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ON THE ERROR ESTIMATE OF LINEAR FINITE ELEMENT APPROXIMATION TO THE ELASTIC CONTACT PROBLEM WITH CURVED CONTACT BOUNDARY*1)

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Abstract

In this paper, the linear finite element approximation to the elastic contact problem with curved contact boundary is considered. The error bound $O(h^{\frac{1}{2}})$ is obtained with requirements of two times continuously differentiable for contact boundary and the usual regular triangulation, while I.Hlavacek et. al. obtained the error bound $O(h^{\frac{3}{4}})$ with requirements of three times continuously differentiable for contact boundary and extra regularities of triangulation (c.f. [2]).

Key words: Contact problem, Finite element approximation.

1. Preliminary

The error estimate of linear finite element approximation to the elastic contact problem with curved contact boundary was considered in [2], in which the authors obtained the error bound of $O(h^{\frac{3}{4}})$ with a much complex proof, requirement of three times continuously differentiable for contact boundary and extra regularities of triangulation (c. f. [2, Theorem 3.3, p.149]). In this paper, we obtained the error bound of $O(h^{\frac{1}{2}})$ with only requirement of two times continuously differentiable for contact boundary and the usual regular triangulation (c.f. [1]).

According to the notations in [2], let $\Omega = \Omega' \cup \Omega''$.

$$\mathcal{H}^1(\Omega) = \{v = (v', v'') : v' \in [H^1(\Omega')]^2, v'' \in [H^1(\Omega'')]^2\},$$

$$V = \{v \in \mathcal{H}^1(\Omega) : v' = 0 \ \text{ on } \ \Gamma_u, v''_n = 0 \ \text{ on } \ \Gamma_0\},$$

$$K = \{v \in V : v'_n + v''_n \leq 0 \ \text{ on } \ \Gamma_k\},$$

where $v_n = v_i n_i$ the normal component of the displacement, then the elastic contact problem with curved contact boundary is as follows (c.f.Fig.1):

$$\begin{cases} \text{to find} \qquad u \in K, \quad \text{such that} \\ A(u, v-u) \geq L(v-u) \qquad \forall v \in K, \end{cases} \tag{1.1}$$

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where

$$A(u,v)=\int_{\Omega}\sigma_{ij}(u)e_{ij}(v)dx,$$

$$L(v)=\int_{\Omega}F_iv_idx+\int_{\Gamma_\sigma}P_iv_ids,$$

$$e_{ij}(v)=\frac{1}{2}(\partial_jv_i+\partial_iv_j),i,j=1,2,-\text{the tensor field of strain,}$$

$$\sigma_{ij}=c_{ijkm}e_{km}(v),i,j=1,2,-\text{ the tensor field of stress ditermined by the generalized Hook's Law,}$$

and $c_{ijkm}=c_{jikm}=c_{kmij}$,

$$c_{ijkm}(x)e_{ij}e_{km}\geq c_0e_{ij}e_{ij}, \tag{1.2}$$

holds for all symmetric matrices $(e_{ij})_{1\leq i,j\leq 2}$ and all $x\in\Omega$. It is well known that the equivalent boundary value problem of (1.1) is as follows (c.f.[2]):

$$-\partial_j\sigma_{ij}(u)=F_i,\qquad \text{in }\ \Omega=\Omega'\cup\Omega''; \tag{1.3}$$

$$\begin{cases} u=0 \qquad \text{on }\ \Gamma_u,\\ \sigma_{ij}^M(u)n_j^M=P_i^M, M=','', \qquad \text{on }\ \Gamma_\sigma^M\subset\partial\Omega^M,\\ u_n=0,T_t=0, \qquad \text{on }\ \Gamma_0; \end{cases} \tag{1.4}$$

$$\begin{cases} u_n'+u_n''\leq 0,\quad T_n'=T_n''\leq 0,\\ (u_n'+u_n'')T_n'=0, \qquad \text{on }\ \Gamma_k,\\ T_t'=T_t''=0, \end{cases} \tag{1.5}$$

where $T_n=\sigma_{ij}n_jn_i, T_t=\sigma_{ij}n_jt_i, n^M=(n_1^M,n_2^M)$ and $t^M=(t_1^M,t_2^M)$ are the outer unit normal and the corresponding unit tangential to $\partial\Omega^M$.

Here and what follows a repeated index always means summation over the number 1, 2.

Consider the linear finite element approximation to the problem (1.1). Let $\mathcal{T}_h'$ and $\mathcal{T}_h''$ be the regular triangulations of Ω' and Ω'' with consistency, which means that the node on Γ_k is the common node of $\mathcal{T}_h'$ and $\mathcal{T}_h''$ (c.f. Fig.2). Let V_h be the linear finite element space corresponding to V, which particularly means that $v_h'=0$ on Γ_u and $v_{hn}''=0$ on Γ_0 for $v_h\in V_h$, and

$$K_h=\{v_h\in V_h:(v_{hn}'+v_{hn}'')(P)\leq 0 \qquad \forall\ \text{ nodes }\ P\in\Gamma_k\}, \tag{1.6}$$

then the linear finite element approximation to the problem (1.1) is as follows:

$$\begin{cases} \text{to find} \qquad u_h\in K_h, \qquad \text{such that}\\ A(u_h,v_h-u_h)\geq L(v_h-u_h) \qquad \forall\ \ v_h\in K_h. \end{cases} \tag{1.7}$$

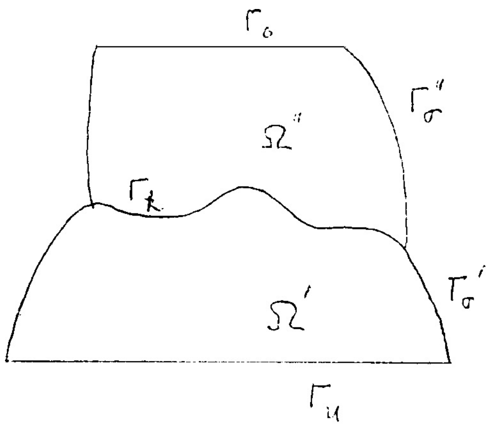

Fig.1

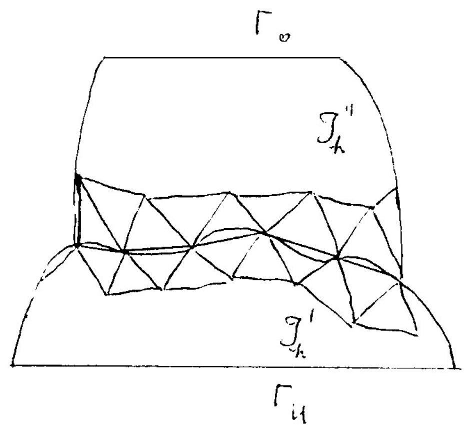

Fig.2

2. Error Estimate

In this section, we present the error estimate of the approximate problem (1.7). First the abstract error estimate is the following (c.f. [2]).

Lemma 1. *Let u and u_h be the solutions of (1.1) and (1.7) respectively, then*

$$|u-u_h|^2 \le C \inf_{v_h \in K_h} \{|u-v_h|^2 + \int_{\Gamma_k} T'_n(u')n'[(v'_h - u'_h) - (v''_h - u''_h)]ds\}, \tag{2.1}$$

where $C = Const. > 0$ independent of u and h, and

$$|v|^2 = A(v,v) \tag{2.2}$$

Noting that v'_h is defined on $\gamma \subset \Gamma_k$ with the natural extension of $v'_h|_{\tau'}$ as follows: let $\widetilde{\tau}'$ be the curved triangle consisted of $\overline{a_1a_2}, \overline{a_1a_3}$ and $\gamma = \widehat{a_2a_3}$, then $v'_h|_{\widetilde{\tau}'} \in P_1(\widetilde{\tau}')$ and $v'_h|_{\tau'}$ remains the original (c.f.Fig.3).

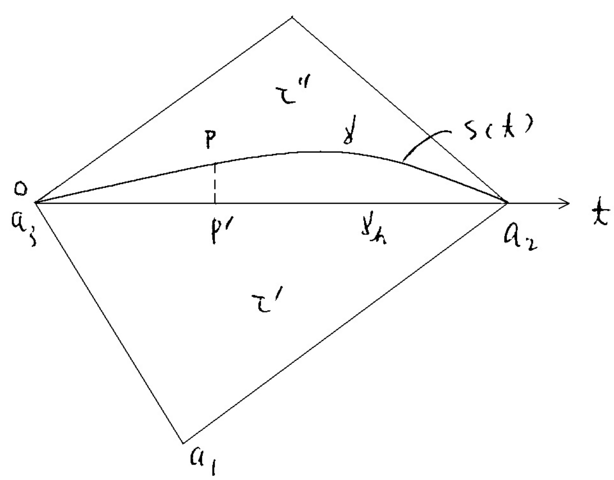

Fig.3

Lemma 2. *(c.f.[3])* $\forall \tau \in \mathcal{T}_h$,

$$\int_{\partial\tau} |w|^2 ds \leq C\{h^{-1}\|w\|^2_{0,\tau} + h|w|^2_{1,\tau}\} \qquad \forall \ \ w \in H^1(\tau). \tag{2.3}$$

We have the following error estimate

Theorem 3. *Assume that the curved contact boundary Γ_k is two times continuously differentiable, the solution $u \in \mathcal{H}^2(\Omega)$ of the problem (1.1), and the triangulation $\mathcal{T}_h$ of $\Omega = \Omega' \cup \Omega''$ is regular and consistence. Then for the approximate problem (1.7), the following error estimate holds*

$$|u - u_h|^2 \leq Ch\|T_n'(u')\|_{0,\Gamma_k}\{|u|_{2,\Omega} + \sum_{i=1}^{2}(\|F_i\|_{0,\Omega} + \|P_i\|_{0,\Gamma_\sigma})\}, \tag{2.4}$$

where $C = Const. > 0$ independent of u and h.

Proof. Let $v_h = \Pi_h u$, the piece wise linear interpolation of u with respect to $\mathcal{T}_h$, then

$$|u - u_h|^2 \leq C\{|u - \Pi_h u|^2 + \int_{\Gamma_k} T_n'(u')n'[(\Pi_h u' - u_h') - (\Pi_h u'' - u_h'')]ds\}. \tag{2.5}$$

By the error estimate of piece wise linear interpolation, we have

$$|u - \Pi_h u|^2 \leq Ch^2|u|^2_{2,\Omega}. \tag{2.6}$$

We now estimate the second term on the right hand side of (2.5) as follows

(i) First we have

$$\begin{aligned} I &= \int_{\Gamma_k} T_n'(u')n'[(\Pi_h u' - u_h') - (\Pi_h u'' - u_h'')]ds \\ &= \int_{\Gamma_k} T_n'(u')n'[(\Pi_h u' - u') - (\Pi_h u'' - u'')]ds \\ &+ \int_{\Gamma_k} T_n'(u')n'[(u' - u_h') - (u'' - u_h'')]ds = I_1 + I_2. \end{aligned} \tag{2.7}$$

Also using the error estimate of piece wise linear interpolation and Lemma 2, we have

$$I_1 \leq Ch^{\frac{3}{2}}\|T_n'(u')\|_{0,\Gamma_k}|u|_{2,\Omega}. \tag{2.8}$$

(ii) By the conditions (1.5), the second term I_2 on the right hand side of (2.7) can be written as follows

$$\begin{aligned} I_2 &= -\int_{\Gamma_k} T_n'(u')n'(u_h' - u_h'')ds \\ &= -\sum_{\gamma\subset\Gamma_k}\int_\gamma T_n'(u')n'(u_h' - u_h'')ds = \sum_{\gamma\subset\Gamma_k} I_{2,\gamma}. \end{aligned} \tag{2.9}$$

Again taking account of the conditions (1.5) and $u_h \in K_h$, we have

$$\begin{aligned} I_{2,\gamma} &= -\int_\gamma T_n'(u')\{n'(u_h' - u_h'') - [n'(u_h' - u_h'')]|_{\gamma_h}\}ds - \int_\gamma T_n'(u')[n'(u_h' - u_h'')]|_{\gamma_h} ds \\ &\leq -\int_\gamma T_n'(u')\{n'(u_h' - u_h'') - [n'(u_h' - u_h'')]|_{\gamma_h}\}ds, \end{aligned} \tag{2.10}$$

where $[n'(u_h' - u_h'')]|_{\gamma_h}$ denotes the restriction of $n'(u_h' - u_h'')$ on $\gamma_h = \overline{a_2 a_3}$ (c.f.Fig.3). Then

$$\begin{aligned} I_{2,\gamma} &\le - \int_\gamma T_n'(u')(n_\gamma' - n_{\gamma_h}')(u_h' - u_h'')ds - \int_\gamma T_n'(u') n_{\gamma_h}' [(u_h' - u_h'')|_\gamma - (u_h' - u_h'')|_{\gamma_h}]ds \\ &= I_{2,\gamma}^1 + I_{2,\gamma}^2. \end{aligned} \tag{2.11}$$

(iii) We now estimate the term $I_{2,\gamma}^1$ (c.f.Fig.3). Let the chord $\widehat{a_3 a_2} = \gamma$ be denoted by $s = s(t), 0 \le t \le \gamma_h$, which is two times continuously differentiable by the assumption of the theorem, and $s(0) = s(\gamma_h) = 0$. Then by the Taylor expansion, it can be seen that

$$|\frac{ds(t)}{dt}| \le Ch, \quad t \in [0, \gamma_h],$$

and from which we have (c.f.Appendix)

$$|n_\gamma' - n_{\gamma_h}'| \le Ch. \tag{2.12}$$

Thus, with use of Lemma 2,

$$I_{2,\gamma}^1 \le Ch \|T_n'(u')\|_{0,\gamma} \|u_h\|_{0,\gamma} \tag{2.13}$$

Next, we estimate the term $I_{2,\gamma}^2$ as follows. We have (c.f.Fig.3)

$$|u_h'(P) - u_h'(P')| \le |\overline{PP'}||\nabla u_h'| \le Ch^2 |\nabla u_h'|, \tag{2.14}$$

and

$$|u_h''(P) - u_h''(P')| \le Ch^2 |\nabla u_h''|, \tag{2.15}$$

then, with use of Lemma 2,

$$\begin{aligned} I_{2,\gamma}^2 &\le Ch^2 \int_\gamma |T_n'(u')(|\nabla u_h'| + |\nabla u_h''|)ds \\ &\le Ch^2 \|T_n'(u')\|_{0,\gamma} (\|\nabla u_h'\|_{0,\gamma} + \|\nabla u''\|_{0,\gamma}) \\ &\le Ch^{\frac{3}{2}} \|T_n'(u')\|_{0,\gamma} (|u_h'|_{1,\tau'}^2 + |u_h''|_{1,\tau''}^2)^{\frac{1}{2}}. \end{aligned} \tag{2.16}$$

From (2.9), (2.11), (2.13) and (2.16), and using the trace theorem, it can be seen that

$$\begin{aligned} I_2 &\le Ch \|T_n'(u')\|_{0,\Gamma_k} (\|u_h\|_{1,\Omega} + \|u_h'\|_{0,\Gamma_k} + \|u_h''\|_{0,\Gamma_k}) \\ &\le Ch \|T_n'(u')\|_{0,\Gamma_k} \|u_h\|_{1,\Omega} \end{aligned} \tag{2.17}$$

(iv) Finally $\|u_h\|_{1,\Omega}$ can be estimated as follows. In (1.7), taking $v_h = 0$ implies that

$$|u_h|^2 \le L(u_h) \le \sum_{i=1}^2 (\|F_i\|_{0,\Omega} + \|P_i\|_{0,\Gamma_\sigma}) \|u_h\|_{1,\Omega}, \tag{2.18}$$

then with use of Korn's inequality and Poincare inequality, it can be seen that

$$\|u_h\|_{1,\Omega} \le C|u_h|, \tag{2.19}$$

from which we have

$$\|u_h\|_{1,\Omega} \le C \sum_{i=1}^{2} \{\|F_i\|_{0,\Omega} + \|P_i\|_{0,\Gamma_\sigma}\}. \tag{2.20}$$

From (2.17) and (2.20), we have

$$I_2 \le Ch\|T_n'(u')\|_{0,\Gamma_k} \sum_{i=1}^{2} \{\|F_i\|_{0,\Omega} + \|P_i\|_{0,\Gamma_\sigma}\}. \tag{2.21}$$

From (2.7), (2.8) and (2.21), the proof is completed.

Appendix. Noting the Fig.3, it can be seen that

$$\overrightarrow{n}|_\gamma = (-s'(t), 1)/\Delta, \qquad \overrightarrow{n}|_{\gamma_h} = (0, 1),$$

then

$$\|\overrightarrow{n}|_\gamma - \overrightarrow{n}|_{\gamma_h}\|^2 = 2\frac{\Delta - 1}{\Delta},$$

where

$$\Delta = \sqrt{s'(t)^2 + 1}.$$

Using the inequality (2.12), we can easily obtain that

$$\|\overrightarrow{n}|_\gamma - \overrightarrow{n}|_{\gamma_h}\|^2 \le Ch^2.$$

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