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A MODIFIED ALGORITHM OF FINDING AN ELEMENT OF CLARKE GENERALIZED GRADIENT FOR A SMOOTH COMPOSITION OF MAX-TYPE FUNCTIONS*1)

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Abstract

This paper refers to Clarke generalized gradient for a smooth composition of max-type functions of the form: $f(x) = g(x, \max_{j\in J_1} f_{1j}(x), \cdots, \max_{j\in J_m} f_{mj}(x))$, where $x \in \mathbf{R}^n$, J_i, $i = 1, \cdots, m$ are finite index sets, g and $f_{ij}, j \in J_i$, $i = 1, \cdots, m$, are continuously differentiable on $\mathbf{R}^{m+n}$ and $\mathbf{R}^n$, respectively. In a previous paper, we proposed an algorithm of finding an element of Clarke generalized gradient for f, at a point. In that paper, finding an element of Clarke generalized gradient for f, at a point, is implemented by determining the compatibilities of systems of linear inequalities many times. So its computational amount is very expensive. In this paper, we will modify the algorithm to reduce the times that the compatibilities of systems of linear inequalities have to be determined.

Key words: Nonsmooth optimization, Clarke generalized gradient, Max-type function.

1. Introduction

The smooth composition of max-type functions plays an important role in nonsmooth optimization. The general form of this kind of functions is :

$$f(x) = g(x, \max_{j\in J_1} f_{1j}(x), \cdots, \max_{j\in J_m} f_{mj}(x)), \tag{1.1}$$

where $x \in \mathbf{R}^n, J_i, i = 1, \cdots, m$ are finite index sets, g and $f_{ij}, j \in J_i$, $i = 1, \cdots, m$ are continuously differentiable on $\mathbf{R}^{m+n}$ and $\mathbf{R}^n$, respectively. Many publications deal with the problem related to minimizing this class of functions, see for instance [3, 8, 9]. However, authors have to restrict themselves to considering particular cases about f, or take f as a quasidifferentiable function, in the sense of Demyanov and Rubinov [4]. For instance, $g(x, y_1, \cdots, y_m)$, where $x \in \mathbf{R}^n$, is supposed to be nondecreasing with respect to each y_i for $i = 1, \cdots, m$ in [8] and f is taken as a quasidifferentiable function in [3, 9]. Until now, no papers on minimizing f in general case by using the technology of Clarke generalized gradient appear. The present situation may be caused from that one has not found a way to obtain an element of Clarke generalized gradient of f.

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The definition of Clarke generalized gradient for a locally Lipschitzian function and its properties see [1, 2].

Gao and Xia [5] proposed two algorithms of finding an element of Clarke generalized gradient of f, at a point. The algorithms proposed in [5] could be taken as subalgorithms embedded in some algorithms of minimizing f, for instance, bundle methods [7, 10], such that these algorithms become implementable ones. However, the computational amount of the algorithms proposed in [5] is very expensive. One has to determine the compatibilities of systems of linear inequalities $\prod_{i=1}^{m} \mathrm{card} J_i(x)$ times to find an element of Clarke generalized gradient of f, at the point x, where card denotes cardinality, the definition of $J_i(x)$ see (1.2) below. It is this reason why one does apply them to minimizing f immediately.

In this paper, we intend to modify the algorithms proposed in [5] to reduce the computational amount. Now, we go back to [5]. Let $\overline{x} \in \mathbf{R}^n$. Denote

$$J_i(\overline{x}) = \{j \in J_i \mid f_{ij}(\overline{x}) = \max_{j \in J_i} f_{ij}(\overline{x})\}, i = 1, \cdots, m. \tag{1.2}$$

Given each set of indices $j_i \in J_i(\overline{x})$ for $i = 1, \cdots, m$, one construct the following system of linear inequalities

$$\mathrm{L}_{j_1 \cdots j_m} \qquad (\nabla f_{it_i}(\overline{x}) - \nabla f_{ij_i}(\overline{x}))^T y < 0, y \in \mathbf{R}^n, \forall t_i \in J_i(\overline{x}) \setminus \{j_i\}, i = 1, \cdots, m.$$

It is easy to see that $\mathrm{L}_{j_1 \cdots j_m}$ is a system of $\sum_{i=1}^{m} (\mathrm{card} J_i(\overline{x}) - 1)$ strictly linear inequalities with n variables. One has the following two theorems.

Theorem1.1.[Th.2,5] *Suppose there exists a set of indices $j_i \in J_i(\overline{x})$ for $i = 1, \cdots, m$, such that the system of linear inequalities $\mathrm{L}_{j_1 \cdots j_m}$ is consistent. Then $\nabla g(x, f_{1j_1}(x), \cdots, f_{mj_m}(x))|_{x=\overline{x}} \in \partial f(\overline{x})$, where $\partial f(\overline{x})$ refers to Clarke generalized gradient of f, at $\overline{x}$.*

Theorem1.2.[Th.3,5] *Suppose $j, k \in J_i(\overline{x}), j \neq k$ implies $\nabla f_{ij}(x) \neq \nabla f_{ik}(\overline{x})$ for $i = 1, \cdots m$. Then there exists at least one set of indices $j_i \in J_i(\overline{x})$ for $i = 1, \cdots, m$ such that its related system of linear inequalities $\mathrm{L}_{j_1 \cdots j_m}$ is consistent.*

In the light of Theorems 1.1 and 1.2, an algorithm of finding an element of Clarke generalized gradient of f, at $\overline{x}$, is constructed in [5]. The algorithm works under the hypothesis in Theorem 1.2. The general outline of the algorithm is: For each set of indices $j_i \in J_i(\overline{x}), i = 1, \cdots, m$, one determines the compatibility of the system of linear inequalities $\mathrm{L}_{j_1 \cdots j_m}$. If $\mathrm{L}_{j_1 \cdots j_m}$ is consistent, calculate $\xi = \nabla g(x, f_{1j_1}(x), \cdots, f_{mj_m}(x))|_{x=\overline{x}}$, which is an element of Clarke generalized gradient for f, at $\overline{x}$.

For dealing with more general case in which hypothesis in Theorem 1.2 may be not satisfied, a modified algorithm that works without the hypothesis: $\nabla f_{ij}(\overline{x}) \neq \nabla f_{ik}(\overline{x}), \forall j, k \in J_i(\overline{x}), j \neq k, i = 1, \cdots, m$, is presented. The general outline of the modified algorithm is: Determine index sets $\overline{J}_i(\overline{x})$ for $i = 1, \cdots, m$ according to the rule below:

$$\overline{J}_i(\overline{x}) \subset J_i(\overline{x}), \;\; i = 1, \cdots, m,$$

$$\nabla f_{ij}(\overline{x}) \neq \nabla f_{ik}(\overline{x}), \forall j, k \in \overline{J}_i(\overline{x}), j \neq k, i = 1, \cdots, m,$$

$$\forall \; t_i \in J_i(\overline{x}), \exists \; k_i \in \overline{J}_i(\overline{x}) \;\text{ such that }\; \nabla f_{it_i}(\overline{x}) = \nabla f_{ik_i}(\overline{x}), i = 1, \cdots, m.$$

Evidently, the index set $\overline{J}_i(\overline{x})$ is a subset of $J_i(\overline{x})$. Particularly, if $J_i(\overline{x})$ satisfies that $\nabla f_{ij}(\overline{x}) \neq \nabla f_{ik}(\overline{x}), \forall j,k \in J_i(\overline{x}), j \neq k$, then $\overline{J}_i(\overline{x}) = J_i(\overline{x})$. Actually, $\overline{J}_i(\overline{x})$ can be determined by reserving only one index among this kind of indices that $j,k \in J_i(\overline{x})$ with $\nabla f_{ij}(\overline{x}) = \nabla f_{ik}(\overline{x})$. Given each set of indices $j_i \in \overline{J}_i(\overline{x})$ for $i = 1, \cdots, m$, we construct a related system of linear inequalities as follows:

$$\overline{\mathrm{L}}_{j_1 \cdots j_m} \qquad (\nabla f_{it_i}(\overline{x}) - \nabla f_{ij_i}(\overline{x}))^T y < 0, y \in \mathbf{R}^n, \forall t_i \in \overline{J}_i(\overline{x}) \setminus \{j_i\}, i = 1, \cdots, m.$$

Then we have the following theorem:

Theorem1.3.[5] *Suppose there exists a set of indices $j_i \in \overline{J}_i(\overline{x})$ for $i = 1, \cdots, m$, such that the system of linear inequalities $\overline{\mathrm{L}}_{j_1 \cdots j_m}$ is consistent. Then $\nabla g(x, f_{1j_1}(x), \cdots, f_{mj_m}(x))|_{x=\overline{x}} \in \partial f(\overline{x})$.*

Note that substituting $\overline{J}_i(\overline{x})$ and $\overline{\mathrm{L}}_{j_1 \cdots j_m}$ for $J_i(\overline{x})$ and $\mathrm{L}_{j_1 \cdots j_m}$, respectively in Theorem 1.2, the theorem also holds. The lines above, Theorems 1.2 and 1.3 tell us that at least one element of Clarke generalized gradients of f, at $\overline{x}$, can be obtained by determining compatibility of $\overline{\mathrm{L}}_{j_1 \cdots j_m}$, for each set of indices $j_i \in \overline{J}_i(x), i = 1, \cdots, m$.

As we know, the main work of two algorithms, as mentioned above, is due to determining the compatibility of system of linear inequalities $\mathrm{L}_{j_1 \cdots j_m}$ (or $\overline{\mathrm{L}}_{j_1 \cdots j_m}$) for each set of indices $j_1, \cdots, j_m, j_i \in J_i(\overline{x})$ (or $j_i \in \overline{J}_i(\overline{x})$). For instance, by the modified algorithm, one has to determine the compatibilities of systems of linear inequalities for $\prod_{i=1}^{m} \mathrm{card} \overline{J}_i(x)$ times. This cost is too expensive to bare for applications. Hence, reducing the computational amount of algorithms in [5] will be a meaning work. In this paper, we will present an algorithm that could be used to find a set of indices $j_i \in \overline{J}_i(\overline{x})$ for $i = 1, \cdots, m$ such that $\overline{\mathrm{L}}_{j_1 \cdots j_m}$ is consistent, by determining the compatibility of systems of linear inequalities less than $\sum_{i=1}^{m-1} \mathrm{card} \overline{J}_i(\overline{x})$ times.

2. Algorithm and Some Theorems

In this section, we will present an algorithm of calculating an element of Clarke generalized gradient for f, at $\overline{x}$. In the algorithm, we first find a set of indices $j_i \in \overline{J}_i(\overline{x})$ for $i = 1, \cdots, m$ such its related system of linear inequalities $\overline{\mathrm{L}}_{j_1 \cdots j_m}$ is consistent. Then, we calculate an element of Clarke generalized gradient for f, at $\overline{x}$, $\nabla g(x, f_{1j_1}(x), \cdots, f_{mj_m}(x))|_{x=\overline{x}}$. We now present two lemmas and one theorem, which will be used to prove that the algorithm is well defined and implementable.

Definition 2.1. *Let C be a closed convex set in R^n. We say $x \in C$ is an extreme, if there is no convex combination $x = \sum_{i=1}^{k} \lambda_i x_i$, where $x_i \in C, \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0$, other than $x_1 = \cdots = x_k = x$.*

Lemma 2.1. *Suppose $a_i \in \mathbf{R}^n, i = 1, \cdots, m$ with $a_i \neq a_j, \forall i,j = 1, \cdots, m, i \neq j$. If there exists an index $i_0 \in \{1, \cdots, m\}$ such that a_{i_0} is an extreme of the convex set $\mathrm{co}\{a_i \mid i = 1, \cdots, m\}$, then the following system of linear inequalities*

$$a_i^T y < a_{i_0}^T y, \quad y \in \mathbf{R}^n, \forall \ i \in \{1, \cdots, m\} \setminus \{i_0\} \tag{2.1}$$

is consistent and its solution set is an open convex cone in $\mathbf{R}^n$.

Proof. Assume that the system of linear inequalities (2.1) is inconsistent. Then we have that

$$\begin{aligned}\delta^*(y \mid \{a_{i_0}\}) &= a_{i_0}^T y \\ &\leq \max\{a_i^T y \mid i \in \{1,\cdots,m\} \setminus \{i_0\}\} \\ &= \delta^*(y \mid \text{co}\{a_i \mid i \in \{1,\cdots,m\} \setminus \{i_0\}\}), \forall y \in \mathbf{R}^n,\end{aligned} \tag{2.2}$$

where $\delta^*(\cdot \mid S)$ denotes the support function of the set S. According to the correspondence between a closed convex set and its support function, we have that $a_{i_0} \in \text{co}\{a_i \mid i \in \{1,\cdots,m\} \setminus \{i_0\}\}$. This contradicts that a_{i_0} is an extreme of the set $\text{co}\{a_i \mid i = 1,\cdots,m\}$. Hence the system of linear inequalities (2.1) is consistent. On the other hand, (2.1) is a system of strictly linear inequalities, so its solution set is an open convex cone.

Lemma 2.2. *Let $a_i \in \mathbf{R}^n$ for $i = 1,\cdots,m$. If there exists an index $i_0 \in \{1,\cdots,m\}$ with $\| a_{i_0} \|^2 = \max\{\| a_i \|^2 \mid i = 1,\cdots,m\}$, then a_{i_0} is an extreme of the convex set $\text{co}\{a_i \mid i = 1,\cdots,m\}$.*

Proof. It is easy to see that

$$\begin{aligned}\| a_{i_0} \|^2 &= \max\{\| a_i \|^2 \mid i = 1,\cdots,m\} \\ &= \max\{\| v \|^2 \mid v \in \text{co}\{a_i \mid i = 1,\cdots,m\}\}.\end{aligned} \tag{2.3}$$

According to the proof of Proposition 2.33 in [6], a_{i_0} is an extreme of the set $\text{co}\{a_i \mid i = 1,\cdots,m\}$.

Theorem 2.1. *Let Γ be an open convex cone in $\mathbf{R}^n$ with $\text{int}\Gamma \neq \emptyset$ and $a_i \in \mathbf{R}^n, i = 1,\cdots,m$ satisfy $a_i \neq a_j, \forall 1 \leq i,j \leq m, i \neq j$. Then there exist an index $i_0 \in \{1,\cdots,m\}$ such that the following system of linear inequalities*

$$a_i^T y < a_{i_0}^T y, \quad y \in \Gamma, \forall i \in \{1,\cdots,m\} \setminus \{i_0\} \tag{2.4}$$

is consistent and its solution set is an open convex cone in $\mathbf{R}^n$.

Proof. Denote

$$\Gamma_i = \{y \in \mathbf{R}^n \mid a_j^T y < a_i^T y, \forall j \in \{1,\cdots,m\} \setminus \{i\}\}, i = 1,\cdots,m, \tag{2.5}$$

$$B_{jk} = \{y \in \mathbf{R}^n \mid a_j^T y = a_k^T y, y \neq 0\}, j,k = 1,\cdots,m, j \neq k. \tag{2.6}$$

We first prove that the following relation holds:

$$(\bigcup_{1\leq i\leq m} \Gamma_i) \bigcup (\bigcup_{1\leq j,k\leq m, j\neq k} B_{jk}) = \mathbf{R}^n \setminus \{0\}. \tag{2.7}$$

It is true that

$$(\bigcup_{1\leq i\leq m} \Gamma_i) \bigcup (\bigcup_{1\leq j,k\leq m, j\neq k} B_{jk}) \subset \mathbf{R}^n \setminus \{0\}. \tag{2.8}$$

Therefore, we need only to prove the following relation holds:

$$\mathbf{R}^n \setminus \{0\} \subset (\bigcup_{1\leq i\leq m} \Gamma_i) \bigcup (\bigcup_{1\leq j,k\leq m, j\neq k} B_{jk}). \tag{2.9}$$

For each vector $\overline{y} \in \mathbf{R}^n \setminus \{0\}$, denote

$$I(\overline{y}) = \{i \in \{1,\cdots,m\} \mid a_i^T \overline{y} = \max_{j=1,\cdots,m} a_j^T \overline{y}\}.$$

As we know, there are two cases about the index set $I(\overline{y})$. In one case, $I(\overline{y})$ is a singleton, i.e., there is an index $i_0 \in \{1,\cdots,m\}$ such that $\{i_0\} = I(\overline{y})$. Then one has that $a_j^T y < a_{i_0}^T y, \forall j \in \{1,\cdots,m\}\setminus\{i_0\}$. This implies $\overline{y} \in \Gamma_{i_0}$. In the sequel, (2.9) holds. In another case, $I(\overline{y})$ is not a singleton. Then, there exist $j,k \in I(\overline{y}), j \neq k$ such that $a_j^T \overline{y} = a_k^T \overline{y}$. This leads to $\overline{y} \in B_{jk}$. Formula (2.9) also holds. Here one has completes the proof of (2.7). In the light of (2.7), one has that

$$\Gamma \setminus \{0\} \subset (\bigcup_{1\leq i\leq m} \Gamma_i)\bigcup(\bigcup_{1\leq j,k\leq m, j\neq k} B_{jk}). \tag{2.10}$$

According to our hypothesis that $a_j \neq a_k, j \neq k, j,k = 1,\cdots,m$, one has that each $B_{jk}, j \neq k, j,k = 1,\cdots,m$ is a subset of $(n-1)$-dimensional subspace of $\mathbf{R}^n$. In addition, $\text{int}(\bigcup_{1\leq j,k\leq m,j\neq k} B_{jk}) = \emptyset$. On the other hand, Γ is an open convex cone in $\mathbf{R}^n$ with $\text{int}\Gamma \neq \emptyset$. Then it follows from (2.10) that there exists a $y \in \Gamma, y \neq 0$ such that $y \in \bigcup_{1\leq i\leq m} \Gamma_i$. In other worlds, there exists an index $i_0 \in \{1,\cdots,m\}$ with $y \in \Gamma_{i_0}$. That is to say the system (2.4) is consistent. Evidently, its solution set is an open convex cone in $\mathbf{R}^n$. This completes the proof of the theorem.

For convenience of the statement, denote

$$\overline{\mathrm{L}}_{j_1\cdots j_r} \qquad (\nabla f_{it_i}(\overline{x}) - \nabla f_{ij_i}(\overline{x}))^T y < 0, y \in \mathbf{R}^n, \forall t_i \in \overline{J}_i(\overline{x}) \setminus \{j_i\}, i = 1,\cdots,r,$$

where $1 \leq r \leq m$. We now present the algorithm of calculating an element of Clarke generalized gradients of f at $\overline{x}$.

Algorithm 2.1.

Step 0 Given a point $\overline{x} \in \mathbf{R}^n$ and let $r = 2$.

Step 1 Compute values of $f_{ij}(\overline{x}), j \in J_i, i = 1,\cdots,m$.

Step 2 Determine index sets $J_i(\overline{x}), i = 1,\cdots,m$, compute values of $\nabla f_{ij}(\overline{x}), j \in J_i(\overline{x}), i = 1,\cdots,m$ and determine index sets $\overline{J}_i(\overline{x}), i = 1,\cdots,m$.

Step 3 If all of index sets $\overline{J}_i(\overline{x}), i = 1,\cdots,m$ are singletons, calculate an element of Clarke generalized gradient $\xi = \nabla g(x, f_{1j_1}(x),\cdots,f_{mj_m}(x)) \mid_{x=\overline{x}}$, where $J_i(\overline{x}) = \{j_i\}, i = 1,\cdots,m$ (in this case, f is differentiable at $\overline{x}$) and stop, otherwise go to Step 4.

Step 4 Finding an index $j_1 \in \overline{J}_1(\overline{x})$ such that $\| \nabla f_{1j_1}(\overline{x}) \| = \max\{\| \nabla f_{1j}(\overline{x}) \| \mid j \in \overline{J}_1(\overline{x})\}$.

Step 5 For each index $t_r \in \overline{J}(\overline{x})$, determine the compatibility of the system $\overline{\mathrm{L}}_{j_1\cdots j_{r-1}t_r}$ to find an index $j_r \in \overline{J}_r(\overline{x})$ such that $\overline{\mathrm{L}}_{j_1\cdots j_r}$ is consistent.

Step 6 If $r = m$, go to Step 7, otherwise, set $r = r+1$ and go to Step 5.

Step 7 Calculate an element of Clarke generalized gradient $\xi = \nabla g(x, f_{1j_1}(x),\cdots,f_{mj_m}(x)) \mid_{x=\overline{x}}$ and stop.

Theorem 2.2. *Algorithm 2.1 is well-defined and implementable. We can obtain one element of Clarke generalized gradient of f, at $\overline{x}$, by executing it.*

Proof. We first prove that the algorithm is implementable. It is sufficient to prove that Step 5 is implementable. We will use mathematical induction. By virtue of Lemma 2.2, $\nabla f_{1j_1}(\overline{x})$ is an extreme of the convex set $\text{co}\{\nabla f_{1j}(\overline{x}) \mid j \in \overline{J}_1(\overline{x})\}$. According to Lemma 2.1, the system $\overline{\mathrm{L}}_{j_1}$, i.e.,

$$\overline{\mathrm{L}}_{j_1} \qquad (\nabla f_{1t_1}(\overline{x}) - \nabla f_{1j_1}(\overline{x}))^T y < 0, y \in \mathbf{R}^n, \forall t_1 \in \overline{J}_1(\overline{x}) \setminus \{j_1\}$$

is consistent and its solution set is an open convex cone in $\mathbf{R}^n$. Suppose $\overline{\mathrm{L}}_{j_1\cdots j_r}(2 \le r \le m-1)$ is consistent and its solution set is a non-empty open convex cone in $\mathbf{R}^n$, denoted by (Arg $\overline{\mathrm{L}}_{j_1\cdots j_r}$). According to Theorem 2.1, we can find an index $j_{r+1} \in \overline{J}_{r+1}(\overline{x})$ such that the system

$$(\nabla f_{r+1\ t_{r+1}}(\overline{x}) - \nabla f_{r+1\ j_{r+1}}(\overline{x}))^T < 0,\ y \in \text{Arg } (\overline{\mathrm{L}}_{j_1\cdots j_r}),\ \forall t_{r+1} \in \overline{J}_{r+1}(\overline{x}) \setminus \{j_{r+1}\} \tag{2.11}$$

is consistent. Evidently, the system above happen to be $\overline{\mathrm{L}}_{j_1\cdots j_{r+1}}$. That is to say $\overline{\mathrm{L}}_{j_1\cdots j_{r+1}}$ is consistent. Thus it follows from Theorem 1.3 that $\nabla g(x, f_{1j_1}(x), \cdots, f_{mj_m}(x)) \mid_{x=\overline{x}}$ is an element of Clarke generalized gradient of f, at $\overline{x}$. This completes the proof of the theorem.

Remark. One should determine the compatibilities of systems of linear inequalities no more than $\sum_{i=1}^{m-1} \text{card}\overline{J}_i(\overline{x})$ times by executing Algorithm 2.1.

Executing Algorithm 2.1, one has to determine the compatibilities of linear inequalities for many times. The proposition below tells us that determining the compatibility of a system of linear inequalities can be transformed into solving an auxiliary linear programming:

Proposition2.1.[5] Let A be an $m \times n$ matrix. Then the linear system $Ay < 0, y \in \mathbf{R}^n$ is consistent if and only if the minimum value of the objective function of the following linear programming

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} z_j \\ \text{subject to} & A^T p + (z_1, \cdots, z_n)^T \ge 0 \\ & \sum_{i=1}^{m} p_i = 1 \\ & p_i \ge 0, i = 1, \cdots, m, z_j \ge 0, j = 1, \cdots, n \end{array}$$

is non-zero, where p_i is the $i-$th component of p.

With a slight modification, Algorithm 2.1 can be used to find more than one elements of Clarke generalized gradient of f, at $\overline{x}$. Actually, for any $t_1 \in \overline{J}_1(\overline{x})$ that $\nabla f_{1t_1}(\overline{x})$ is an extreme of the set $\text{co}\{\nabla f_{1j}(\overline{x}) \mid j \in \overline{J}_1(\overline{x})\}$, set $j_1 = t_1$ in Step 4 and execute Steps 5-7 of Algorithm 2.1 once, we can obtain M elements of Clarke generalized gradient of f, at $\overline{x}$, where M denotes the number of extremes of the set $\text{co}\{\nabla f_{1j}(\overline{x}) \mid j \in \overline{J}_1(\overline{x})\}$.

Example 2.1. Let

$$\begin{aligned} f(x) &= g(\max_{j=1,2} f_{1j}(x), \max_{j=1,2,3} f_{2j}(x), \max_{j=1,2,3} f_{3j}(x)) \\ &= \max_{j=1,2} f_{1j}(x) - \max_{j=1,2,3} f_{2j}(x) \max_{j=1,2,3} f_{3j}(x), \end{aligned}$$

where

$$\begin{aligned} &x = (x_1, x_2, x_3)^T, \\ &f_{11}(x) = x_1 + x_2, f_{12}(x) = x_1 - x_2, \\ &f_{21}(x) = x_1, f_{22}(x) = x_2, f_{23}(x) = x_3, \\ &f_{31}(x) = \sin x_1, f_{32}(x) = x_1^2 + \sin x_2, f_{33}(x) = x_2 + \cos x_3 - 1. \end{aligned}$$

Take $\overline{x} = 0 \in \mathbf{R}^3$. Obviously, $J_1(0) = \{1, 2\}, J_2(0) = \{1, 2, 3\}, J_3(0) = \{1, 2, 3\}$. It follows that

$$\begin{aligned}
&\nabla f_{11}(0) = (1,1,0)^T, \nabla f_{12}(0) = (1,-1,0)^T,\\
&\nabla f_{21}(0) = (1,0,0)^T, \nabla f_{22}(0) = (0,1,0)^T, \nabla f_{23}(0) = (0,0,1)^T,\\
&\nabla f_{31}(0) = (1,0,0)^T, \nabla f_{32}(0) = (0,1,0)^T, \nabla f_{33}(0) = (0,1,0)^T.
\end{aligned}$$

We then have $\overline{J}_1(0) = J_1(0) = \{1,2\}, \overline{J}_2(0) = J_2(0) = \{1,2,3\}, \overline{J}_3(0) = \{1,2\}$. Executing the algorithm in [5], we have to determine the following systems of linear inequalities: $\overline{\mathrm{L}}_{111}, \overline{\mathrm{L}}_{112}, \overline{\mathrm{L}}_{121}, \overline{\mathrm{L}}_{122}, \overline{\mathrm{L}}_{131}, \overline{\mathrm{L}}_{132}, \overline{\mathrm{L}}_{211}, \overline{\mathrm{L}}_{212}, \overline{\mathrm{L}}_{221}, \overline{\mathrm{L}}_{222}, \overline{\mathrm{L}}_{231}, \overline{\mathrm{L}}_{232}$. For instance, $\overline{\mathrm{L}}_{111}$ has the following form:

$$\overline{\mathrm{L}}_{111} \quad \begin{aligned}
&(\nabla f_{12}(0) - \nabla f_{11}(0))^T y < 0\\
&(\nabla f_{22}(0) - \nabla f_{21}(0))^T y < 0\\
&(\nabla f_{23}(0) - \nabla f_{21}(0))^T y < 0\\
&(\nabla f_{32}(0) - \nabla f_{31}(0))^T y < 0
\end{aligned}$$

We now execute Algorithm 2.1. Evidently,

$$\| \nabla f_{11}(0) \| = \| \nabla f_{12}(0) \| = \max\{\| \nabla f_{11}(0) \|, \| \nabla f_{12}(0) \|\}.$$

Set $j_1 = 1$. We next turn to determine the compatibilities of the following systems:

$$\overline{\mathrm{L}}_{11} \quad \begin{aligned}
&(\nabla f_{12}(0) - \nabla f_{11}(0))^T y < 0\\
&(\nabla f_{22}(0) - \nabla f_{21}(0))^T y < 0\\
&(\nabla f_{23}(0) - \nabla f_{21}(0))^T y < 0
\end{aligned}$$

$$\overline{\mathrm{L}}_{12} \quad \begin{aligned}
&(\nabla f_{12}(0) - \nabla f_{11}(0))^T y < 0\\
&(\nabla f_{21}(0) - \nabla f_{22}(0))^T y < 0\\
&(\nabla f_{23}(0) - \nabla f_{22}(0))^T y < 0
\end{aligned}$$

$$\overline{\mathrm{L}}_{13} \quad \begin{aligned}
&(\nabla f_{12}(0) - \nabla f_{11}(0))^T y < 0\\
&(\nabla f_{21}(0) - \nabla f_{23}(0))^T y < 0\\
&(\nabla f_{22}(0) - \nabla f_{23}(0))^T y < 0
\end{aligned}$$

The systems above can be rewritten as follows:

$$\overline{\mathrm{L}}_{11} \quad \begin{aligned}
-2y_2 &< 0\\
-y_1 + y_2 &< 0\\
-y_1 + y_3 &< 0
\end{aligned}$$

$$\overline{\mathrm{L}}_{12} \quad \begin{aligned}
-2y_2 &< 0\\
y_1 - y_2 &< 0\\
-y_2 + y_3 &< 0
\end{aligned}$$

$$\overline{\mathrm{L}}_{13} \quad \begin{aligned}
-2y_2 &< 0\\
y_1 - y_3 &< 0\\
y_2 - y_3 &< 0
\end{aligned}$$

It is easy to see that $\overline{\mathrm{L}}_{11}, \overline{\mathrm{L}}_{12}, \overline{\mathrm{L}}_{13}$ are consistent. Setting $j_2 = 2$, we then determine the compatibilities the following systems:

$$\overline{\mathrm{L}}_{121} \quad \begin{aligned}
&(\nabla f_{12}(0) - \nabla f_{11}(0))^T y < 0\\
&(\nabla f_{21}(0) - \nabla f_{22}(0))^T y < 0\\
&(\nabla f_{23}(0) - \nabla f_{22}(0))^T y < 0\\
&(\nabla f_{32}(0) - \nabla f_{31}(0))^T y < 0
\end{aligned}$$

$$\overline{\mathrm{L}}_{122}\quad \begin{array}{l}(\nabla f_{12}(0)-\nabla f_{11}(0))^T y<0\\(\nabla f_{21}(0)-\nabla f_{22}(0))^T y<0\\(\nabla f_{23}(0)-\nabla f_{22}(0))^T y<0\\(\nabla f_{31}(0)-\nabla f_{32}(0))^T y<0\end{array}$$

The systems above can be expressed by

$$\overline{\mathrm{L}}_{121}\quad \begin{array}{r}-2y_2<0\\y_1-y_2<0\\-y_1+y_3<0\\-y_1+y_2<0\end{array}$$

$$\overline{\mathrm{L}}_{122}\quad \begin{array}{r}-2y_2<0\\y_1-y_2<0\\-y_2+y_3<0\\y_1-y_2<0\end{array}$$

Obviously, $\overline{\mathrm{L}}_{121}$ is inconsistent and $\overline{\mathrm{L}}_{122}$ is consistent. Hence

$$\begin{aligned}\xi &= \nabla g(f_{11}(x),f_{22}(x),f_{32}(x))|_{x=0}\\&= \nabla(f_{11}(x)-f_{22}(x)f_{32}(x))|_{x=0}\\&= \nabla(x_1+x_2-x_2(x_1^2+\sin x_2))|_{x=0}\\&= (1,1,0)^T\end{aligned}$$

is an element of Clarke generalized gradient of f, at $x=0$.

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