_n) \in A$ is connected in A to $(x_{m'}, y_{n'}) \in A$ if there exist $\{(m_k, n_k)\}_{k=1,2,\cdots,K}$ $(K \le \infty)$ such that $(x_{m_k}, y_{n_k}) \in A$, $(x_m, y_n) \in N_2(x_{m_1}, y_{n_1})$, $(x_{m_k}, y_{n_k}) \in N_2(x_{m_{k+1}}, y_{n_{k+1}})$ $(k = 1, 2, \cdots, K-1.)$, and $(x_{m_{K-1}}, y_{n_{K-1}}) \in N_2(x_{m'}, y_{n'})$.*

Remark 3.4. By definition, if $(x_m, y_n) \in A$ is connected in A to $(x_{m'}, y_{n'}) \in A$, then $(x_{m'}, y_{n'})$ is connected in A to (x_m, y_n).

Definition 3.5. *A is connected if A consists of only one element, or if A consists of more than two elements and for any $(x_m, y_n), (x_{m'}, y_{n'}) \in A$, (x_m, y_n) is connected in A to $(x_{m'}, y_{n'})$.*

Definition 3.6. *A is disconnected from B if $(A \cup \partial A) \cap B$ is empty, or equivalently if $(B \cup \partial B) \cap A$ is empty.*

Remark 3.7. By definition, if A is disconnected from B, then B is disconnected from A.

Let $\lambda_d^{(D)}$ and $\Phi_{m,n}^{(D)}$ be the first eigenvalue and corresponding eigenfunction of the discrete Dirichlet problem for discrete Laplacian on the bounded connected set D, that is,

$$\begin{cases} -\triangle_d \Phi_{m,n}^{(D)} = \lambda_d^{(D)} \Phi_{m,n}^{(D)}, & (x_m, y_n) \in D, \\ \Phi_{m,n}^{(D)} = 0, & (x_m, y_n) \in \partial D. \end{cases}$$

We can assume that $\Phi_{m,n}^{(D)}$ is positive in D and $\langle \Phi_{m,n}^{(D)}, 1 \rangle_D = 1$, where $\langle a_{m,n}, b_{m,n} \rangle_D$ denotes

$$\langle a_{m,n}, b_{m,n} \rangle_D = h_x h_y \sum_{(x_m, y_n) \in D} a_{m,n} b_{m,n}.$$

For nonnegative initial date, we can also prove the blow-up of solutions.

Theorem 3.8. *Let $\delta \ge 1$ and Ω_0 be connected subset of Ω_h. Suppose that $u_{0_{m,n}} \ge 0$ in Ω_h and $u_{0_{m,n}} > 0$ in Ω_0. If $\lambda_d^{(\Omega_0)} < \mu$, then the solution blows up in a finite time T_d, where*

$$T_d \le \begin{cases} \dfrac{1}{\delta(\mu - \lambda_d^{(\Omega_0)})} \langle u_{0_{m,n}}^{1-\delta}, \Phi_{m,n}^{(\Omega_0)} \rangle_{\Omega_0}^{\frac{\delta}{\delta-1}}, & \text{if } \delta > 1, \\ \dfrac{1}{\mu - \lambda_d} \exp(-\langle \log\, u_{0_{m,n}}, \Phi_{m,n} \rangle_{\Omega_0}) & \text{if } \delta = 1. \end{cases}$$

The proof of the above theorem is almost the same as the proof of theorem 3.1 and we omit it.

4. Blow-Up Rate and Asymptotic

In this section we discuss the behavior of semidiscrete solutions.

Lemma 4.1. *Let $\delta \ge 1$ and T_{max} is the maximal existence time. If $u_{0_{m,n}}$ is nonnegative and satisfies $\triangle_d u_{0_{m,n}} + \mu u_{0_{m,n}} \ge 0$, then*

$$\frac{d}{dt} u_{m,n}(t) \ge 0 \qquad \text{for} \quad t \in [0, T_{max}).$$

Proof. Take $T_0 < T_{max}$, fixed. Putting $z_{m,n}(t) = \frac{d}{dt}u_{m,n}(t)$, we have

$$\frac{d}{dt}z_{m,n} = \delta u_{m,n}^{\delta-1}(\triangle_d u_{m,n} + \mu u_{m,n})z_{m,n} + u_{m,n}^{\delta}(\triangle_d z_{m,n} + \mu z_{m,n}). \tag{18}$$

Let K be a positive constant satisfying

$$K > \max_{\substack{m,n \\ 0\le t\le T_0}} \Big(\delta u_{m,n}^{\delta-1}(\triangle_d u_{m,n} + \mu u_{m,n}) + \mu u_{m,n}^{\delta}\Big).$$

Consider the function $w_{m,n}(t) = z_{m,n}(t)e^{-Kt}$. From eq. (18), we have

$$\begin{aligned}\frac{d}{dt}w_{m,n}(t) &= z_{m,n}(t)e^{-Kt}\{-K + \delta u_{m,n}^{\delta-1}(t)(\triangle_d u_{m,n}(t) + \mu u_{m,n}(t)) + \mu u_{m,n}^{\delta}(t)\} \\ &\quad + u_{m,n}^{\delta}(t)e^{-Kt}\triangle_d z_{m,n}\end{aligned} \tag{19}$$

Note that $w_{m,n}(0) \ge 0$ because $z_{m,n}(0) \ge 0$.

Assume that there exist $t_0 > 0$ and (m_0, n_0) such that $w_{m_0,n_0}(t_0)$ is a negative minimum over $\{t|\ 0 \le t \le t_0\} \times \{(m,n)|\ 0 \le m \le M,\ 0 \le n \le N\}$.

From eq. (19), we have

$$\frac{d}{dt}w_{m_0,n_0}(t)|_{t=t_0} \ge z_{m_0,n_0}e^{-Kt_0}\{-K + \delta u_{m_0,n_0}^{\delta-1}(\triangle_d u_{m_0,n_0} + \mu u_{m_0,n_0}) + \mu u_{m_0,n_0}^{\delta}\}, \tag{20}$$

since $\triangle_d z_{m_0,n_0}$ is nonnegative.

The l.h.s. of (20) is non-positive and the r.h.s. of (20) is positive. This is a contradiction and we conclude that

$$z_{m,n}(t) \ge 0.$$

From now on, we treat only blowing-up solutions and always assume the following:

(A1) $u_{0_{m,n}}$ *is a positive bounded function in* Ω_h *and satisfies*

$$\triangle_d u_{0_{m,n}} + \mu u_{0_{m,n}} \ge 0 \quad \text{in } \Omega_h.$$

Remark 4.2. Now, we consider the blow-up case, thus, $\triangle_d u_{0_{m,n}} + \mu u_{0_{m,n}}$ is not identically zero in Ω_h. Because, if $\triangle_d u_{0_{m,n}} + \mu u_{0_{m,n}} \equiv 0$, that is, $\lambda_d = \mu$, then the solution does not blows up in a finite time.

We have the following result on blow-up rate.

Theorem 4.3. *Suppose that* $0 < \delta < 2$. *Then there exists a positive constant* C *such that*

$$u_{m,n}(t) \le C(T_d - t)^{-1/\delta}, \quad 0 \le t < T_d.$$

Proof. Put

$$J(t) = \frac{1}{2-\delta}\ \langle u_{m,n}^{2-\delta}, 1\,\rangle.$$

Differentiating by t and making use of (4), we have

$$J'(t) = \langle u_{m,n}^{1-\delta}, \frac{d}{dt}u_{m,n}\,\rangle$$

and

$$J''(t) = 2\ \langle u_{m,n}^{-\delta}, (\frac{d}{dt}u_{m,n})^2\,\rangle.$$

Notice that $J'(t) > 0$ for any $t \in (0, T_d)$. Since $0 < \delta < 2$, Schwarz's inequality gives

$$(J'(t))^2 \le \frac{2-\delta}{2} J''(t)J(t),$$

from which it follows that

$$J'(t)J(t)^{-2/(2-\delta)} \ge J'(0)J(0)^{-2/(2-\delta)} \equiv C_1 > 0.$$

Hence

$$(J(t)^{-\delta/(2-\delta)})' \le -\frac{\delta}{2-\delta} C_1 = -C_2. \tag{21}$$

Integration over (t, T_d) yields

$$J(t) \le C_3 \ (T_d - t)^{-(2-\delta)/\delta}.$$

Hence we have

$$u_{m,n}(t)^{2-\delta} \le C_4 (T_d - t)^{-(2-\delta)/\delta}, \quad 0 \le t < T_d.$$

Thus we have the assertion.

We define the function $M(t)$ by

$$M(t) = \max_{m,n} u_{m,n}(t).$$

Now we characterize blowing-up solutions by the blow-up rate.

Definition 4.4. *The blowing-up solution is called "Type I" if the blowing-up solution satisfies the following inequality:*

$$(T_d - t)^{1/\delta} M(t) < \infty, \quad \text{as } t \to T_d.$$

The blowing-up solution is called "Type II" if the blowing-up solution satisfies the following inequality:

$$(T_d - t)^{1/\delta} M(t) \to \infty, \quad \text{as } t \to T_d.$$

Remark 4.6. From the above theorem, when $0 < \delta < 2$, we have

$$M(t) \le C(T_d - t)^{-1/\delta}. \tag{22}$$

Thus, only Type I blow-up occurs in the case $0 < \delta < 2$. On the other hand, when $\delta \ge 2$, we do not have any upper estimate for blow-up rate. A. Friedman and B. McLeod [5], K. Anada, I. Fukuda and M. Tsutsumi [1] and M. Tsutsumi and T. Ishiwata [13] show that Type II blow-up may occur for special initial data when $\delta \ge 2$. Numerical work [7] suggests that Type II blow-up may occurs in the case $\delta \ge 2$ for more general initial data. The question whether only Type II blow-up occurs in the case $\delta \ge 2$, or not, is still open.

In theorems 3.1 and 3.8, the blow-up time T_d is estimated by initial data and eigenfunction. For the case $0 < \delta < 2$, we can get the upper bound of blow-up time which depend on only initial data.

Theorem 4.7. (*upper bound of blow-up time*) *Suppose that* $0 < \delta < 2$. *Then*

$$T_d \le \frac{2-\delta}{\delta} \frac{J(0)}{J'(0)},$$

where

$$J'(0)=\langle u_{0_{m,n}},\triangle_d u_{0_{m,n}}+\mu u_{0_{m,n}}\rangle.$$

Proof. From eq.(21) we have

$$J^{-\frac{\delta}{2-\delta}}(t)-J^{-\frac{\delta}{2-\delta}}(0)\le -C_2 t.$$

Letting $t\to T_d$, we get

$$-J^{-\frac{\delta}{2-\delta}}(0)\le -C_2T_d.$$

Since $C_2=\frac{\delta}{2-\delta}J'(0)J^{-2/(2-\delta)}(0)$, we obtain

$$T_d\le \frac{2-\delta}{\delta}\frac{J(0)}{J'(0)}.$$

We also obtain the lower bound of blow-up time for the case $\delta>0$.

Theorem 4.8. *Suppose that $\delta>0$. Then, we have following estimate:*

$$T_d>\frac{1}{\delta}(\max_{x\in\Omega_h}u_{0_{m,n}})^{-\delta}.$$

Proof. Put $w_{m,n}(t)=v(t)-u_{m,n}(t)$, where $v(t)$ is a solution of the following ordinary equation:

$$\text{(ODE)}\qquad \begin{cases} \dfrac{d}{dt}v=v^{\delta+1}, & t>0, \quad (23)\\ v_0=v(0)=\max_{x\in\Omega_h}u_{0_{m,n}}+\varepsilon. & \quad (24)\end{cases}$$

We have

$$\frac{d}{dt}w_{m,n}=u_{m,n}^{\delta}\triangle_d w_{m,n}+v^{\delta+1}-u_{m,n}^{\delta+1}. \tag{25}$$

Note that $w_{0_{m,n}}=w_{m,n}(0)\ge\varepsilon$. Since $w_{m,n}$ is continuous in $t\ge 0$ (moreover, smooth in $t>0$.) , there exists a small positive τ such that $w>0$ for $t\in[0,\tau)$ and any $(x_m,y_n)\in\Omega_h$. Assume that there exist $(x_{m_0},y_{n_0})\in\Omega_h$ and $t_0\in\mathbb{R}_+$ such that $w_{m_0,n_0}(t_0)=0$ and $w_{m,n}(t)=0$ for $t<t_0$ and any $\in(x_m,y_n)\Omega_h$. Let $M_t=\{(x_m,y_n)\in\Omega_h|w_(m,n)(t)$ is local minimum at each fixed$t\in R_+\}$. Obviously, $(x_{m_0},y_{n_0})\in M_{t_0}$. By assumption, there exist $(x_{m_1},y_{n_1})\in\Omega_h$ and $t_1<t_0$ such that $(x_{m_1},y_{n_1})\in M_{t_1}$ and

$$\frac{d}{dt}w_{m_1,n_1}(t_1)<0. \tag{26}$$

However, at (x_{m_1},u_{n_1},t_1), we see that $\triangle_d w_{m,n}(t)\ge 0$, $u_{m,n}(t)>0$ and $v^{\delta+1}-u_{m,n}^{\delta+1}>0$ since $w>0$ for $t<t_0$. Then, (r.h.s) of (25) at (x_{m_1},y_{n_1},t_1) is positive. This contradicts with the fact (26). Then, we get $w_{m,n}(t)>0$, that is, $v(t)$ always is grater than $u_{m,n}(t)$ whenever they exist. As $v(t)$ blows up at $t=v_0^{-\delta}/\delta$, we have the assertion.

To analysis the asymptotic behavior near the blow-up time, we define the following:

Definition 4.9. *A grid point (x_m,y_n) is called a blow-up point for blowing-up semidiscrete solution if there are sequences $t_k\uparrow T_d$ such that $u_{m,n}(t_k)\to\infty$. We denote the set of all blow-up points, namely the blow-up set, by S_d^*,*

$$S_d^*=\{(x_m,y_n)|\ \exists t_k \text{ s.t. } t_k\to T_d \ \text{ and } \ u_{m,n}(t_k)\to\infty\}.$$

Definition 4.10. *$w_{m,n}(t)$ is a normalized function defined by*

$$w_{m,n}(t) = \frac{u_{m,n}(t)}{M(t)}.$$

Definition 4.11. *S_d is the blow-up core defined by*

$$S_d = \{(x_m, y_n) \mid \exists t_k \text{ and } \exists w_{m,n} \text{ s.t. } t_k \to T_d, \quad w_{m,n}(t_k) \to w_{m,n} \quad \text{and} \quad w_{m,n} > 0\}.$$

Here we call $w_{m,n}$ "limit function."

Remark 4.12. By definition, $S_d \subseteq S_d^*$.

Theorem 4.13. *Suppose that $0 < \delta < 2$. Then, there exists a sequence $t_k \uparrow T_d$ such that*

$$(T - t_k)^{1/\delta} u_{m,n}(t_k) \longrightarrow \begin{cases} z_{m,n} & \text{for } (x_m, y_n) \in S_d, \\ 0 & \text{for } (x_m, y_n) \in S_d^c \end{cases} \tag{27}$$

as $t_k \to T_d$, where $z_{m,n}$ is the solution of the boundary value problem

$$\triangle_d z_{m,n} + \mu z_{m,n} = \frac{1}{\delta} z_{m,n}^{1-\delta}, \qquad (x_m, y_n) \in S_d, \tag{28}$$

$$z_{m,n} = 0, \qquad (x_m, y_n) \in \partial S_d. \tag{29}$$

Proof. Introducing the new variable

$$s = -\log(1 - \frac{t}{T_d}) \ ; \quad [0, T_d) \to (0, \infty),$$

we define

$$\phi_{m,n}(s) = (T_d - t)^{1/\delta} u_{m,n}(t).$$

Theorem 4.3 shows that there exists a positive constant C_1 such that $0 \le \phi_{m,n}(s) \le C_1$. Then $\phi_{m,n}(s)$ solves the initial-boundary value problem (RP1):

$$\frac{d}{ds}\phi_{m,n} = -\frac{1}{\delta}\phi_{m,n} + \phi_{m,n}^{\delta}(\triangle_d \phi_{m,n} + \mu \phi_{m,n}), \quad (x_m, y_n) \in \Omega_h, t > 0, \tag{30}$$

$$\phi_{m,n}(t) = 0, \qquad (x_m, y_n) \in \partial\Omega_h, t > 0, \tag{31}$$

$$\phi_{m,n}(0) = \phi_0(x) = T_d^{1/\delta} u_{0_{m,n}}, \qquad (x_m, y_n) \in \Omega_h. \tag{32}$$

Taking the inner product of the both sides of (30) with $\phi_{m,n}^{-\delta} \frac{d}{ds}\phi_{m,n}$, then we have

$$\begin{aligned} \langle \phi_{m,n}^{-\delta}, (\frac{d}{ds}\phi_{m,n})^2 \rangle &= -\frac{1}{\delta(2-\delta)} \frac{d}{ds} \langle \phi_{m,n}^{2-\delta}, 1 \rangle + \frac{\mu}{2} \frac{d}{ds} \langle \phi_{m,n}, \phi_{m,n} \rangle \\ &- \frac{1}{2}\frac{d}{ds}\Big(\sum_{n=1}^{N-1} \sum_{m=1}^{M} hk(D_{\bar{x}}\phi_{m,n})^2 + \sum_{m=1}^{M-1} \sum_{n=1}^{N} hk(D_{\bar{y}}\phi_{m,n})^2 \Big). \end{aligned}$$

Integrating from 0 to s, we have

$$\begin{aligned} &\int_0^s \langle \phi_{m,n}^{-\delta}(\xi), (\frac{d}{d\xi}\phi_{m,n}(\xi))^2 \rangle d\xi + \frac{1}{\delta(2-\delta)} \langle \phi_{m,n}^{2-\delta}(s), 1 \rangle \\ &\qquad + \frac{1}{2}\Big(\sum_{n=1}^{N-1} \sum_{m=1}^{M} hk(D_{\bar{x}}\phi_{m,n}(s))^2 + \sum_{m=1}^{M-1} \sum_{n=1}^{N} hk(D_{\bar{y}}\phi_{m,n}(s))^2 \Big) \\ &= \frac{\mu}{2} \langle \phi_{m,n}(s), \phi_{m,n}(s) \rangle - \frac{\mu}{2} \langle \phi_{m,n}(0), \phi_{m,n}(0) \rangle + \frac{1}{\delta(2-\delta)} \langle \phi_{m,n}^{2-\delta}(0), 1 \rangle \end{aligned}$$

$$+\frac{1}{2}\Big(\sum_{n=1}^{N-1}\sum_{m=1}^{M} hk(D_{\bar{x}}\phi_{m,n}(0))^2+\sum_{m=1}^{M-1}\sum_{n=1}^{N} hk(D_{\bar{y}}\phi_{m,n}(0))^2\Big)$$
$$\le \exists C_2<\infty,\quad \forall s\ge 0,$$

since $\phi_{m,n}(s)\le C_1$ and $\phi_{m,n}(0)$ is bounded. Hence we obtain

$$\begin{aligned}\int_0^\infty \sum_{n=1}^{N-1}\sum_{m=1}^{M-1}\phi_{m,n}^{-\delta}(s)\Big(\frac{d}{ds}\phi_{m,n}(s)\Big)^2 ds &= \Big(\frac{2}{2-\delta}\Big)^2\sum_{n=1}^{N-1}\sum_{m=1}^{M-1}\int_0^\infty\Big(\frac{d}{ds}\phi_{m,n}(s)^{1-\frac{\delta}{2}}\Big)^2 ds\\ &< +\infty. \end{aligned}\tag{33}$$

Taking any sequence $\{s_j\}\subset \mathbb{R}_+$ such that $s_j\nearrow\infty$ as $j\to\infty$, we put

$$\phi_{m,n}^{(j)}=\phi_{m,n}(s_j).$$

Since $\{\phi_{m,n}^{(j)}\}$ is bounded, there exist a subsequence $\{j'\}\subset\{j\}$ and $\{\psi_{m,n}\}$ such that

$$\phi_{m,n}^{(j')}\longrightarrow\psi_{m,n}\quad\text{as } j'\to\infty \text{ for each } m,n.$$

In the view of (33) we may assume that

$$\frac{d}{ds}\phi_{m,n}^{1-\frac{\delta}{2}}(s_j')=\frac{2-\delta}{2}\phi_{m,n}^{-\frac{\delta}{2}}(s_j')\frac{d}{ds}\phi_{m,n}(s_j')\longrightarrow 0\quad\text{as}\quad s_j'\to\infty.$$

Notice that

$$\phi_{m,n}^{-\frac{\delta}{2}}\frac{d}{ds}\phi_{m,n}=-\frac{1}{\delta}\phi_{m,n}^{1-\frac{\delta}{2}}+\phi_{m,n}^{\frac{\delta}{2}}(\triangle_d\phi_{m,n}+\mu\phi_{m,n}).$$

Taking $s=s_{j'}$ and letting $j'\to\infty$, we have

$$\psi_{m,n}^{\frac{\delta}{2}}(\triangle_d\psi_{m,n}+\mu\psi_{m,n})=\frac{1}{\delta}\psi_{m,n}^{1-\frac{\delta}{2}}.$$

Since $\psi_{m,n}>0$ on the blow-up core S_d, we obtain

$$\triangle_d\psi_{m,n}+\mu\psi_{m,n}=\frac{1}{\delta}\psi_{m,n}^{1-\delta}\quad\text{in } S_d.\tag{34}$$

On the other hand, for $(x_m,y_n)\in S_d^c$, the rate of $u_{m,n}(t)$ is slower than the rate of $M(t)$ if (x_m,y_n) is blow-up point, or $u_{m,n}(t)$ remains bounded as $t\to T_d$. When $0<\delta<2$, there is the estimate (22). Thus, we have

$$(T_d-t)^{1/\delta}u_{m,n}(t)\longrightarrow 0\quad\text{as } j'\to\infty$$

for $(x_m,y_n)\in S_d^c$. Hence we have the assertion.

Now, S_d may be irregular, but we can express S_d as the following direct sum decomposition:

$$S_d=\bigcup_{k=0}^{N_0}S_d^{(k)},$$

where $S_d^{(k)}$ are connected subdomain of S_d and disconnected each other.

Theorem 4.14. *Under the assumption (A1), we have*

$$\lambda_d^{(S_d^{(k)})}\le\mu.$$

Proof. Take any sequence $t_j \uparrow T_d$ and let $w_{m,n}^{(j)} = w_{m,n}(t_j)$. Note that $0 < w_{m,n}^{(j)} \leq 1$ in Ω_h and $\triangle_d w_{m,n}^{(j)} + \mu w_{m,n}^{(j)} \geq 0$ in Ω_h. Then there exist a subsequence $\{j'\}$ and $w_{m,n}$ such that $w_{m,n}^{(j)} \to w_{m,n}$ as $j' \to \infty$. Consequently, we have $0 \leq w_{m,n} \leq 1$ and

$$\triangle_d w_{m,n} + \mu w_{m,n} \geq 0.$$

Taking the inner product with $\Phi_{m,n}^{(S_d^{(k)})}$ in $S_d^{(k)}$, we get

$$\langle \triangle_d w_{m,n} + \mu w_{m,n}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}} \geq 0,$$

from which we deduce

$$(\mu - \lambda_d^{(S_d^{(k)})}) \langle w_{m,n}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}} \geq 0.$$

Since $w_{m,n}$ is strictly positive in blow-up core and $\Phi_{m,n}^{(S_d^{(k)})}$ is also strictly positive in $S_d^{(k)}$, then we have

$$\mu \geq \lambda_d^{(S_d^{(k)})}.$$

Theorem 4.15. *Suppose that* $0 < \delta < 2$. *Then we have*

$$\lambda_d^{(S_d^{(k)})} < \mu.$$

Proof. Taking the inner product of the both side of (34) with $\Phi_{m,n}^{(S_d^{(k)})}$ in $S_d^{(k)}$, we have

$$\langle \triangle_d \psi_{m,n} + \mu \psi_{m,n}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}} = \frac{1}{\delta} \langle \psi_{m,n}^{1-\delta}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}} > 0.$$

Then we have

$$(\mu - \lambda_d^{(S_d^{(k)})}) \langle \psi_{m,n}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}} > 0.$$

Since

$$\langle \psi_{m,n}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}} > 0,$$

thus we get

$$\mu > \lambda_d^{(S_d^{(k)})}.$$

Theorem 4.16. *Assume that* $\delta \geq 2$ *and (A1). If Type II blow-up occurs, then there exist a sequence* $t_j \uparrow T_d$ *such that*

$$\lambda_d(S_d^{(k)}) = \mu$$

and

$$w_{m,n}(t_j) \longrightarrow \begin{cases} z_{m,n} & \text{for } (x_m, y_n) \in S_d, \\ 0 & \text{for } (x_m, y_n) \in S_d^c \end{cases} \tag{35}$$

as $t_j \to T_d$, *where* $z_{m,n}$ *is the solution of the boundary value problem:*

$$\triangle_d z_{m,n} + \mu z_{m,n} = 0, \qquad (x_m, y_n) \in S_d, \tag{36}$$

$$z_{m,n} = 0, \qquad (x_m, y_n) \in \partial S_d, \tag{37}$$

and satisfies $\max_{m,n} z_{m,n} = 1$.

Proof. Let

$$\varphi^{(\lambda)}_{m,n}(s) = \lambda^{1/\delta} u_{m,n}(t), \quad 0 < t = \lambda s + t^* < T_d, \tag{38}$$

then we have

$$\begin{aligned} \frac{d}{ds}\varphi^{(\lambda)}_{m,n}(s) &= \lambda^{\frac{1}{\delta}+1}\frac{d}{dt}u_{m,n}(\lambda s + t^*) \\ &= \lambda^{\frac{1}{\delta}+1}u^{\delta}_{m,n}(\triangle_d u_{m,n} + \mu u_{m,n}) \\ &= (\lambda^{\frac{1}{\delta}}u_{m,n})^{\delta}(\triangle_d(\lambda^{\frac{1}{\delta}}u_{m,n}) + \lambda^{\frac{1}{\delta}}\mu u_{m,n}) \\ &= (\varphi^{(\lambda)}_{m,n})^{\delta}(\triangle_d\varphi^{(\lambda)}_{m,n} + \mu\varphi^{(\lambda)}_{m,n}), \qquad -\frac{t^*}{\lambda} < s < \frac{T_d - t^*}{\lambda}. \end{aligned}$$

Therefore, $\varphi^{(\lambda)}_{m,n}$ solves the initial boundary value problem

$$(\mathrm{P}1_\lambda) \quad \begin{cases} \dfrac{d}{ds}\varphi^{(\lambda)}_{m,n}(s) = (\varphi^{(\lambda)}_{m,n})^{\delta}(\triangle_d\varphi^{(\lambda)}_{m,n} + \mu\varphi^{(\lambda)}_{m,n}), & (x_m, y_n) \in \Omega_h,\ s > 0, \quad (39)\\ \varphi^{(\lambda)}_{m,n}(s) = 0, & (x_m, y_n) \in \partial\Omega_h,\ s > 0 \quad (40)\\ \varphi^{(\lambda)}_{m,n}(0) = \lambda^{1/\delta}u_{m,n}(t^*), & (x_m, y_n) \in \Omega_h, \quad (41) \end{cases}$$

and exists for all $s \in [0, (T_d - t^*)/\lambda)$.

Take any sequence $t^*_j \uparrow T_d$ as $j \to \infty$ and let $\lambda_j^{1/\delta} = 1/M(t^*_j)$. Notice that there exist $\hat{m}$ and $\hat{n}$ such that $\varphi^{(\lambda_j)}_{\hat{m},\hat{n}}(0) = 1$,

$$\varphi^{(\lambda_j)}_{m,n}(0) = w_{m,n}(t^*_j),$$

$$0 < \varphi^{(\lambda_j)}_{m,n}(0) \le 1,$$

and

$$\triangle_d\varphi^{(\lambda_j)}_{m,n}(0) + \mu\varphi^{(\lambda_j)}_{m,n}(0) \ge 0.$$

Since $\varphi^{(\lambda_j)}_{m,n}(0)$ is bounded, there exist a subsequence $\{j\}$ (we also denote $\{j\}$.) and $\varphi_{m,n}$ such that

$$\varphi^{(\lambda_j)}_{m,n}(0) \to \varphi_{m,n} \quad \text{as } j \to \infty \text{ for each } m, n.$$

Notice that

$$0 \le \varphi_{m,n} \le 1, \tag{42}$$

$$\max_{m,n}\varphi_{m,n} = 1, \tag{43}$$

and

$$\triangle_d\varphi_{m,n} + \mu\varphi_{m,n} \ge 0. \tag{44}$$

By the definition of blow-up core, it may be given by

$$S_d = \{(x_m, y_n) \in \Omega_h \ \mid \varphi_{m,n} > 0\}.$$

and it is decomposed as the following:

$$S_d = \bigcup_{k=0}^{N_0} S_d^{(k)},$$

where $S_d^{(k)}$ are connected subdomain of S_d and disconnected each other.

By theorem 4.14, we already have $\lambda_d^{(S_d^{(k)})} \le \mu$. Now we assume that $\lambda_d^{(S_d^{(k)})} < \mu$. By theorem 3.8, the solution of $(\mathrm{P1}_{\lambda_j})$ blows up in a finite time T_{d_j}, which is bounded by

$$T_{d_j} \le \frac{1}{\delta(\mu - \lambda_d^{(S_d^{(k)})})} \langle \varphi_{m,n}^{(\lambda_j)}(0)^{1-\delta}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}}^{\frac{\delta}{\delta-1}} \equiv J_j.$$

Since the limit function $\varphi_{m,n}$ is strictly positive in S_d, thus there exists $J_\infty < +\infty$ such that

$$J_\infty \equiv \lim_{j\to\infty} J_j.$$

Then, for any $\varepsilon > 0$ there exists $j_1 \in \mathbb{N}$ such that $J_j \le J_\infty + \varepsilon$ for $j \ge j_1$. Namely, if $j \ge j_1$, the solution $\varphi_{m,n}^{(\lambda_j)}(s)$ does not exist for $s > J_\infty + \varepsilon$.

On the other hand, the solution of $(\mathrm{P1}_{\lambda_j})$ exists for all $s \in [0, S_j)$, where $S_j = (T_d - t_j^*)/\lambda_j$.

Since $u_{m,n}(t)$ is Type II blowing-up solution, we have

$$S_j \to \infty \quad \text{as} \quad j \to \infty.$$

Moreover, there exists subsequence $\{j\}$ (also denote $\{j\}$.) such that $S_j \uparrow \infty$ as $j \to \infty$. Then, there exists $j_2 \in \mathbb{N}$ such that $S_j > J_\infty + \varepsilon$ for $j \ge j_2$, which leads to the contradiction. Then we have $\lambda_d^{(S_d^{(k)})} = \mu$.

Next, we prove $\triangle_d \varphi_{m,n} + \mu\varphi_{m,n} = 0$. From (44), we already have $\triangle_d \varphi_{m,n} + \mu\varphi_{m,n} \ge 0$. Assume that $\triangle_d \varphi_{m,n} + \mu\varphi_{m,n} \not\equiv 0$ in $S_d^{(k)}$. Multiplying by $\Phi_{m,n}^{(S_d^{(k)})}$, we have

$$\langle \triangle_d \varphi_{m,n} + \mu\varphi_{m,n}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}} > 0,$$

since $\triangle_d \varphi_{m,n} + \mu\varphi_{m,n} \ge 0, \not\equiv 0$ and $\Phi_{m,n}^{(S_d^{(k)})} > 0$i n $S_d^{(k)}$.

Then we have

$$(\mu - \lambda_d^{(S_d^{(k)})}) \langle \varphi_{m,n}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}} > 0.$$

Note that $\langle \varphi_{m,n}, \Phi_{m,n}^{(S_d^{(k)})} \rangle_{S_d^{(k)}} > 0$, thus, we deduce

$$\mu - \lambda_d^{(S_d^{(k)})} > 0.$$

This is a contradiction and we get

$$\triangle_d \varphi_{m,n} + \mu\varphi_{m,n} \equiv 0.$$

Dedicated to Professor Hideo Kawarada on his 60th birthday

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