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SOLVING INTEGRAL EQUATIONS WITH LOGARITHMIC KERNEL BY USING PERIODIC QUASI-WAVELET*1)

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Abstract

In solving integral equations with logarithmic kernel which arises from the boundary integral equation reformulation of some boundary value problems for the two dimensional Helmholtz equation, we combine the Galerkin method with Beylkin's ([2]) approach, series of dense and nonsymmetric matrices may appear if we use traditional method. By appealing the so-called periodic quasi-wavelet (PQW in abbr.) ([5]), some of these matrices become diagonal, therefore we can find a algorithm with only $O(K(m)^2)$ arithmetic operations where m is the highest level. The Galerkin approximation has a polynomial rate of convergence.

Key words: Periodic Quasi-Wavelet, Integral equation, Multiscale.

1. Introduction

We try to solve the following integral equation

$$u(x) = \int_0^{2\pi} u(y)(a_0 \log|2\sin\frac{x-y}{2}| + b(x,y))dy = g(x), x \in [0, 2\pi] \qquad (1.1)$$

where a_0 is a constant, and $b(x,y)$ is a continuous function of (x,y) and is 2π periodic in each variable, which appears in exterior boundary value problems for the two-dimensional Helmholtz equation (see [9], [13], [14], [12], [24]). We want to solve the equation by using wavelets. The most important method on solving integral equations was introduced in [3], but the method introduced in [3] can not be applied directly

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to this equation. Recently, Beylkin and Brewster introduced a new method called Multiscale Strategy in [2]. But when we apply Beylkin's ([2]) method, there appears dense and nonsymmetric matrices, which leads to large complexity. We therefore appeal to the so-called PQW, some of the matrices become diagonal.

Our idea of construction of PQW traces back to the sources of multiresolution analysis and the orthogonal periodic spline functions (see [15]). In [15] the author constructed periodic orthonormal splines (the scaling function), but they did not give the wavelets. Koh, Lee and Tan ([11]) and Tasche [22] constructed periodic wavelets by using Fourier coefficients of some functions and by using some special techniques. For instance, in [22], the author, constructed the periodic wavelets by appealing the Euler Frobenius function. Our construction is manipulating the periodic B-spline directly, and the construction of mother wavelets is very simple such that the decomposition and reconstruction formulas for the coefficient involve only two non-zero terms, that behave as the Haar basis. By using this wavelet, the complexity in solving the integral equation is much smaller. Since our wavelet has no localization, we call it quasi-wavelet [5].

In recent years, this equation and it's numerical solution have received much attention in the literature. A considerable part of the research on the numerical solution of this integral equation is concerned with the application of Galerkin methods, collocation methods and qualocation methods and their error analysis. For details, we refer the reader to [13], [23], [10] and the references therein.

Wavelets, which are originally developed for signal and image processing ([7]), has been applied in solving partial differential equations([1], [8]) and integral equations ([3], [18], [19], [21]). The latest paper that we received was written by Chen, Micchelli and Xu (see [6]) which deal with second kind of integral equations with singular kernel by using multiwavelet. In this paper, we introduce periodic quasi-wavelets and its application to solving equation (1.1). Here, FFT and Multiscale Strategy introduced in [2] are the key techniques. Multiresolution analysis (MRA) was introduced by Meyer [17] and Mallat [16] as a general framework for construction of the wavelet bases. Using MRA, the notion of the non–standard representation of operators was introduced in

[3]. For a wide class of operators, the non–standard form is sparse and permits fast algorithms for evaluation of functions (see also [19]). But the method can't applied to (1.1) since the deduced matrix is not sparse. The PQW joins the DFT and spline together to make the singular part of the operator diagonal. To get a fast numerical method, the idea of Multiscale Strategy is also used. Because PQW is based on B-Spline functions, the Galerkin approximation has polynomial rate of convergence. We need $O(K(m)^2)$ arithmetic operations to solve the equation where m is the highest level.

The paper is organized as follows:

In section 2, PQW are introduced and some properties are given.

In sections 3 , Quasi-Wavelet procedure is introduced. The convergence of Galerkin method based on PQW is introduced in section 4 and the error is estimated in section 5.

2. Periodic Quasi-Wavelets

Let $n \geq 1$ be an odd integer. K is an positive integer, and T is a positive real number. Let $h := T/K$, and for a positive integer m, $h_m := h/2^m$, then the point set $\{y_\nu^m\}$ are defined as $y_\nu^m = (\nu - \frac{n+1}{2})h_m$. We will also denote $K(m) := 2^m K$.

The so called $B - spline$ is defined by

$$\begin{aligned} B_j^n(x, h_m) &= (-1)^{n+1}(y_{n+1+j}^m - y_j^m)[y_j^m, \cdots, y_{j+n+1}^m]_y (x-y)_+^n \\ &= \frac{1}{n!h^n} \sum_{k=0}^{n+1} (-1)^k \begin{pmatrix} n+1 \\ k \end{pmatrix} (x - y_j^m - kh_m)_+^n \end{aligned} \tag{2.1}$$

The family of all such splines with knots $\{\nu h_m\}_{\nu \in \mathbb{Z}}$ will be denoted by $S_n(h_m)$, where h_m is the length of step.

Now we define the class of periodic spline functions in $S_n(h_m)$. Let $\widetilde{S}_n(h_m) := \widetilde{S}_n(h_m, [0,T]) = \{f \mid f$ is a polynomial of degree n on each interval $[jh_m, (j+1)h_m)$, $j = 0, 1, \cdots, K(m) - 1;\ f \in C^{n-1}[0,T]$; and $S^{(i)}(0) = S^{(i)}(T),\ i = 0, 1, \cdots, n-1\}$.

Of course, each function in $\widetilde{S}_n(h_m)$ can be extended periodically to the whole real axis. The collection of the extended periodic functions in $\widetilde{S}_n(h_m)$ is denoted by $\overset{\circ}{S}_n(h_m)$.

Evidently, $\widetilde{S}_n(h_m)$ is the restriction of $\overset{\circ}{S}_n\ (h_m)$ on the interval $[0,T]$.

$$\overset{\circ}{S}_n\ (h_m) = \{f \mid f(x) \in \widetilde{S}_n(h_m), x \in [0,T]; f(x) = f(x+T), x \in R\}.$$

Since the dimension of $\widetilde{S}_n(h_m)$ is $K(m)$ (see [20]), we have to find $K(m)$ functions which form a basis of $\widetilde{S}_n(h_m)$.

Evidently, for $x \in [0,T]$, the function $\widetilde{B_j^n}(x,h_m) := B_j^n(x,h_m) + B_{j+K(m)}^n(x,h_m)$ belongs to $\widetilde{S}_n(h_m)$, $\quad j = -n_0, \cdots, K(m) - n_0 - 1$, where $n_0 = 1 + [\frac{n}{2}]$.

It is easy to prove the following proposition :

Proposition 2.1. *The system of functions* $\{\widetilde{B}_j^n(x,h_m)\}_{-n_0}^{K-n_0-1}$ *constitutes a basis of* $\widetilde{S}_n(h_m)$.

We define the inner product of two functions f and g by $< f, g >= \frac{1}{T}\int_0^T f(x)\overline{g(x)}dx$. The Fourier expansion of $\overset{\circ}{B}{}_0^n$ has the following form

$$\overset{\circ}{B}{}_0^n\ (x,h_m) = (K(m))^n \sum_{l\in\mathbb{Z}} \left(\frac{\sin\dfrac{l\pi}{K(m)}}{l\pi} \right)^{n+1} \exp\left(\frac{i2\pi lx}{T}\right). \tag{2.2}$$

Define the function space V_m by $V_m := \overset{\circ}{S}_n\ (h_m)$. From the basic properties of spline functions, we have the following

Proposition 2.2. *For* $m \geq 0$, $-n_0 \leq j \leq K(m) - 1 - n_0$, *the function* $\widetilde{B}_j^n(x,h_m)$ *satisfies the two scale equation*

$$\widetilde{B}_j^n(x,h_m) = \sum_{-n_0-1}^{n_0+1} f_{n,\nu}\widetilde{B}_{\nu+2j}^n(x,h_{m+1}), \tag{2.3}$$

where

$$f_{n,\nu} = \begin{cases} \dfrac{1}{2^n}\begin{pmatrix} n+1 \\ n_0+1-\nu \end{pmatrix} & ,\nu = n_0 - n, \cdots, n_0+1, \\ 0 & ,\textit{otherwise}. \end{cases} \tag{2.4}$$

From Proposition 2.2, we conclude that

$$V_m \subset V_{m+1} \subset \cdots, \tag{2.5}$$

and $\{V_m\}$ is dense in $\overset{\circ}{L}_2\ [0,T]$, (refers to [20]), that is,

$$\overline{\bigcup_{\substack{m\in\mathbb{Z}\\ m\geq 0}} V_m} = \overset{\circ}{L}_2\ [0,T]. \tag{2.6}$$

Define $A_k^{n,j}(x)$ as follows:

$$A_a^{n,j}(x) = C_a^{n,j} \sum_{l=0}^{K(j)-1} \exp(2\pi i l a / K(j)) \overset{\circ}{B}{}_0^{n} (x - lh_j, h_j), \tag{2.7}$$

where

$$\begin{aligned} C_a^{n,j} &= [t_0 + 2\sum_{\lambda=1}^{n} t_\lambda \cos(\lambda a h_j)]^{-\frac{1}{2}}, \\ t_\lambda &= B_0^{2n+1}(\lambda, 1), B_0^{2n+1}(\cdot, 1) \in S_{2n+1}(1). \end{aligned} \tag{2.8}$$

From (2.7), we can prove that $A_\nu^{n,m}(x)$ has the following Fourier expansion

$$\begin{aligned} A_\nu^{n,m}(x) = \quad & C_\nu^{n,m}(K(m))^{n+1} \sum_{\lambda \in \mathbb{Z}} \left(\frac{\sin \dfrac{\nu\pi}{K(m)}}{(\nu + \lambda K(m))\pi} \right)^{n+1} \\ & \cdot \exp\left(\frac{2\pi i(\nu + \lambda K(m))x}{T} \right). \end{aligned} \tag{2.9}$$

From the Fourier expansion of $A_\nu^{n,m}(x)$, we have

Lemma 2.3. $\{A_k^{n,j}(x)\}_{k=0}^{K(j)-1}$ *is an orthonormal basis of* $V_j = \overset{\circ}{S}_n (h_j)$,

$$\langle A_{k_1}^{n,j}(x), A_{k_2}^{n,j}(x) \rangle = \delta_{k_1,k_2}, \tag{2.10}$$

where $0 \le k_1, k_2 \le K(j) - 1$.

Since $\overset{\circ}{B}{}_0^{n} (x - lh_j, h_j)$ is refinable, $A_\nu^{n,m}(x)$ are also refinable, moreover, their two-scale equation has a very elegant form, it reflects the intrinsic property of this functions.

Theorem 2.4. $A_\nu^{n,m}(x)$ *satisfies the following two-scale equation:*

$$A_\nu^{n,m}(x) = a_\nu^{n,m+1} A_\nu^{n,m+1}(x) + b_\nu^{n,m+1} A_{\nu+K(m)}^{n,m+1}(x), \tag{2.11}$$

where $a_\nu^{n,m+1}$ *and* $b_\nu^{n,m+1}$ *are constants.*

$$a_\nu^{n,m+1} = C_\nu^{n,m}(\cos \frac{\nu\pi}{K(m+1)})^{n+1} / C_\nu^{n,m+1}, \tag{2.12}$$

$$b_\nu^{n,m+1} = C_\nu^{n,m}(\sin \frac{\nu\pi}{K(m+1)})^{n+1} / C_{\nu+K(m)}^{n,m+1}, \tag{2.13}$$

$0 \le \nu \le K(m) - 1$ *where* $C_\nu^{n,m}$ *is defined as in (2.8).*

Proof. From (2.7) and (2.9), we have

$$A_\nu^{n,m}(x) = C_\nu^{n,m} \sum_{l=0}^{K(m)-1} \exp(2\pi i l\nu/K(m)) \overset{\circ}{B}{}_0^{n} (x - lh_m, h_m)$$

$$= C_\nu^{n,m} \sum_{l=0}^{K(m)-1} \exp(2\pi i l\nu/K(m)) \sum_{k=-n_0-1}^{n_0+1} f_{n,k} \overset{\circ}{B}{}_0^{n} (x - (k+2l)h_{m+1}, h_{m+1})$$

$$= C_\nu^{n,m} \sum_{l=0}^{K(m)-1} \exp(2\pi i l\nu/K(m)) \cdot \sum_{k=-n_0-1}^{n_0+1} f_{n,k} \widetilde{B}_0^n (x - (k+2l)h_{m+1}, h_{m+1})$$

$$= C_\nu^{n,m} \sum_{l=0}^{K(m)-1} \exp(2\pi i l\nu/K(m)) \sum_{k=-n_0-1}^{n_0+1} f_{n,k}$$
$$\sum_{\mu=0}^{K(m+1)-1} d_\mu A_\mu^{n,m+1}(x - \frac{(k+2l)h}{2^{m+1}})$$

$$= C_\nu^{n,m} \sum_{l=0}^{K(m)-1} \exp(2\pi i l\nu/K(m)) \sum_{k=-n_0-1}^{n_0+1} f_{n,k} \sum_{\mu=0}^{K(m+1)-1} d_\mu C_\mu^{n,m+1}$$
$$\cdot (K(m+1))^{n+1} \sum_{\lambda\in\mathbb{Z}} \left(\frac{\sin\dfrac{\mu\pi}{K(m+1)}}{(\mu+\lambda K(m+1))\pi} \right)^{n+1}$$
$$\cdot \exp\left(\frac{2\pi i(\mu+\lambda K(m+1))x}{T}\right) \exp\left(\frac{-2\pi i(\mu+\lambda K(m+1))(k+2l)}{K(m+1)}\right)$$

$$= C_\nu^{n,m} \cdot \sum_{k=-n_0-1}^{n_0+1} f_{n,k} \sum_{\mu=0}^{K(m+1)-1} d_\mu C_\mu^{n,m+1} (K(m+1))^{n+1}$$
$$\sum_{\lambda\in\mathbb{Z}} \left(\frac{\sin\dfrac{\mu\pi}{K(m+1)}}{(\mu+\lambda K(m+1))\pi} \right)^{n+1} \cdot \exp\left(\frac{2\pi i(\mu+\lambda K(m+1))x}{T}\right)$$
$$\cdot \exp\left(\frac{-2\pi i\mu k}{K(m+1)}\right) \cdot \sum_{l=0}^{K(m)-1} \exp\left(\frac{2\pi i l(\nu-\mu)}{K(m)}\right)$$

$$= a_\nu^{n,m+1} A_\nu^{n,m+1}(x) + b_\nu^{n,m+1} A_{\nu+K(m)}^{n,m+1}(x),$$

Define

$$D_\nu^{n,m}(x) = b_\nu^{n,m+1} A_\nu^{n,m+1}(x) - a_\nu^{n,m+1} A_{\nu+K(m)}^{n,m+1}(x), \tag{2.14}$$

$\nu = 0, \cdots, K(m) - 1$. These functions have the following properties :

$$\langle D_{\nu_1}^{n,m}, D_{\nu_2}^{n,m} \rangle = \delta_{\nu_1,\nu_2} \quad \text{for } 0 \le \nu_1, \nu_2 \le K(m) - 1 \tag{2.15}$$

$$D_{\nu}^{n,m} \in V_{m+1} \quad \text{for } 0 \le \nu \le K(m) - 1 \tag{2.16}$$

$$\langle D_{\nu_1}^{n,m}, A_{\nu_2}^{n,m} \rangle = 0 \quad \text{for } 0 \le \nu_1, \nu_2 \le K(m) - 1 \tag{2.17}$$

Let $W_m = \text{Span } \{D_{\nu}^{n,m} \mid \nu = 0, \cdots, K(m) - 1\}$ (2.18)

Then we have the following

Theorem 2.5. *The class of functions* $\{D_{\nu}^{n,m}\}_{\nu=0}^{K(m)-1}$ *is an orthonormal basis of the function space* W_m, *and* $V_{m+1} = V_m \oplus W_m$.

We call $A_{\nu}^{n,m}$ *father quasi-wavelet*; $D_{\nu}^{n,m}$ the *mother quasi-wavelet*. We put the prefix 'quasi' before the word wavelet, because it differs from wavelet in the sense of Meyer's.

Let P_m, Q_m be the projection operators which maps $\overset{\circ}{L}_2\ [0, T]$ onto V_m and W_m respectively, define

$$\alpha_{\nu}^{m} = \langle f, A_{\nu}^{n,m} \rangle, \quad \beta_{\nu}^{m} = \langle f, D_{\nu}^{n,m} \rangle. \tag{2.19}$$

Theorem 2.6. *For the coefficients* $\{\alpha_{\nu}^{m}\}$ *and* $\{\beta_{\nu}^{m}\}$ *defined in (2.19), we have the decomposition formulas*

$$\alpha_{\nu}^{m} = a_{\nu}^{n,m+1} \alpha_{\nu}^{m+1} + b_{\nu}^{n,m+1} \alpha_{\nu+K(m)}^{m+1}, \tag{2.20}$$

$$\beta_{\nu}^{m} = b_{\nu}^{n,m+1} \alpha_{\nu}^{m+1} - a_{\nu}^{n,m+1} \alpha_{\nu+K(m)}^{m+1}, \tag{2.21}$$

$\nu = 0, \cdots, K(m) - 1$.

The reconstruction formulas are

$$\alpha_{\nu}^{m+1} = a_{\nu}^{n,m+1} \alpha_{\nu}^{m} + b_{\nu}^{n,m+1} \beta_{\nu}^{m}, \tag{2.22}$$

$$\alpha_{\nu+K(m)}^{m+1} = b_{\nu}^{n,m+1} \alpha_{\nu}^{m} - a_{\nu}^{n,m+1} \beta_{\nu}^{m}, \tag{2.23}$$

$\nu = 0, \cdots, K(m) - 1$.

Let $\alpha^m := (\alpha_0^m, \cdots, \alpha_{K(m)-1}^m)^T$, $\beta^m := (\beta_0^m, \cdots, \beta_{K(m)-1}^m)^T$, where the notation $(\cdot)^T$ means the transpose of $(\cdot)$. In order to express our problems clear, we use the following matrices.

Define

$$W_m = \begin{pmatrix} a_0^{n,m} & 0 & \cdots & b_0^{n,m} & 0 & \cdots \\ 0 & a_1^{n,m} & \cdots & 0 & b_1^{n,m} & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \\ b_0^{n,m} & 0 & \cdots & -a_0^{n,m} & 0 & \cdots \\ 0 & b_1^{n,m} & \cdots & 0 & -a_1^{n,m} & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \end{pmatrix}, \tag{2.24}$$

then W_m is a real matrix of size $K(m) \times K(m)$.

Denote the upper half of W_m by L_m, the other half by H_m, i.e.

$$W_m = \begin{pmatrix} L_m \\ H_m \end{pmatrix}.$$

Then (2.20) and (2.21) can be written into

$$\begin{pmatrix} \alpha^m \\ \beta^m \end{pmatrix} = \begin{pmatrix} L_{m+1} \\ H_{m+1} \end{pmatrix} \alpha^{m+1}, \tag{2.25}$$

and (2.22), (2.23) are equivalent to

$$\begin{pmatrix} \alpha^{m+1} \\ \beta^{m+1} \end{pmatrix} = (L_{m+1}^T, H_{m+1}^T) \begin{pmatrix} \alpha^m \\ \beta^m \end{pmatrix}. \tag{2.26}$$

For the contents $C_k^{n,j}$, we have the following estimates which will be helpful in our error analysis.

Lemma 2.7. *$C_k^{n,j}$ satisfies the following inequality*

$$1 \leq |C_a^{n,j}| \leq (\frac{\pi}{2})^{n+1}, \quad \textit{for } a = 0, \cdots, K(j)-1. \tag{2.27}$$

Proof. Set

$$E_a = \sum_{\nu=0}^{K(j)-1} \exp(2\pi i \nu a / K(j)) \overset{\circ}{B}{}_0^{2n+1} (\nu h_j, h_j), \tag{2.28}$$

then from (2.8), $C_a^{n,j} = |E_a|^{-\frac{1}{2}}$, for $a = 0, 1 \cdots, K(j)-1$.

Now we estimate $|E_a|$.

Since $\overset{\circ}{B}{}_0^{2n+1}((\nu h_j, h_j) \geq 0$, therefore from (2.28)

$$\max_{0\leq a\leq K(j)-1} |E_a| \leq |E_0| = \sum_{\nu=0}^{K(j)-1} \overset{\circ}{B}{}_0^{2n+1}(\nu h_j, h_j) = 1$$

For the lower bound, we write E_a into another form in

$$\begin{aligned} E_a &= K(j)^{2n+2} \sum_{\lambda\in\mathbb{Z}} \left(\frac{\sin\dfrac{a\pi}{K(j)}}{(a+\lambda K(j))\pi} \right)^{2n+2} \\ &\geq K(j)^{2n+2} \left(\frac{\sin\dfrac{a\pi}{K(j)}}{a\pi} \right)^{2n+2} \geq (\tfrac{2}{\pi})^{2n+2}. \end{aligned}$$

In the following theorem, we estimate the quantity $\|(f-P_nf)^{(s)}\|_q$.

Theorem 2.8. *For every* $f \in \overset{\circ}{C}{}^n[0,T]$,

$$\||(f-P_nf)^{(s)}\|_\infty \leq A(n,s)\ h_n^{n-s}\ \omega(f^{(n)}; \frac{n+1}{2}h_n),$$

where $A(n,s)$ *is a constant depending on* n *and* s, ω *is the periodic modulus of smoothness.*

Theorem 2.9. *Suppose* $1 \leq p,q < \infty$. *Then there exists a constant* $B(n,s,p,q)$ *such that for all* $f \in L_p^{n+1}[0,T]$,

$$\||(f-P_nf)^{(s)}\|_{\overset{\circ}{L}_q} \leq B(n,s,p,q)\ h^{n+1-s-\frac{1}{p}+\frac{1}{q}}\ \|f^{(n+1)}\|_{\overset{\circ}{L}_p}. \tag{2.29}$$

The proof of above two theorems is not difficult by using the same method developed by de Boor and Fix [4] based on quasi-interpolant.

It is well known that B-spline functions can approximate smooth function very well, but it is unfortunately that the integral kernel of the integral equation concerned is weakly singular. The following theorem deals with the problem to give a estimate the approximating rate of $\log|2\sin\frac{x}{2}|$ from V_m. The method used here will be reused in the error analysis of our algorithm.

Theorem 2.10. *let* $f(x) = -2\log|2\sin\frac{x}{2}|$, *then*

$$\|f-P_mf\|_{L_2} \leq C_1 h_m^{\frac{1}{2}}, \tag{2.30}$$

where C_1 *is independent of* m.

Proof. From the definition of $A_j^{n,m}$, we obtain

$$A_j^{n,m}(x) = C_j^m (K(m))^{n+1} \sum_{\lambda \in \mathbb{Z}} \left(\frac{\sin \frac{j\pi}{K(m)}}{(j+\lambda K(m))\pi} \right)^{n+1} \exp(i(j+\lambda K(m))x). \quad (2.31)$$

It is also known that

$$f(x) = \sum_{m \in \mathbb{Z}, m \neq 0} \frac{1}{|m|} e^{imx}, \quad (2.32)$$

then

$$< f, A_j^{n,m} >= C_j^{n,m} (K(m))^{n+1} \sum_{\lambda \in \mathbb{Z}} \left(\frac{\sin \dfrac{j\pi}{K(m)}}{(j+\lambda K(m))\pi} \right)^{n+1} \cdot \frac{1}{j+\lambda K(m)}. \quad (2.33)$$

From (2.12) and (2.13), using the inequality: $|\sin(x)| \leq |x|$, it is immediately to see that for all $0 < j < K(m)$.

$$|a_j^{n,m+1}| \leq C_3 \frac{K(m)-j}{K(m+1)}, \quad (2.34)$$

$$|b_j^{n,m+1}| \leq C_4 \frac{j}{K(m+1)}. \quad (2.35)$$

From (2.27) and (2.33), calculating the summation directly, we have

$$\begin{aligned} | < f, A_j^{n,m+1} > | &\leq C_0 \left(\frac{1}{j} + (K(m))^{n+1} \sum_{\lambda \in \mathbb{Z}, \lambda \neq 0} \frac{1}{(j+\lambda K(m))^{n+1}} \right) \\ &\leq C_5 \frac{1}{j}. \end{aligned} \quad (2.36)$$

Similarly we have

$$| < f, A_{j+K(m)}^{n,m+1} > | \leq C_6 \frac{1}{K(m)-j}. \quad (2.37)$$

Since we have

$$\begin{aligned} < f, D_j^{n,m} > &=< f, b_j^{n,m+1} A_j^{n,m+1} - a_j^{n,m+1} A_{j+K(m)}^{n,m+1} > \\ &= b_j^{n,m+1} < f, A_j^{n,m+1} > - a_j^{n,m+1} < f, A_{j+K(m)}^{n,m+1} >, \end{aligned}$$

such that

$$| < f, D_j^{n,m} > | \leq C_7 \frac{1}{K(m+1)}. \quad (2.38)$$

In fact, for $j = 0$, (2.38) is also true.

Note that

$$\|f - P_m f\|^2 = \sum_{l \geq m} \sum_{j=0}^{K(l)-1} | < f, D_j^{l,m} > |^2 \leq \sum_{l \geq m} C_7^2 \frac{K_l}{K_{l+1}^2} \leq C \cdot \frac{1}{K(m)},$$

where $C_1, \cdots, C_7$ are all independent of j and m.

Theorem 2.10 follows immediately.

Remark. We were introduced by the refree's report that the inequality of this theorem can be rewritten as follows:

$$\|f - P_m\|_{L^2} \leq ch^{1-\epsilon} \|f\|_{H^{1-\epsilon}} \quad (\epsilon > 0).$$

3. Processing Integral Equation by Quasi–Wavelet Procedure

In order to introduce our algorithm clearly, we expose our idea in following several subsections.

3.1. Discretization: Projection into V_m

In this part, we begin to discuss the method to solve (1.1). Rewrite (1.1) in the operator form:

$$u = Tu + g. \tag{3.1}$$

The first step is discretize the integral equation to obtain a linear system. Usually, Galerkin approximation is used to deal with (3.1). Let P_m be the projective operator of $\mathring{L}_2\ [0, 2\pi]$ onto V_m. Then the following new equation is an approximate version of (3.1).

$$u_m = P_m T u_m + P_m g, \tag{3.2}$$

where $u_m \in V_m$.

Because $\{A_j^{n,m}\}$ is an orthonormal basis of V_m, suppose that

$$u_m = \sum_{j=0}^{K(m)-1} s_j^m A_j^{n,m}, \quad P_m g = \sum_{j=0}^{K(m)-1} g_j^m A_j^{n,m}. \tag{3.3}$$

Substitute (3.3) into (3.2) to obtain:

$$s_j^m = \sum_{k=0}^{K(m)-1} \beta_{jk} s_k^m + g_j^m \quad (0 \leq j \leq K(m) - 1), \tag{3.4}$$

where

$$\beta_{jk}^m =< TA_k^{n,m}, A_j^{n,m} > . \tag{3.5}$$

Now we have to solve linear system (3.4).

Denote

$$a(x-y) = a_0 \log|2\sin\frac{x-y}{2}|. \tag{3.6}$$

Then

$$\begin{aligned}\beta_{jk}^m &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}(a(x-y)+b(x,y))A_k^{n,m}(y)\overline{A_j^{n,m}(x)}dxdy \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} a(x-y)A_k^{n,m}(y)\overline{A_j^{n,m}(x)}dxdy \\ &+\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} b(x,y)A_k^{n,m}(y)\overline{A_j^{n,m}(x)}dxdy \\ &= e_{jk}^m + f_{jk}^m \quad (0 \le j \le K(m)-1).\end{aligned} \tag{3.7}$$

For simplicity, we drop the indices.

Let

$$E^m = (e_{jk}^m), \quad F^m = (f_{jk}^m), \quad s^m = (s_j^m), \quad g^m = (g_j^m), \tag{3.8}$$

where $0 \le j,k \le K(m)-1$, that is to say, E^m and F^m are matrices of size $K(m)\times K(m)$, s^m and g^m are column vectors of length $K(m)$.

Then (3.4) is equivalent to

$$s^m = (E^m + F^m)s^m + g^m. \tag{3.9}$$

3.2 Splitting the Linear Equation

Before discussing the algorithm, we will analyze the matrix E^m first.

Theorem 3.1.

$$E^m = \operatorname{diag}(e_{00}^m, \cdots, e_{K(m)-1,K(m)-1}^m). \tag{3.10}$$

Proof. Assume that

$$\begin{aligned}A_k^{n,m}(x) &= \sum_{l\in\mathbb{Z}} q_{k+lK(m)}e^{i(k+lK(m))x}, \\ a(x-y) &= \sum_{k\in\mathbb{Z}} p_k e^{ikx},\end{aligned}$$

where the former formula can be proved from the definition of $A_k^{n,m}$.

$$\begin{aligned} e_{\mu,\nu}^m &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} a(y-t)A_\nu^{n,m}(t)\overline{A}_\mu^{n,m}(y)dtdy \\ &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} p_k \sum_{\lambda_1\lambda_2\in\mathbb{Z}} q_{\lambda_1}\overline{q_{\lambda_2}}\cdot \\ &\quad \int_0^{2\pi}\int_0^{2\pi} e^{i(k-\mu-\lambda_2K(m))y+i(\nu+\lambda_1K(m)-k)t}dydt \\ &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} p_k \sum_{\lambda_1,\lambda_2\in\mathbb{Z}} q_{\lambda_1}\overline{q_{\lambda_2}}(2\pi)^2\cdot\delta_{k,\mu+\lambda_2K(m)}\delta_{k,\nu+\lambda_1K(m)} \\ &= (2\pi\cdot\sum_{\lambda\in\mathbb{Z}} p_{\mu+\lambda K(m)}|q_\lambda|^2)\delta_{\nu,\mu} = e_{\mu,\mu}^m\delta_{\nu,\mu}. \end{aligned}$$

The theorem is proved.

We will use the periodic quasi–wavelet transform to solve (3.9).

Let W_m be the matrix given in (2.24), then we have

$$W_m^TW_m = H_m^TH_m + L_m^TL_m = I_m, \tag{3.12}$$

and

$$H_mH_m^T = I_{m-1}, \quad L_mL_m^T = I_{m-1}, \tag{3.13}$$

where I_m denote the unit matrix of size $K(m)\times K(m)$.

Apply W_m to both sides of (3.9) and split into a pair of equations to obtain

$$\begin{aligned} &L_ms^m = L_m(E^m+F^m)s^m + L_mg^m \\ &= L_m(E^m+F^m)L_m^TL_ms^m + L_m(E^m+F^m)H_m^TH_ms^m + L_mg^m, \end{aligned} \tag{3.14}$$

and

$$H_ms^m = H_m(E^m+F^m)L_m^TL_ms^m + H_m(E^m+F^m)H_m^TH_ms^m + H_mg^m \tag{3.15}$$

Denote

$$L_ms^m = s_{m-1}^m, \quad H_ms^m = d_{m-1}^m, \tag{3.16}$$

$$L_mg^m = g_s^{m-1}, \quad H_mg^m = g_d^{m-1}, \tag{3.17}$$

$$L_mE^mL_m^T = E_{ss}^{m-1}, \quad L_mE^mH_m^T = E_{sd}^{m-1}, \tag{3.18}$$

$$H_mE^mL_m^T = E_{ds}^{m-1}, \quad H_mE^mH_m^T = E_{dd}^{m-1}, \tag{3.19}$$

$$L_mF^mL_m^T = F_{ss}^{m-1}, \quad L_mF^mH_m^T = F_{sd}^{m-1}, \tag{3.20}$$

$$H_m F^m L_m^T = F_{ds}^{m-1}, \quad H_m F^m H_m^T = F_{dd}^{m-1}. \tag{3.21}$$

Then the linear systems (3.14) and (3.15) can be rewritten into

$$s_{m-1}^m = (E_{ss}^{m-1} + F_{ss}^{m-1})s_{m-1}^m + (E_{sd}^{m-1} + F_{sd}^{m-1})d_{m-1}^m + g_s^{m-1} \tag{3.22}$$

$$d_{m-1}^m = (E_{ds}^{m-1} + F_{ds}^{m-1})s_{m-1}^m + (E_{dd}^{m-1} + F_{dd}^{m-1})d_{m-1}^m + g_d^{m-1} \tag{3.23}$$

3.3 Approximate Version of Multiscale Strategy

We are going to solve (3.9) by using Multiscale Strategy([2]). If we can solve s_{m-1}^m, d_{m-1}^m from (3.22) and (3.23), then the solution s^m of (3.9) can be obtained by using the reconstruction formulas (2.22) and (2.23). According to the idea of Multiscale Strategy, we want to solve out d_{m-1}^m from (3.23) then substitute it into (3.22). But in general, it is difficult to do so, since the matrix F_{dd}^{m-1} may be dense. Even if we use Daubechies' Wavelet, the induced F_{dd}^{m-1} may be dense, and E_{dd}^{m-1} is not diagonal. Since $b(x,y)$ is smooth, the norm of F_{dd}^{m-1} is very small if m is large enough. We can not solve d_{m-1}^m precisely, but we can obtain an approximate version which can be easily obtained. According to this instruction, move the term $E_{dd}^{m-1}d_{m-1}^m$ to the left side of (3.23), multiply both side by $\gamma^{m-1} := (I - E_{dd}^{m-1})^{-1}$,

$$d_{m-1}^m = \gamma^{m-1}(E_{ds}^{m-1} + F_{ds}^{m-1})s_{m-1}^m + \gamma^{m-1}F_{dd}^{m-1}d_{m-1}^m + \gamma^{m-1}g_d^{m-1}, \tag{3.24}$$

where we assume γ^{m-1} exists for all large m.

Substitute (3.24) into (3.22), we obtain

$$\begin{aligned} s_{m-1}^m = \; & [(E_{ss}^{m-1} + E_{sd}^{m-1}\gamma^{m-1}E_{ds}^{m-1}) + \\ & +(F_{ss}^{m-1} + E_{sd}^{m-1}\gamma^{m-1}F_{ds}^{m-1} + F_{sd}^{m-1}\gamma^{m-1}E_{ds}^{m-1})]s_{m-1}^m + \\ & +F_{sd}^{m-1}\gamma^{m-1}F_{ds}^{m-1}s_{m-1}^m + F_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}d_{m-1}^m + \\ & +E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}d_{m-1}^m + (E_{sd}^{m-1} + F_{sd}^{m-1})\gamma^{m-1}g_d^{m-1} + g_s^{m-1}. \end{aligned} \tag{3.25}$$

It is clear that all E's are diagonal and all F's are dense. If we want to avoid the multiplication of dense matrix, we should treat them skillfully. Note that the operator norm of F_{sd}^{m-1}, F_{ds}^{m-1} and F_{dd}^{m-1} are very small because of the smoothness of $b(x,y)$. We will show in section 4 that the norm of F_{sd}^{m-1}, F_{ds}^{m-1} and F_{dd}^{m-1} have the order h_m^s when m tends to infinity, where s is the smooth order of $b(x,y)$, but the norm of E_{sd}^{m-1},

E_{ds}^{m-1} and E_{dd}^{m-1} have the order h_m. Hence we can throw away the terms which have the order h_m^{2s} to ensure that our new algorithm can approximate the true solution as well as possible. The terms $F_{sd}^{m-1}\gamma^{m-1}F_{ds}^{m-1}s_{m-1}^m$ and $F_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}d_{m-1}^m$ can be thrown away. But we must deal with $E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}d_{m-1}^m$, because it has order larger then h_m^{2s}. Act $E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}$ on both sides of (3.24) to obtain

$$\begin{aligned}&E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}d_{m-1}^m\\&=E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}\gamma^{m-1}E_{ds}^{m-1}s_{m-1}^m+E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}\gamma^{m-1}g_d^{m-1}\\&+E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}\gamma^{m-1}F_{ds}^{m-1}s_{m-1}^m+E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}d_{m-1}^m.\end{aligned}\tag{3.26}$$

Note that the last two terms contain two small factors, i.e., F_{dd}^{m-1}, F_{ds}^{m-1}, so we can throw them away. Substitute (3.26) into (3.25).

$$s_{m-1}^m=(\tilde{E}^{m-1}+\tilde{F}^{m-1})s_{m-1}^m+\rho^{m-1}+\tilde{g}^{m-1},\tag{3.27}$$

where

$$\tilde{E}^{m-1}=E_{ss}^{m-1}+E_{sd}^{m-1}\gamma^{m-1}E_{ds}^{m-1},\tag{3.28}$$

$$\tilde{F}^{m-1}=F_{ss}^{m-1}+E_{sd}^{m-1}\gamma^{m-1}F_{ds}^{m-1}+F_{sd}^{m-1}\gamma^{m-1}E_{ds}^{m-1}+E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}\gamma^{m-1}E_{ds}^{m-1}\tag{3.29}$$

$$\begin{aligned}\rho^{m-1}&=F_{sd}^{m-1}\gamma^{m-1}F_{ds}^{m-1}s_{m-1}^m+E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}\gamma^{m-1}F_{ds}^{m-1}s_{m-1}^m\\&+E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}d_{m-1}^m+F_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}d_{m-1}^m,\end{aligned}\tag{3.30}$$

and

$$\tilde{g}^{m-1}=g_s^{m-1}+(E_{sd}^{m-1}+F_{sd}^{m-1})\gamma^{m-1}g_d^{m-1}+E_{sd}^{m-1}\gamma^{m-1}F_{dd}^{m-1}\gamma^{m-1}g_d^{m-1}.\tag{3.31}$$

We shall prove that ρ^{m-1} really has very small norm, so we only need to solve out $\tilde{s}^{m-1}$ from the following equation:

$$\tilde{s}^{m-1}=(\tilde{E}^{m-1}+\tilde{F}^{m-1})\,\tilde{s}^{m-1}+\tilde{g}^{m-1}.\tag{3.32}$$

3.4 Algorithm

In section 5 we will prove $\tilde{s}^{m-1}$ and s_{m-1}^m is slightly different, but the unknowns in (3.32) is only half of that in (3.9). If we have already solved $\tilde{s}^{m-1}$, then we simply substitute it into (3.24) instead of s_{m-1}^m to solve out d_{m-1}^m. It is also easy to obtain d_{m-1}^m

because the norm of F_{dd}^{m-1} is very small, iterative method can work. We also note that (3.32) is similar to (3.9), such that the above method can be repeated, reducing the unknowns by half at each step. But we can not reach a scalar equation finally, for we can't ensure that the solution approximates the true solution well enough. In fact, we will stop at level m_1. At m_1^{th} level, we can solve equations like (3.32) by any solver such as Guassian Elimination. Because this direct method needs $O(K_{m_1}^3)$ operations which is nearly $O(K(m)^{3m_1/m})$. At k^{th} level, we need $O(K(k)^2)$ operations. Hence the whole operations that we need is $O(K(m)^2) + O(K(m)^{3m_1/m})$. To ensure the complexity is as small as possible, it is sufficient that $m_1 = [\frac{2}{3}m]$, where $[\cdot]$ denotes the integer part of the number in it. Before giving our algorithm, some notations must be introduced.

For $k < m$, let $\tilde{s}^k$ be the true solution of the equation

$$\tilde{s}^k = (\tilde{E}^k + \tilde{F}^k)\,\tilde{s}^k + \tilde{g}^k, \tag{3.33}$$

and define $\tilde{s}_k^{k+1} = L_k\,\tilde{s}^{k+1}$, then from (3.27) we can see that $\tilde{s}_k^{k+1}$ satisfies

$$\tilde{s}_k^{k+1} = (\tilde{E}^k + \tilde{F}^k)\,\tilde{s}_k^{k+1} + \tilde{g}^k + \rho^k, \tag{3.34}$$

where we denote

$$\tilde{E}^k = \tilde{E}_{ss}^k + \tilde{E}_{sd}^k \tilde{\gamma}^k \tilde{E}_{ds}^k, \tag{3.35}$$

$$\tilde{F}^k = \tilde{F}_{ss}^k + \tilde{E}_{sd}^k \tilde{\gamma}^k \tilde{F}_{ds}^k + \tilde{F}_{sd}^k \tilde{\gamma}^k \tilde{E}_{ds}^k + \tilde{E}_{sd}^k \tilde{\gamma}^k \tilde{F}_{dd}^k \tilde{\gamma}^k \tilde{E}_{ds}^k, \tag{3.36}$$

$$\rho^k = \tilde{F}_{sd}^k \tilde{\gamma}^k \tilde{F}_{ds}^k \tilde{s}_k^{k+1} + \tilde{E}_{sd}^k \tilde{\gamma}^k \tilde{F}_{dd}^k \tilde{\gamma}^k \tilde{F}_{ds}^k \tilde{s}_k^{k+1}$$

$$+ \tilde{E}_{sd}^k\, \gamma^k\, \tilde{F}_{dd}^k \tilde{\gamma}^k \tilde{F}_{dd}^k \tilde{d}_k^{k+1} + \tilde{F}_{sd}^k \tilde{\gamma}^k \tilde{F}_{dd}^k \tilde{d}_k^{k+1}, \tag{3.37}$$

$$\tilde{g}^k = \tilde{g}_s^k + (\tilde{E}_{sd}^k + \tilde{F}_{sd}^k)\,\tilde{\gamma}^k \tilde{g}_d^k + \tilde{E}_{sd}^k \tilde{\gamma}^k \tilde{F}_{dd}^k \tilde{\gamma}^k \tilde{g}_d^k, \tag{3.38}$$

$$L_{k+1}\,\tilde{g}^{k+1} = \tilde{g}_s^k, \quad H_{k+1}\,\tilde{g}^{k+1} = \tilde{g}_d^k, \tag{3.39}$$

$$L_{k+1}\,\tilde{E}^{k+1}\,L_{k+1}^T = \tilde{E}_{ss}^k, \quad L_{k+1}\,\tilde{E}^{k+1}\,H_{k+1}^T = \tilde{E}_{sd}^k, \tag{3.40}$$

$$H_{k+1}\,\tilde{E}^{k+1}\,L_{k+1}^T = \tilde{E}_{ds}^k, \quad H_{k+1}\,\tilde{E}^{k+1}\,H_{k+1}^T = \tilde{E}_{dd}^k, \tag{3.41}$$

$$L_{k+1}\,\tilde{F}^{k+1}\,L_{k+1}^T = \tilde{F}_{ss}^k, \quad L_{k+1}\,\tilde{F}^{k+1}\,H_{k+1}^T = \tilde{F}_{sd}^k, \tag{3.42}$$

$$H_{k+1}\tilde{F}^{k+1}L_{k+1}^{T}=\tilde{F}_{ds}^{k},\quad H_{k+1}\tilde{F}^{k+1}H_{k+1}^{T}=\tilde{F}_{dd}^{k},\tag{3.43}$$

$$\overset{\sim}{\gamma}^{k}=(I-\tilde{E}_{dd}^{k})^{-1}.\tag{3.44}$$

These notations are defined analogous to those in (3.16)–(3.21). We denote $\tilde{E}^{m}=E^{m}$, $\tilde{F}^{m}=F^{m}$, $\tilde{s}^{m}=s^{m}$, $\tilde{g}^{m}=g^{m}$. Let $\overline{s}^{k}$ be the reconstructive result of $\overline{s}^{k-1}$ and $\overline{d}^{k-1}$, that is

$$\overline{s}^{k}=L_{k}^{T}\overline{s}^{k-1}+H_{k}^{T}\overline{d}^{k-1}.\tag{3.45}$$

When $k=m_1$, we define $\overline{s}^{m_1}=\tilde{s}^{m_1}$. Let $\overline{d}^{k}$ be the solution of the function

$$\overline{d}^{k}=(\tilde{E}_{ds}^{k}+\tilde{F}_{ds}^{k})\overline{s}^{k}+(\tilde{E}_{dd}^{k}+\tilde{F}_{dd}^{k})\overline{d}^{k}+\tilde{g}_{d}^{k}\tag{3.46}$$

Thus we obtain the following algorithm.

Algorithm.

Step 1. Compute previously g^m, E^m and F^m (see (3.8)), where m is large enough for our purpose. Let $k=m$

Step 2. Decompose $\tilde{g}^{k}$ by using (3.17) to obtain $\tilde{g}_{s}^{k-1}$ and $\tilde{g}_{d}^{k-1}$. Decompose $\tilde{E}^{k}$ and $\tilde{F}^{k}$ by using (3.40) – (3.43) to obtain $\tilde{E}_{l}^{k-1}$, $\tilde{F}_{l}^{k-1}$ when l represents ss, sd, ds, dd respectively.

Step 3. Compute $\tilde{E}^{k-1}$, $\tilde{F}^{k-1}$, $\tilde{g}^{k-1}$ using (3.35),(3.36) and (3.38) respectively, and $\overset{\sim}{\gamma}^{k-1}:=(I-\tilde{E}_{dd}^{k-1})^{-1}$. Let $k-1\to k$.

Step 4. l Goto Step 2 until $k=m_1:=[\frac{2}{3}m]$.

Step 5. Solve $\overline{s}^{k}$ from the equation

$$\overline{s}^{k}=(\tilde{E}^{k}+\tilde{F}^{k})\overline{s}^{k}+\tilde{g}^{k}\tag{3.47}$$

using any solver such as Gaussian Elimination.

Step 6. Using the iterative method to solve $\overline{d}^{k}$. The iterative procedure from former value $\overline{d}_{-}^{k}$ to new value $\overline{d}_{+}^{k}$ is as follows :

$$\overline{d}_{+}^{k}=\overset{\sim}{\gamma}^{k}(\tilde{E}_{ds}^{k}+\tilde{F}_{ds}^{k})\overline{s}^{k}+\overset{\sim}{\gamma}^{k}\tilde{F}_{dd}^{k}\overline{d}_{-}^{k}+\overset{\sim}{\gamma}^{k}\tilde{g}_{d}^{k}\tag{3.48}$$

where the initial value may be chosen arbitrarily.

Step 7. Reconstruct $(\tilde{s}^{k},\tilde{d}^{k})\to\overline{s}^{k+1}$ using (3.45). Let $k+1\to k$.

Step 8. Goto Step 6 until $k=m$.

Step 9. Compute the approximate solution of (3.2) using (3.3) with $\overline{s}^m$ instead of $\tilde{s}^m$.

4. Convergence of Galerkin Approximation

In this section, we will discuss the difference between the solution of (3.2) and the solution of (3.1). For our purpose, we quote a lemma from [20]. The norm used here is $\overset{\circ}{L}_2$-norm.

Lemma 4.1. *For $f \in \overset{\circ}{C}^s [0,2\pi]\times[0,2\pi], s \le n$ where n is the degree of B-spline function, f_m be the projection of f onto $V_m \otimes V_m$, then we have the following estimation*

$$\|f - f_m\| \le C h_m^s \tag{4.1}$$

where C is a constant.

Lemma 4.2. *For $u \in \overset{\circ}{C}^s [0,2\pi], s \le n$, let $(T_1 u)(x) = \int_0^{2\pi} \log|2\sin\frac{x-y}{2}|u(y)dy$, then*

$$\|T_1 u - P_m T_1 u\| \le C h_m^s \tag{4.2}$$

where C is an absolute constant depending on u.

Proof. Denote $\tilde{u}_m(x,y)$ the projection of $u(x-y)$ onto $V_m \otimes V_m$, that is,

$$\tilde{u}_m(x,y) = \sum_{i,j} u_{ij} A_i^{n,m}(x)\overline{A_j^{n,m}(y)},$$

where $u_{ij} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} u(x-y)\overline{A_i^{n,m}(x)}A_i^{n,m}(y)dxdy$. Then

$$\begin{aligned} \|T_1 u - P_m T_1 u\| &\le \|\textstyle\int_0^{2\pi} \log|2\sin\frac{y}{2}|(u(x-y) - \tilde{u}_m(x,y))dy\| \\ &\le C\|u(x-y) - \tilde{u}_m(x,y)\| \\ &\le C' h_m^s, \end{aligned} \tag{4.3}$$

where the last conclusion in (4.3) can be obtained from Lemma 4.1.

Theorem 4.3. *Assume u_m be the solution of (3.2) , u be the solution of (3.1). We also assume $I-T$ has bounded inverse and $b \in \overset{\circ}{C}^s [0,2\pi]\otimes[0,2\pi], g \in \overset{\circ}{C}^s [0,2\pi], u \in \overset{\circ}{C}^s [0,2\pi]$, where $s \le n$. Then*

$$\|u - u_m\| \le C h_m^s \tag{4.4}$$

where C is an constant independent of n.

Proof. Because u and u_m satisfies (3.1) and (3.2) respectively, such that

$$\begin{aligned} u - u_m &= Tu - P_mTu_m + g - P_mg \\ &= Tu - P_mTu + P_mTu - P_mTu_m + g - P_mg. \end{aligned}$$

From a result in [12] (Th. 10.1 P.142), we know $I - P_mT$ is invertible for large m, and for all large m, their inverse have same bound. By using Th. 2.8, Lemma 4.1 and Lemma 4.2, we have

$$\|u - u_m\| = \|(I - P_mT)^{-1}[(Tu - P_mTu) + (g - P_mg)]\| \leq Ch_m^s \tag{4.5}$$

where we use the smoothness of $b(x, y)$ in applying Lemma 4.1. The theorem follows immediately.

Remark. The assumption that $I - P_mT$ is invertible for large m can be found in [12].

5. Error Analysis

In this section, we will analyze the computational error in the algorithm. We will use all the notations defined in previous sections.

Let s^m be the true solution of the equation

$$s^m = (E^m + F^m)s^m + g^m, \tag{5.1}$$

where E^m, F^m and g^m are defined in (3.8).

Let $\overline{s}^m$ be the approximate solution of s^m, which is defined in (3.45).

We only need to consider the error $\|s^m - \overline{s}^m\|$, where the norm is l^2-norm. Then from the orthonormality of discrete periodic spline wavelet transform, we have

$$\begin{aligned} \|s^m - \overline{s}^m\|^2 &= \|L_m^T(s_{m-1}^m - \overline{s}^{m-1}) + H_m^T(d_{m-1}^m - \overline{d}^{m-1})\|^2 \text{ (from (3.45))} \\ &= \|s_{m-1}^m - \overline{s}^{m-1}\|^2 + \|d_{m-1}^m - \overline{d}^{m-1}\|^2, \end{aligned} \tag{5.2}$$

where s_{m-1}^m satisfies (3.22). Now we want to estimate $s^m - \overline{s}^m$ by using iterative method, that is, we will estimate $s^{m-1} - \overline{s}^{m-1}$. For our purpose, we have to establish some propositions.

Lemma 5.1. *Under the condition of Theorem 4.3, we have for all $k > 0$*

$$\|E_{\&}^{k-1}\| \leq M_e h_{k-1}, \tag{5.3}$$

$$\|F_{\&}^{k-1}\| \leq M_f h_{k-1}^s, \tag{5.4}$$

for $\& = sd, ds, dd$, where M_e and M_f are constants independent of k.

Proof. (5.4) is obviously by using lemma 4.1 because we have the following inequality

$$\|F_{\&}^{k-1}\| \leq \|b_k(x,y) - b_{k-1}(x,y)\|$$

for $\& = sd, ds, dd$, where $b_k(x,y)$ is the projection of $b(x,y)$ onto $V_k \otimes V_k$.

For (5.3), we only prove the lemma for $\& = sd$, others are similar.

Here the norm of a matrix is defined by it's operator norm while the norm of a vector is l^2-norm. Then from classical linear algebraic theory, the norm of a matrix is the radius of it's spectrum, i.e., for a matrix A,

$$\|A\| = \max\{\lambda | \lambda \text{ is a eigenvalue of } A\}$$

From the (3.7) and (3.8), we have

$$\begin{aligned} E_{sd}^{k-1}(j,j) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} a(x-y) A_j^{n,k}(x) \overline{D_j^{n,k}(y)} dx dy \\ &= \frac{a_j^k b_j^k}{2\pi} \int_0^{2\pi} \int_0^{2\pi} a(x-y) (A_j^{n,k+1}(x) \overline{A_j^{n,k+1}(y)} \\ &\qquad - A_{j+K(k-1)}^{n,k+1}(x) \overline{A_{j+K(k-1)}^{n,k+1}(y)}) dx dy. \end{aligned} \tag{5.5}$$

Using the same method in Theorem 2.10, we can prove

$$\left| \int_0^{2\pi} \int_0^{2\pi} a(x-y) A_j^{n,k}(x) \overline{A_j^{n,k}(y)} dx dy \right| \leq C_0 \frac{1}{j}, \tag{5.6}$$

and similarly,

$$\left| \int_0^{2\pi} \int_0^{2\pi} a(x-y) A_{j+K(k-1)}^{n,k}(x) \overline{A_{j+K(k-1)}^{n,k}(y)} dx dy \right| \leq C_0 \frac{1}{K(k-1) - j}, \tag{5.7}$$

for $0 < j < K(k-1)$.

From (2.32), (2.33), (5.6) and (5.7), we have for $0 < j < K(k-1)$,

$$\begin{aligned} E_{sd}^{k-1}(j,j) &\leq C\frac{K(k-1)-j}{K(k)}\frac{j}{K(k)}(\frac{1}{j}+\frac{1}{K(k-1)-j}) \\ &\leq C\frac{1}{K(k)} \leq M_e h_{k-1}. \end{aligned}$$

For $j = 0$, it is true as well. Hence we have $\|E_{sd}^{k-1}\| \leq \max_{0\leq j<K(k-1)}\{E_{sd}^{k-1}(j,j)\}$. The result is proved.

In this section, all conclusions are under the condition in Theorem 4.3 if we do not propose new conditions.

Beside $E_{\&}^k$ and $F_{\&}^k$, we should also estimate the norm of $\tilde{E}{}_{\&}^{k}$ and $\tilde{F}{}_{\&}^{k}$ for & represents sd, ds, dd respectively.

Lemma 5.2. *For M_e in Lemma 5.1, suppose m_0 satisfies that for $k \geq m_0$*

$$M_e h_k \leq 0.5. \tag{5.8}$$

Then when m is large enough, i.e. $3m_0 \leq m$, then for $[\frac{2m}{3}] \leq k < m$ and $\& = sd, ds, dd$

$$\| \tilde{E}{}_{\&}^{k} \| \leq 2^{m-k-1} M_e h_k, \tag{5.9}$$

$$\| \tilde{F}{}_{\&}^{k} \| \leq 2^{m-k-1} M_f h_k, \tag{5.10}$$

Denote $\overset{\sim}{\gamma}{}^{k} = (I - \tilde{E}{}_{dd}^{k})^{-1}$, then from (5.8) and (5.9),

$$\| \overset{\sim}{\gamma}{}^{k} \| \leq 2. \tag{5.11}$$

Proof. We will use induction method to verify the lemma. For $k = m-1$, from the notation in (5.8)–(5.11), we have $\tilde{E}{}_{\&}^{m-1} = E_{\&}^{m-1}$ and $\tilde{F}{}_{\&}^{m-1} = F_{\&}^{m-1}$, hence from Lemma 5.1, the conclusion has been proved. Assume that for k, the lemma is valid. Now we prove (5.9) for $\& = sd$, others are similar.

From the definition of $\overset{\sim k}{E}_{sd}$ in (5.8), we have

$$\|\overset{\sim k-1}{E}_{sd}\| = \|L_k \overset{\sim k}{E} H_k^T\|$$

$$= \|L_k \overset{\sim k}{E}_{ss} H_k^T + L_k \overset{\sim k}{E}_{sd} \gamma^k \overset{\sim k}{E}_{ds} H_k^T\| \quad \text{(by (3.35)}$$

$$\leq \|L_k \overset{\sim k}{E}_{ss} H_k^T\| + \|\overset{\sim k}{E}_{sd}\|\|\overset{\sim k}{\gamma}\|\|\overset{\sim k}{E}_{ds}\| (\text{since } \|L_k\| \leq 1,\ \|H_k\| \leq 1)$$

$$\leq \|L_k \overset{\sim k}{E}_{ss} H_k^T\| + 2(2^{m-k-1} M_e h_k)^2 \quad \text{(by assumption of induction)}$$

$$\leq \|L_k L_{k+1} \overset{\sim k+1}{E} L_{k+1}^T H_k^T\| + 2(2^{m-k-1} M_e h_k)^2 \quad \text{(from (3.40))}$$

$$\leq \|L_k L_{k+1} \overset{\sim k+1}{E}_{ss} L_{k+1}^T H_k^T\| + 2(2^{m-k-2} M_e h_{k+1})^2 + 2(2^{m-k-1} M_e h_k)^2$$

$$\leq \|L_k \cdots L_m E^m L_m^T \cdots L_{k+1}^T H_k^T\| + 4(2^{m-k-1} M_e h_k)^2$$

$$\leq \|L_k E^k H_k^T\| + 4(2^{m-k-1} M_e h_k)^2 \leq \|E_{sd}^{k-1}\| + 4(2^{m-k-1} M_e h_k)^2$$

$$\leq M_e h_{k-1} + (2^{m-k-1} M_e h_{k-1})^2 \quad \text{(from Lemma 5.1)}$$

$$= M_e h_{k-1}(1 + 2^{m-k-1} M_e h_{k-1} 2^{m-k-1}) = M_e h_{k-1}(1 + M_e h_{2k-m} 2^{m-k-1})$$

$$\leq M_e h_{k-1}(1 + 2^{m-k-1}) \leq 2^{m-k} M_e h_{k-1}.$$

(5.9) follows.

The proof of (5.10) is similar. (5.11) is the immediate result of the condition and (5.9).

Because our algorithm omits ρ^k at each level, we should estimate ρ^k. From Lemma 5.1 and (3.37) it is immediately to obtain

Lemma 5.3. *For $m \geq 3m_0$, then*

$$\|\rho^{m-1}\| \leq 8M_f^2 \|s^m\| h_{m-1}^{2s}, \tag{5.12}$$

where m_0 is defined in Lemma 5.2 and M_f is in Lemma 5.1.

In each step of the algorithm, we should ensure that the matrices $I - E^k - F^k$ is invertible. This is the following lemma.

Lemma 5.4. *Suppose that $I-T$ has bounded inverse, and assume that for $k \geq 2m_0$, $I - E^k - F^k$ is invertible. Then for $[\frac{2}{3}m] < k < m$, we have*

$$\|(I - \overset{\sim k}{E} - \overset{\sim k}{F})^{-1}\| \leq M_c, \tag{5.13}$$

where M_c is independent of k, m.

Proof. Under the condition of the lemma, We assume that for $k \geq 2m_0$, there exists a constant M,

$$\|(I - E^k - F^k)^{-1}\| \leq M.$$

From Lemma 5.2,

$$\begin{aligned}
&\| \overset{\sim k}{E} - E^k\| = \| \overset{\sim k}{E}_{ss} + \overset{\sim k}{E}_{sd} \gamma^k \overset{\sim k}{E}_{ds} - E^k\| \quad \text{(from (3.35)}\\
&\leq \| \overset{\sim k}{E}_{ss} - E^k\| + \| \overset{\sim k}{E}_{sd} \| \| \| \overset{\sim k}{\gamma} \| \| \| \overset{\sim k}{E}_{ds} \| \\
&\leq \| \overset{\sim k}{E}_{ss} - E^k\| + 2(2^{m-k-1} M_e h_k)^2 \quad \text{(from Lemma 5.2)}\\
&\leq \|L_{k+1} \overset{\sim k+1}{E} L_{k+1}^T - E^k\| + 2(2^{m-k-1} M_e h_k)^2 \quad \text{(from (3.40))}\\
&\leq \|L_{k+1} \overset{\sim k+1}{E}_{ss} L_{k+1}^T - E^k\| + 2(2^{m-k-2} M_e h_{k+1})^2 + 2(2^{m-k-1} M_e h_k)^2\\
&\leq \|L_{k+1} \cdots L_m \overset{\sim m}{E} L_m^T \cdots L_{k+1}^T - E^k\| + 4(2^{m-k-1} M_e h_k)^2\\
&\leq \|E^k - E^k\| + 4(2^{m-k-1} M_e h_k)^2 \quad (\overset{\sim m}{E} = E^m).
\end{aligned}$$

For $\frac{2}{3}m < k$, we have $\lim_{k\to\infty} \|E^k - \overset{\sim k}{E}\| = 0$. Similarly, we can prove that $\lim_{k\to\infty} \|F^k - \overset{\sim k}{F}\| = 0$, i.e.

$$\lim_{k\to\infty} \|E^k + F^k - \overset{\sim k}{E} - \overset{\sim k}{F}\| = 0.$$

(5.13) follows immediately.

The most feature of our algorithm is throwing away ρ^k at each step so that we can reduce the complexity. We will prove in Lemma 5.6 that all thrown ρ^k are very small.

Denote $\tilde{s}{}^{k}_{k-1} = L_k \tilde{s}{}^{k}$,

Lemma 5.5. *For all $2m_0 < \frac{2}{3}m < k < m$, there exists a constant M_s independent of k, such that*

$$\| \tilde{s}{}^{k} \| \leq M_s, \tag{5.14}$$

when the smoothness degree $s \geq 1$.

Proof.

$$\begin{aligned}
\| \tilde{s}{}^{k} \| &\leq \| \tilde{s}{}^{k} - \tilde{s}{}^{k+1}_{k} \| + \| \tilde{s}{}^{k+1}_{k} \| \\
&\leq M_c \|\rho^k\| + \| \tilde{s}{}^{k+1}_{k} \| \quad \text{(from (3.27))} \\
&\leq [4M_c M_f^2 h_k^{2s} 2^{2m-2k-2}(1 + 2M_e 2^{m-k-1} h_k) + 1] \| \tilde{s}{}^{k+1} \| \text{(from (3.37))} \\
&\leq (1 + 8M_c M_f^2 h_k^{2s} 2^{2m-2k-2}) \| \tilde{s}{}^{k+1} \| \\
&\leq \Pi_{j=k}^{m-1} (1 + 8M_c M_f^2 h_j^{2s} 2^{2m-2j-2}) \| \tilde{s}{}^{m} \| \\
&\leq \exp\{\sum_{j=k}^{m} 8M_c M_f^2 2^{2m-2j-2} h_j^{2s}\} \|s^m\| \\
&\leq \exp\{M_1 2^{2m-2k-2sk}\} \|s^m\| \\
&\leq \exp\{M_1 2^{2m-2k-2sk}\} M_c \|g^m\| \\
&\leq M_2 \exp\{M_1 2^{2m-2k-2sk}\}.
\end{aligned}$$

When $k \geq \frac{2}{3}m$ and $s \geq 1$, $2m - 2k - 2sk < 0$, hence the result is immediate.

From Lemma 5.5 and Lemma 5.2, we have

Lemma 5.6.

$$\|\rho^k\| \leq M_\rho 2^{2m-2k} h_k^{2s}, \tag{5.15}$$

for $m_0 < k < m$.

We now turn to our main goal, we shall estimate the error of the algorithm.

Let s^m be the true solution of linear system

$$s^m = (E^m + F^m) s^m + g^m, \tag{5.16}$$

and $\overline{s}^m$ be defined as (3.45) where $k = m$.

We obtain our main conclusion.

Theorem 5.7. *For $m \geq 3m_0$, we have*

$$\|s^m - \overline{s}^m\| \leq M_a h_m^s, \tag{5.17}$$

where M_a is a constant independent of m

Proof. Let $\delta^k = (I - \overset{\sim k}{E}_{dd} - \overset{\sim k}{F}_{dd})^{-1}$, and $\xi^k = \overset{\sim k}{E}_{ds} + \overset{\sim k}{F}_{ds}$, and $q^k = (1 + \|\delta^k\|\|\xi^k\|)$, then

$$\begin{aligned}
\|s^m - \overline{s}^m\| &\leq \|s_{m-1}^m - \overline{s}^{m-1}\| + \|d_{m-1}^m - \overline{d}^{m-1}\| \\
&\leq (1 + \|\delta^{m-1}\|\|\xi^{m-1}\|)(\|s_{m-1}^m - \overset{\sim m-1}{s}\| + \|\overset{\sim m-1}{s} - \overline{s}^{m-1}\|) \\
&\leq q^{m-1}(\|\overset{\sim m-1}{s} - \overline{s}^{m-1}\| + M_c\|\rho^{m-1}\|) \quad \text{(from (3.27),(3.32))} \\
&\leq M_c \sum_{k=m_1}^{m-1} \Pi_{l=k}^{m-1}\|q^l\|\|\rho^k\| \leq M_1 M_c \sum_{k=m_1}^{m-1} \|\rho^k\| \quad \text{(from Lemma 5.2)} \\
&\leq M_2 h_{m_1}^{2s} 2^{2m-2m_1} \leq M_a h_m^s 2^{2m-2m_1+sm-2sm_1}.
\end{aligned}$$

If $s \geq 2$, we can see (5.30) is true.

Define

$$\overline{u}_m = \sum_{\nu=0}^{K(m)-1} \overline{s}^m A_\nu^{n,m}$$

Because the members of s^k are the coefficients of u_k on the basis of V_k, we have the following conclusion.

Corollary 5.8. *Assume that $\overline{u}_m$ be the approximate solution using the algorithm. For large m, and $s \geq 2$, there exists a constant M independent of m*

$$\|u - \overline{u}_m\| \leq M h_m^s \tag{5.18}$$

Thus the estimation of the error of the algorithm is completed.

Remark. From the error analysis above and the description in section 3.4, we can see that the complexity of our algorithm is $O(K(m)^2)$, where m is the highest level.

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