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SUPERCONVERGENCE ANALYSIS FOR CUBIC TRIANGULAR ELEMENT OF THE FINITE ELEMENT*[1)]

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Abstract

In this paper, we construct a projection interpolation for cubic triangular element by using othogonal expansion triangular method. We show two fundamental formulas of estimation on a special partion and obtain a superconvergence result of $1-\epsilon$ order higher for the placement function and its tangential derivative on the third order Lobatto points and Gauss points on each edge of triangular element.

Key words: Finite element, Superconvergence, Projection interpolation.

1. Introduction

Although we had proved the superconvergence of quadratic triangular elements before 1985, the superconvergence research of $k(k \geq 3)$-degree triangular elements only has a few advances, e.g.,.Lin , Yan and Zhou (see [15]) prove that the three degree Hermite elements possess superconvergence and Wahlbin (see [5-7]) obtains a rough result by using a fine interior estimation, that is, the placement function or its gradient may have weak superconvergence in the local symmetric points,e.g. the middle points of each edge of a element. In 1989, we have pointed out that Li Bo's example in [8] can be explained that it is difficult to show the superconvergence of higher degree element by the traditional interpolation expansion (see [10]), but it did not show that there is no superconvergence for the higher degree element and we confirm that there is superconvergence for $k(k \geq 3)$-degree triangular elements. In this paper, using orthogonal expansion of triangular element, we construct an projection interpolation for cubic triangular element. After two fundamental formulas of estimation on a special partion are shown, some superconvergence results of the placement functions and their tangential derivatives at the third order Lobatto points and Gauss points on each edge of triangular elements are proved. Moreover we will study the superconvergence of the complicated problem of the finite element method.

For the sake of convenience,we consider the model problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \ \ \forall v \in H_0^1(\Omega),$$

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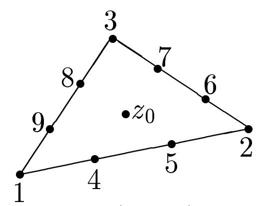
1)The work supported by the Foundation of Natonal Natural Science Of China and the Foundation of Aducation of Hunan Province. The similar results are obtained by Chinese method by Chen[14].

where $a(u,v)=\int_\Omega \nabla u\nabla v dxdy$, $(f,v)=\int_\Omega fvdxdy$, Ω is a smooth or convex polygonal domain . We suppose that Ω is divided into uniform triangles (or local uniform triangles if we consider local superconvergence).Let J^h denote the triangulation, S^h denote the piecewise cubic finite element space over the triangulation J^h. For each triangular element $e\in J^h, e=\triangle z_1z_2z_3$, let $s_i(i=1,2,3)$ denote the opposite side of the vertex z_i of e. h_i is the lenth of the edge s_i $(s_{i+3}=s_i)$, h is the maximum of all lenthes of edges.Let s_i and n_i denote the direction and nomal direction of the edge, respectively and the corresponding directional derivatives are denoted by ∂_i and ∂_{n_i}. $P_k(e)$ and λ_i denote all the k-degree polynomials and area coodinates on element e respectively. $z_i(i=1,\cdots,9)$ are the points of trisection of three edges of element e and z_0 is the barycenter of e (see Figure 1). For all z_i, we may construct basis function $\phi_i\in P_3(e)$ such that $\phi_i(z_j)=\delta_{ij}$.



(Fig.1)

2. Projection Interpolation

Now we construct a new type of interpolation, i.e. projection interpolation like Lin and Zhu in [4] in 1994. It is constructed by projection of a function on polynomial space according to some fashion. Consider firstly the problem of one dimension. Let

$$s=(x_0-h,x_0+h)=(\alpha,\beta).$$

We construct following complete normalizing orthogonal system of polynomials $\{L_n(x)\}$ in the sense of inner product of $L^2(s)$:

$$\begin{aligned}
&L_0(x)=\sqrt{\tfrac{1}{2}}h^{-1/2}\\
&L_1(x)=\sqrt{\tfrac{3}{2}}h^{-3/2}(x-x_0)\\
&L_2(x)=\sqrt{\tfrac{5}{2}}h^{-5/2}[3(x-x_0)^2-h^2]\\
&\cdots\cdots\cdots\cdots\\
&L_i(x)=\sigma_i(\tfrac{d}{dx})^i[A(x)]^i,\ (i\geq 1),\\
&\sigma_i=\sqrt{\tfrac{(2i+1)}{2}}\tfrac{1}{i!}h^{-i-1/2}=O(h^{-i-1/2})\\
&\cdots\cdots\cdots\cdots
\end{aligned}\tag{2.1}$$

where $A(x) = \frac{1}{2}[(x-x_0)^2 - h^2]$. We further construct their primitive function like

$$\begin{aligned}
&\omega_0 = 1,\\
&\omega_1 = \int_{I_x} L_0 dx,\\
&\omega_2 = \int_{I_x} L_1(x) dx,\\
&\cdots\cdots\cdots\cdots\\
&\omega_{i+1} = \int_{I_x} L_i(x) dx = \sigma_i (\tfrac{d}{dx})^{i-1}[A(x)]^i,\ (i \geq 1)\\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{2.2}$$

where $I_x = (x_0 - h, x)$. It is easy to prove

Proposition 1. *$\{L_i(x)\}$ and $\{\omega_i(x)\}$ have following properties:*
1) $|L_i(x)| \leq Ch_e^{-1/2},\ (L_i, L_j) = \delta_{ij},\ |\omega_i(x)| \leq Ch_e^{1/2}$;
2) $\omega_i(x_0 \pm h) = 0,\ i \geq 2$;
3) $\omega_i(x_0 - (x - x_0)) = (-1)^i \omega_i(x_0 + (x - x_0))$;
4) $i \geq 3$, $(\omega_i, p_{i-3}) = 0$, $\forall p_{i-3} \in P_{i-3}(s)$;
5) *if* $i, j \geq 0$ *and* $i + j$ *is a odd number, then* $(\omega_i, \omega_j) = 0$;
where $P_m(s)$ is the set of all m-th polynomials on s and

$$(p, q) = \int_s pq dx.$$

Assume that $u \in H^1(s)$, then u' can be expanded in Fourier series:

$$u' = \sum_{i=0}^{\infty} \alpha_j L_j(x) = \alpha_0 L_0(x) + \alpha_1 L_1(x) + \cdots + \alpha_j L_j(x) + \cdots,\ \ \alpha_j = (u', L_j) \tag{2.3}$$

Using Parserval equality,we have(see Section 1.4 of Chapter 1 in [4]).

Proposition 2. *If $u \in H^1(s)$, then we have expansion in the sence of pointwise*

$$u(x) = \sum_j \beta_j \omega_j(x)$$

where $\beta_0 = u(x_0 - h),\ \beta_1 = \sqrt{1/2} h^{-1/2}[u(x_0 + h) - u(x_0 - h)],\ \beta_j = \alpha_{j-1} = (u', L_{j-1})$, $(j \geq 2)$. *Further more, if $u \in W^{k+1,\infty}(s)$, then*

$$\beta_{k+1} = O(h_e^{k+1-1/2}) \tag{2.4}$$

Now we construct so called projection interpolation:

$$\Pi_k u = \beta_0 + \beta_1 \omega_1(x) + \beta_2 \omega_2(x) + \cdots + \beta_k \omega_k(x).$$

Proposition 3. *The operator Π_k has following properties:*
1) $\Pi_k^2 = \Pi_k$, *and Π_k is a operator from H^1 onto $P_k(s)$.*
2) *if $u \in W^{k+1,p}$ then*

$$\|u - \Pi_k u\|_{m,p,s} = O(h^{k+1-m}) \|u\|_{k+1,p,s}. \tag{2.5}$$

3. Projection Interpolation on Triangular Element

Let $e = \triangle z_1 z_2 z_3$ be any triangular element , the nodes z_1, z_2, z_3 are in order of positive direction (opposite direction of hands of clock).s_i is the edge opposite to z_i (let $s_{i+3} = s_i$). Then we can define a projection interpolation on each edge s_i according to the fashion in Section 2:

$$w_i^{(k)}(s) = \beta_{i0} + \beta_{i1}\omega_{i1}(s) + \beta_{i2}\omega_{i2}(s) + \cdots + \beta_{ik}\omega_{ik}(s), (k \geq 3) \tag{3.1}$$

where s is parameter of aec length of s_i,w_i satisfies

$$\begin{array}{l} w_i^{(k)}(z_j) = u(z_j), (j = i+1, i+2) \\ w_i^{(k)}(z_0) = u(z_0) + O(h^{k+2}), \forall z_0 \in \{z \in s_i : \omega_{i\,k+1}(z) = 0\}. \end{array} . \tag{3.2}$$

$\{z \in s_i : \omega_{i\,k+1}(z) = 0\}$ is the set of all Lobatto points on s_i.

Let

$$\begin{array}{l} E_k = \lambda_k \phi_0, (k = 1, 2, 3) \\ P_4^0(e) = span(E_1, E_2, E_3) = span(E_1, E_2, E_3, \phi_0), \end{array} \tag{3.3}$$

Let

$$\bar{\omega}_{i4} = \sigma_3 \partial_i^2 (\frac{1}{2}\lambda_{i+1}\lambda_{i+2})^3 + \sum_{k=1}^{3} a_{ik} E_k. \quad (i = 1, 2, 3)$$

which satisfies

$$\begin{array}{l} a(\bar{\omega}_{i4}, \phi_0)_e = 0 \\ (\bar{\omega}_{i4}, 1)_e = 0 \end{array} \tag{3.4}$$

It is quite obvious that

$$\bar{\omega}_{i4} \in P_4(e), \forall e \in J^h, \qquad \bar{\omega}_{i4}|_{s_i} = \omega_{i4}, \quad \bar{\omega}_{i4}|s_k = 0, k \neq i. \qquad (3.5).$$

Let

$$l_k = E_k + \delta_{k1} E_3 + \delta_{k2} \phi_0, \; (k = 1, 2)$$

which satisfies

$$\begin{array}{l} a(l_k, \phi_0)_e = 0 \\ (l_k, 1)_e = 0 \end{array} \tag{3.6}$$

Suppose that the folowing condition holds[2]:

(A): $\{\hat{l}_1, \hat{l}_2\}$ is linear independent

where simbol $\hat{l}$ means

$$\hat{l} = \sum_{i+j=4} c_{ij} x^i y^j, \quad if \quad l = \sum_{i+j \leq 4} c_{ij} x^i y^j$$

The function $W \in S_0^h(\Omega)$ is said to be projection interpolation of 3-th order of u,if it satisfies

1)$W|_{s_i} = w_i^{(3)}, i = 1, 2, 3, \forall e \in J^h$;

2)$a(u - W, \phi_0)_e = 0$,$\forall e \in J^h$.

obviously,there exists an unique projection interpolation $W \in S_0^h(\Omega)$ for $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

2 At least, condition (A) holds for equilateral triangle element

Lemma 1. *The projection interpolation W has following properties:*

1) $W|_{\partial e}$ *is the projection interpolation of 3-th order of* $u|_{\partial e}, W \in C(\bar{\Omega})$.

2) W *is interpolation of 3-th degree of* u *in the finite element space* $S^h(\Omega)$*, and there are estimations:*

$$\|u - W\|_{m,p,e} \leq Ch^{4-m}\|u\|_{4,p,e}, (m = 0, 1), \forall e \in J^h. \tag{3.7}$$

3) *The tangential derivative of* $u - W$ *has higher order estimations* $O(h^4)$ *on Gauss points* $\{G\}$ *of 3-th order on side* s_i *of element* e *and* $O(h^5)$ *estimates on Lobatto points* $\{z_0\}$ *of the 3-th order ,i.e*

$$\begin{array}{l} \partial_\tau(u - W)(G) = O(h^4)\|u\|_{5,\infty,e} \\ (u - W)(z_0) = O(h^5)\|u\|_{5,\infty,e} \end{array} . \tag{3.8}$$

Lemma 2. *Let* $e = \triangle z_1z_2z_3$ *be a triangular element, then* $\forall v \in P_3(e)$*, there is a* $v^* \in P^3(e)$ *such that*

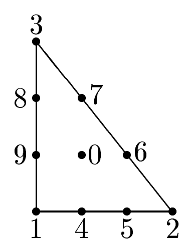
$$v^*|_{\partial e} = v|_{\partial e} \tag{3.9}$$

and the derivative $\partial_i^2\partial_{n_i}v^*$ *of the third order on each side of element* e *can be represented by linear combination of the difference quotient* $\delta_j^3 v$ *(or* $\partial_j^3 v$*) of the third order on the three sides of element* e*, i.e.*

$$\partial_i^2\partial_{n_i}v^* = c_1\delta_j^3 v + c_2\delta_j^3 v + c_3\delta_j^3 v \tag{3.10}$$

where c_i *is independent of* v.

Proof. Because each $\partial_i^2\partial_{n_i}$ is a linear representation of $\partial_3^3, \partial_2^3, \partial_3^2\partial_2, \partial_2^2\partial_3$, we only need to prove that $\partial_3^2\partial_2, \partial_2^2\partial_3$ satisfy the properties. We might as well let e be a right triangle (otherwise using an affine transformation) and z_1 be the right angle vertex and $s_2 = s_3 = 1$(see Fig.2). Then we only need to prove that $\partial_x^2\partial_y, \partial_y^2\partial_x$ satisfy the properties. Let



(Fig.2.)

$$\begin{array}{rl} v^* = & v(z_1)\psi_1 + v(z_2)\psi_2 + v(z_3)\psi_3 + v(z_4)\psi_4 + v(z_5)\psi_5 \\ & +v(z_6)\psi_6 + v(z_7)\psi_7 + v(z_8)\psi_8 + v(z_9)\psi_9 \end{array}$$

where

$$\begin{array}{ll} \psi_1 = -\phi_0/3 + \phi_1 & \phi_0 = -6cxy(x+y-1) \quad c = 9/2 \\ & \phi_1 = -c(x+y-1)(x+y-1/3)(x+y-2/3) \\ \psi_2 = -\phi_0/6 + \phi_2 & \phi_2 = cx(x-1/3)(x-2/3) \\ \psi_3 = \phi_3 & \phi_3 = cy(y-1/3)(y-2/3) \\ \psi_4 = \phi_4 & \phi_4 = 3cx(x+y-1)(x-2/3) \\ \psi_5 = \phi_5 & \phi_5 = -3cx(x+y-1)(x-1/3) \\ \psi_6 = \phi_6 & \phi_6 = -3cxy(y-2/3) \\ \psi_7 = \phi_7 & \phi_7 = 3cxy(y-1/3) \\ \psi_8 = \phi_8 & \phi_8 = -3cy(x+y-1)(y-1/3) \\ \psi_9 = \phi_9 & \phi_9 = 3cy(x+y-1)(y-2/3) \end{array}$$

It is easy to prove (3.9) and we have

$$\partial_x^2 \partial_y v^* = \delta_3^3 v \times 9, \quad \partial_y^2 \partial_x v^* = -(2^{3/2}\delta_1^3 v + \delta_2^3 v) \times 9$$

The proof is completed.

Let $W_4 = \sum_{i=1}^{3} \beta_{i4}\bar{\omega}_{i4}$ .If condition (A) holds, then it is easy to prove that there is a function $Q_4 = \beta_1 l_1 + \beta_2 l_2 \in P_4^0(e)$ such that

$$u - W - W_4 - Q_4 = 0, \forall u \in P_4(e), \forall e \in J^h$$

and

$$\begin{array}{l} |\beta_i(u)| \le Ch^4 |u|_{4,\infty,e} \\ |\beta_i(u)|_e - \beta_i(u)|_{e'}| \le Ch^5 |u|_{5,\infty,e} \end{array} \tag{3.11}$$

for any neibourhood $e, e' \in J^h$. Then we have:

Lemma 3. *Let $W \in S^h(\Omega)$ be the third projection interpolation of u,if condition (A) is valid then we have following decomposition formula:*

$$\begin{array}{l} u - W = \sum_{i=1}^{3} \beta_{i4}\bar{\omega}_{i4} + Q_4 + r_5. \\ h\|Q_4\|_{m,p,e} + \|r_5\|_{m,p,e} = O(h^{5-m})\|u\|_{5,p,e}, \\ (2 \le p \le \infty, m = 0, 1), \quad \forall e \in J^h. \end{array} \tag{3.12}$$

where $Q_4 = \beta_1 l_1 + \beta_2 l_2 \in P_4^0(e)$,$\beta_i = \beta_i(u)$ is a linear functional on $W^{5,p}(e)$,which satisfies formula (3.11)

For each $e \in J^h$,let

$$-\Delta v^* = \alpha_0 + \alpha_1 \mu_1 + \alpha_2 \mu_2,$$

where $\mu_1 = h^{-1}(x - x_0), \mu_2 = h^{-1}(y - y_0), (x_0, y_0) = z_0 \in e$. Using lemma 2,$\alpha_i, i = 1, 2$ can be represented by linear combination $h_j \partial_j^3 v, (j = 1, 2, 3)$.Notice that

$$\partial_j^3 v|_e = -\partial_j^3 v|_{e'}, \tag{3.13}$$

for any neibourhood $e, e' \in J^h$.

Since $S_0^h(\Omega)$ are defined on uniform meshes, due to (3.6),(3.11), (3.13), and $Q_4 \in P_4^0(e)$, we have

$$
\begin{aligned}
a(Q_4,v^*) &= \sum_{e\in J^h}(Q_4,-\Delta v^*)_e = \sum_{e\in J^h}(Q_4,\alpha_1\mu_1+\alpha_2\mu_2)_e \\
&= \sum_{e\in J^h}\sum_{j=1,2,3}\sum_{\bar e\cap\bar e'=s_j}\sum_{i=1,2} c_{ij}h_j\partial_j^3 v(\beta_i|_e-\beta_i|_{e'})mes(e) \\
&= O(h^6)\|u\|_{5,p}\|v\|'_{3,q}, \quad \forall v\in S_0^h(\Omega) \quad \tfrac{1}{p}+\tfrac{1}{q}=1,
\end{aligned}
$$

where $c_{ij}=const.$ are independent of u,v,h,e.

Similarly,using (3.4) we have

$$
\begin{aligned}
&\sum_{e\in J^h}(W_4,-\Delta v^*)_e \\
&= O(h^6)\|u\|_{5,p}\|v\|'_{3,q}, \quad \forall v\in S_0^h(\Omega) \quad \tfrac{1}{p}+\tfrac{1}{q}=1,
\end{aligned}
$$

Then we have:

Lemma 4. *Under the conditions of lemma 2 and lemma 3,we have:*

$$
\begin{aligned}
&a(Q_4,v^*) = O(h^6)\|u\|_{5,p}\|v\|'_{3,q}, \\
&\sum_{e\in J^h}(W_4,-\Delta v^*)_e = O(h^6)\|u\|_{5,p}\|v\|'_{3,q}, \quad \forall v\in S_0^h(\Omega) \quad \tfrac{1}{p}+\tfrac{1}{q}=1, \ \cdot
\end{aligned}
\tag{3.14}
$$

4. Two Basic Estimations and Their Proofs

Theorem 1. *Suppose that the uniform triangular subdivision of Ω is given and W is the projection interpolation of 3-th order of $u\in W^{5,p}(\Omega)\cap H_0^1(\Omega)$. Then we have two basic estimations as follows:*

$$
\begin{aligned}
&|a(W_4+r_5,v^*)| = O(h^{5-m})\|u\|_{5,p}\|v\|'_{2-m,q}, \quad \forall v\in S^h(\Omega) \\
&\quad 1/p+1/q=1,\ m=0,1.
\end{aligned}
$$

where $\|v\|'^q_{s,q}=\sum_e\|v\|^q_{s,q,e}$,*and* $W_4=\sum_{i=1}^{3}\beta_{i4}\bar\omega_{i4}$*(see section 3)*

Proof. For each element e and any $v\in S_0^h$,using (3.7),(3.14) and (3.12) we obtain

$$
\begin{aligned}
&\sum_{e\in J^h}\int_e(W_4,-\Delta v^*)_e \\
&= O(h^{6-m})\|u\|_{5,p}\|v\|'_{3-m,q}, \quad \forall v\in S_0^h(\Omega) \quad \tfrac{1}{p}+\tfrac{1}{q}=1, \ . \\
&\sum_{e\in J^h}\int_e r_5(-\Delta v^*)dx = O(h^5)|u|_{5,p}\|v\|'_{2,q}.
\end{aligned}
$$

By Lemma 1 and Lemma 2 we have

$$
\begin{aligned}
&\sum_{e\in J^h} a(W_4+r_5,v^*)_e \\
&= \sum_{e\in J^h}\oint_{\partial e}(u-W)\partial_n v^* ds + O(h^{6-m})\|u\|_{5,p}\|v\|'_{3-m,q} \\
&= \sum_{e\in J^h}\sum_{i=1,2,3}\int_{s_i}(u-W)\partial_n v^* ds + O(h^{6-m})\|u\|_{5,p}\|v\|'_{3-m,q}.\ (m=0,1,2)
\end{aligned}
\tag{4.1}
$$

where ∂_n is normal derivative, $v^*\in P_3(e)$ satisfies $v^*|_{\partial e}=v|_{\partial e}$. Because W is the projection interpolation of $u|_{s_i}$ and $\partial_n v^*$ is a polynomial of the second order, due to proposision 1 and 2,we have

$$
\begin{aligned}
I_i &\equiv \int_{s_i}(u-W)\partial_n v^* ds = \beta_4 \int_{s_i} \omega_4 \partial_n v^* ds \\
&= \beta_4\sigma_3 \int_{s_i}[A(s)]^3 \partial_i^2 \partial_n v^* ds \\
&= \partial_i^2\partial_n v^* \beta_4\sigma_3 \int_{s_i} [A(s)]^3 ds = \partial_i^2 \partial_n v^* \beta_4 \gamma_3 \\
\gamma_3 &= \sigma_3 \int_{s_i}[A(s)]^3 ds = O(h^{6-3-1/2+1}) = O(h^{3+1/2}), \\
\beta_4 &= \int_{s_i} \partial_i u L_3(s) ds = \int_{s_i} \partial_i^4 u (-1)^3 \sigma_3 [A(s)]^3 ds
\end{aligned} \tag{4.2}
$$

where ∂_i denotes the tangential derivative along the direction of s_i,γ_3 is a constant independent of e, u and v. Using Lemma 2, $\partial_i^2\partial_n v^*$ can be represented by linear combination of the difference quotient of the third order on three sides of element e:

$$\partial_i^2 \partial_n v^* = a_1\delta_1^3 v + a_2 \delta_2^3 v + a_3 \delta_3^3 v \tag{4.3}$$

where a_1, a_2, a_3 are constants. Therefore we have

$$I_i = \beta_4 \cdot \gamma_3 \cdot (a_1\delta_1^3 v + a_2\delta_2^3 v + a_3 \delta_3^3 v) \tag{4.4}$$

For each $j(j=1,2,3)$,there exists an unique neibourhood element e', such that for the corresponding quantities we obtain

$$\delta_j^3 v' = (-1)^3 \delta_j^3 v, \quad \gamma_3' = \gamma_3. \tag{4.5}$$

Packing the corresponding quantities ,we have

$$
\begin{aligned}
&\beta_4' \cdot \gamma_3 \cdot \delta_j^3 v' + \beta_4 \cdot \gamma_3 \cdot \delta_j^3 v \\
&= [(-1)^3 \beta_4' + \beta_4] \cdot \gamma_3 \delta_j^3 v \\
&= (-1)^3 \{(-1)^3 \int_{s_i} \partial_i^4 u \sigma_3 [A(s)]^3 ds + \int_{s_i} \partial_i^4 u \sigma_3 [A(s)]^3 ds\} \cdot \gamma_3 \delta_j^3 v.
\end{aligned} \tag{4.6}
$$

Because s_i and s_i' are parallel, but their directions are opposite, there is a factor $(-1)^4$ between the signs of the derivatives of the 4-th order along directions of s_i and s_i':

$$\partial_i^4 u' = (-1)^4 \partial_i^4 u, \quad \text{on} \quad s_i' \tag{4.7}$$

Therefore (note (4.2)):

$$
\begin{aligned}
&\beta_4' \cdot \gamma_3 \cdot \delta_j^3 v' + \beta_4 \cdot \gamma_3 \cdot \delta_j^3 v \\
&= (-1)^3 \{(-1)^3 \int_{s_i'} \partial_i^4 u \sigma_3 [A(s)]^3 ds + \int_{s_i} \partial_i^4 u \sigma_3 [A(s)]^3 ds\} \cdot \gamma_3 \delta_j^3 v \\
&= (-1)^3 \int_{e \cup e'} \partial \partial_i^4 u \cdot O(h^{6-3-1/2}) dx dy \cdot \gamma_3 \delta_j^3 v \\
&= O(h^6) |u|_{5,p,e\cup e'} |v|_{3,q,e\cup e'}, \quad (1/p + 1/q = 1).
\end{aligned} \tag{4.8}
$$

By using (4.1)–(4.8) and Hoĺder inequality we have

$$a(W_4 + r_5, v^*) = \sum_e a(W_4 + r_5, v^*)_e = O(h^6)|u|_{5,p}|v|_{3,q}', \quad \forall v \in S^h. \tag{4.9}$$

Then by using inverse estimation, we have the first and second basic estimations. The proof is completed.

By (3.12), (3.14),Lemma2,Theorem 1 and inverse estimation we have

$$
\begin{aligned}
a(u-W, v) &= a(u-W, v^*) = a(W_4 + r_5, v^*) + a(Q_4, v^*) \\
&= O(h^{5-m}) \|u\|_{5,p} \|v\|_{2-m,q}', \quad \forall v \in S_0^h(\Omega) \\
&1/p + 1/q = 1,\ m = 0, 1.
\end{aligned} \tag{4.10}
$$

Note that

$$\partial_\tau(u^h - W)(z) = a(u - W, \partial_\tau G_z^h), (u^h - W)(z) = a(u - W, G_z^h). \tag{4.11}$$

where $u^h \in S_0^h(\Omega)$ is the finite element approximation of u.

It is easy to obtain following result using the methods of$_{[1,10]}$:

Theorem 2. *Under the conditions of Lemma 3 and Theorem 1, if u^h is the finite element solution, then we have following superconvergence estimations:*

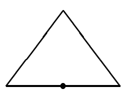
$$\begin{aligned} &\partial_\tau(u^h - W)(z) = O(h^4)\|u\|_{5,\infty}. \\ &(u^h - W)(z) = O(h^5)|\log h|\|u\|_{5,\infty}. \\ &\quad \forall z \in \partial e, \forall e \in J^h \end{aligned}$$

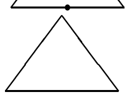
Further more, there are estimations of higher order on the Gauss points $\{G\}$ and the Lobatto points $\{z_0\}$ of 3-th order on each side of element e,(see (3.4)), respectively:

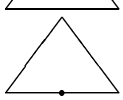
$$\begin{aligned} &\partial_\tau(u - u^h)(G) = (h^4)\|u\|_{5,\infty}, \\ &(u - u^h)(z_0) = O(h^5|\log h|)\|u\|_{5,\infty}. \end{aligned} \tag{4.12}$$

Remark. From the theory of local estimations, if the domain U, which contains the points G or z_0, is covered by the uniform triangulation, then (4.12) is valid as well provided u is sufficient smooth in U.(see $_{[10]}$Ch.5).

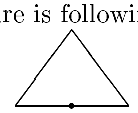
Running through Lagrange triangular element with k=1,2,3, but considering only strictly natural superconvergence for $\partial_x u$,Figure of Wahlbin and etc.[7] is as following:

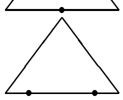
$k = 1$  midpoint of side parallel to $x-$axis

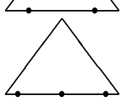
$k = 2$  no superconvergence for $\partial_x u$

$k = 3$  mitpoint of side

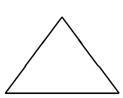
and our figure is following:

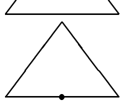
$k = 1$  midpoint of side parallel to $x-$axis

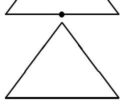
$k = 2$  second order Gauss points

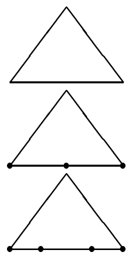
$k = 3$  3-th order Gauss points

Considering only strictly natural superconvergence for u,The figure of Wahlbin and etc. is as following:

$k = 1$  no superconvergence

$k = 2$  mitpoint of side

$k = 3$  No superconvengence for u and our figure is as following:



$k=1$ no superconvergence

$k=2$ second order Lobbato points

$k=3$ 3-th order Lobbato points

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