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ON THE CONVERGENCE OF KING-WERNER ITERATION METHOD IN BANACH SPACE*1)

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Abstract

In this paper , a Kantorovitch-Ostrowski type convergence theorem and an error estimate of $\dfrac{\|f'(z_0)^{-1}f(x_{n+1})\|}{\|f'(z_0)^{-1}f(x_n)\|}$ using the information of higher derivatives at the center between initial points for King-Werner iteration method in Banach space are established.

Key words: Information at the center between initial points, King-Werner iteration method, Convergence, Error estimate.

1. Introduction

Let

$$f(x) = 0 \tag{1.1}$$

where $f : X \to Y$ is a nonlinear operator which maps Banach space X into Banach space Y. The well-known iteration methods for solving (1.1) are the Newton method and very kinds of its improvement methods. One of them is the so called King-Werner method defined by

$$kw(P, x_0, y_0): \quad \begin{cases} z_n = \dfrac{x_n + y_n}{2} \\ x_{n+1} = x_n - f'(z_n)^{-1} f(x_n) \\ y_{n+1} = x_{n+1} - f'(z_n)^{-1} f(x_{n+1}) \end{cases} \quad \forall n \in N_0, \tag{1.2}$$

which is established by King in [7], Werner in [12] in different formulas, respectively. It is interesting that the method (1.2) is of order $1 + \sqrt{2}$ with the same function computation cost and two times combination cost as that of Newton method. Define

$$\omega(x, z) = x - f'(z)^{-1} f(x),$$

then (1.2) can be rewritten as

$$kw(P, x_0, y_0): \quad \begin{cases} z_n = \dfrac{x_n + y_n}{2} \\ x_{n+1} = \omega(x_n, z_n) \\ y_{n+1} = \omega(x_{n+1}, z_n) \end{cases} \quad \forall n \in N_0. \tag{1.3}$$

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There are a number of papers concerning the convergence of Newton method and its improvement methods under the condition of Kantorovitch theorem or relatively close ones (e.g.[2],[6],[8]-[11],[13] etc.). In [4] [5], Kantorovich type convergence theorems and estimates of Newton method and two Newton-like methods using higher derivatives information are proved, respectively, if f has higher derivatives, though they are not used in iteration process. The idea of using higher derivatives at initial points is also used for Halley method in [14], and for a class of parameter based Chebyshev-Halley type methods in [3], where the higher derivatives are used in iteration process.

In this paper, a convergence theorem of Kantorovitch-Ostrowski type using higher derivatives at the center between initial points for King-Werner method (1.2) is established. Also, an error estimate of the decreasing speed of $\dfrac{\|f'(z_0)^{-1}f(x_{n+1})\|}{\|f'(z_0)^{-1}f(x_n)\|}$ is obtained. We put forth the main results in §2 and give the proofs and an example in §3.

2. Main Results

Define $\overline{O(z,t)} = \{x \in X | \|x-z\| \le t\}$, $O(z,t) = \{x \in X | \|x-z\| < t\}$, where $z \in X$.

Theorem 2.1. *Let X, Y be Banach spaces, $f : X \to Y$ have first- and second-order Frechet derivatives, which are bounded linear operators from X to Y and X to $L(X,Y)$, respectively. Suppose $x_0, y_0 \in D \subset X$, a convex subset of X, $z_0 = \frac{x_0+y_0}{2}$, and*

$$\begin{aligned} &\|x_0 - y_0\| \le \tau, \qquad \|x_1 - y_0\| \le \eta, \\ &\|f'(z_0)^{-1} f''(z_0)\| \le \gamma, \\ &\|f'(z_0)^{-1}[f''(x) - f''(y)]\| \le K\|x-y\| \qquad \forall x, y \in D. \end{aligned}$$

If $\overline{O(z_0, t^* - \frac{\tau}{2})} \subset D$,

$$3(\eta + \psi(\tau))\gamma \le \frac{\gamma + 2\sqrt{\gamma^2+2K}}{\gamma + \sqrt{\gamma^2+2K} + K} \tag{2.1}$$

and

$$\frac{K}{2}(\frac{\tau}{2} + \eta)^2 + \gamma(\frac{\tau}{2} + \eta) - 1 < 0, \tag{2.2}$$

where $\psi(\tau) = \frac{1}{48}K\tau^3 - \frac{1}{8}\gamma\tau^2 + \frac{1}{2}\tau$, then

i) *the sequence $kw(f; x_0, y_0)$ defined by (1.2) starting from x_0, y_0 converges to the unique solution of $f(x)$ in $\overline{O(z_0, t^* - \frac{\tau}{2})} \cup O(z_0, t^{**} - \frac{\tau}{2}) \cap D$, where $0 < t^* \le t^{**}$ are two positive zeros of the polynomial*

$$\phi(t) = \frac{K}{6}(t - \frac{\tau}{2})^3 + \frac{1}{2}\gamma(t - \frac{\tau}{2})^2 - (t - \frac{\tau}{2}) + \frac{\tau}{2} + \eta - \frac{\gamma}{8}\tau^2 + \frac{K}{48}\tau^3. \tag{2.3}$$

ii)

$$\|x_n - x^*\| \le t^* - t_n \qquad \|x^* - y_n\| \le t^* - s_n \qquad \forall n \in N_0$$

where $t_n, s_n \in kw(\phi; 0, \tau)$*, which is defined by*

$$kw(\phi;0,\tau): \quad \begin{cases} r_n = \dfrac{t_n + s_n}{2} \\ t_{n+1} = t_n - \phi'(r_n)^{-1}\phi(t_n) \\ s_{n+1} = t_{n+1} - \phi'(r_n)^{-1}\phi(t_{n+1}) \end{cases} \tag{2.4}$$

$t_0 = 0, s_0 = \tau, t_1 = \tau + \eta.$

When $\tau = 0$, that is, $kw(P; x_0, y_0)$ is generated from one point $x_0 = y_0$, it follows that

Theorem 2.2. *Let X, Y be Banach spaces, $f : X \to Y$ have first- and second-order Frechet derivatives, which are bounded linear operators from X to Y and X to $L(X, Y)$, respectively. Suppose $x_0 \in D \subset X$, a convex subset of X, and*

$$\|f'(x_0)^{-1}f(x_0)\| \le \eta, \qquad \|f'(x_0)^{-1}f''(x_0)\| \le \gamma,$$
$$\|f'(x_0)^{-1}[f''(x) - f''(y)]\| \le K\|x - y\| \qquad \forall x, y \in D.$$

If $\overline{O(x_0, t^*)} \subset D$,

$$3\eta\gamma \le \frac{\gamma + 2\sqrt{\gamma^2 + 2K}}{\gamma + \sqrt{\gamma^2 + 2K} + K} \tag{2.5}$$

then

i) *the sequence $kw(f; x_0, x_0)$ defined by (1.2) starting from x_0 converges to the unique solution of $f(x)$ in $\overline{O(x_0, t^*)} \cup O(x_0, t^{**}) \cap D$, where $0 < t^* \le t^{**}$ are two positive zeros of the polynomial*

$$\phi_0(t) = \frac{K}{6}t^3 + \frac{1}{2}\gamma t^2 - t + \eta. \tag{2.6}$$

ii)

$$\|x_n - x^*\| \le t^* - t_n \qquad \|x^* - y_n\| \le t^* - s_n \qquad \forall n \in N_0$$

where $t_n, s_n \in kw(\phi_0; 0, 0)$*, which is defined by*

$$kw(\phi_0;0,\tau) \quad \begin{cases} r_n = \dfrac{t_n + s_n}{2} \\ t_{n+1} = t_n - \phi_0'(r_n)^{-1}\phi_0(t_n) \\ s_{n+1} = t_{n+1} - \phi_0'(r_n)^{-1}\phi_0(t_{n+1}). \end{cases} \tag{2.7}$$

The condition in Theorem 2.2 is just the same as that one obtained for Newton method in [4]. Note that when $\tau = 0$, (2.2) is satisfied, if (2.1) is satisfied (We shall point out in an example that this is not true for $\tau \ne 0$). So, Theorem 2.2 is a special case of Theorem 2.1, and we only proof the Theorem 2.1 in §3.

Further more, we get the following theorem

Theorem 2.3. *Under the condition of Theorem 2.1, we have*

$$\frac{\|f'(z_0)^{-1}f(x_{n+1})\|}{\|f'(z_0)^{-1}f(x_n)\|} \le \frac{\phi(t_{n+1})}{\phi(t_n)} \le \frac{t_{n+2} - t_{n+1}}{t_{n+1} - t_n} \qquad \forall n \in N_0.$$

which says that the error estimates of $\|f'(z_0)^{-1}f(x_n)\|$ can be controlled by itself step by step.

3. Proofs

At first, we use the King-Werner method to determine the smaller positive zero of polynomial (2.3). We have

Lemma 3.1. *Under (2.1) and (2.2), the polynomial (2.3) has two positive zeros satisfying $0 < t^* \le t^{**}$.*

The proof of Lemma 3.1 is as similar as that in [4-5], and is omitted.

Lemma 3.2. *If condition (2.1) and (2.2) are satisfied, $kw(\phi; 0, \tau)$ is defined by (2.4), then*

$$0 = t_0 < s_0 < t_1 < \cdots < t_n < s_n < \cdots < t^* \le t^*$$

and

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = t^* \tag{3.1}$$

Proof. (2.1) and (2.2) imply that $\phi(\tau+\eta) > 0$ and $\phi'(\tau+\eta) < 0$, that is, $\tau+\eta < t^*$. So, (3.1) follows by induction.

Lemma 3.3. *If $x \in D$, then*

$$\begin{aligned} &\|f'(z_0)^{-1}f''(x)\| \le \phi''(t), \qquad && if \quad \|x - z_0\| \le t - \frac{\tau}{2} < t^{**} - \frac{\tau}{2}, \\ &\|[f'(z_0)^{-1}f'(x)]^{-1}\| \le -\phi'(t)^{-1}, && if \quad \|x - z_0\| \le t - \frac{\tau}{2} < t^{*} - \frac{\tau}{2}. \end{aligned} \tag{3.2}$$

Proof. In fact,

$$\begin{aligned} \|f'(z_0)^{-1}f''(x)\| &\le \|f'(z_0)^{-1}f''(z_0)\| + \|f'(z_0)^{-1}[f''(x) - f''(z_0)]\| \\ &\le \phi''(t) \qquad\qquad (\frac{\tau}{2} \le t < t^{**}). \end{aligned}$$

Since $\phi'(t) < 0$ on $[0, t^*)$, we get

$$\begin{aligned} \|I - f'(z_0)^{-1}f'(x)\| &\le \int_0^1 K\theta d\theta \|x - z_0\|^2 + \gamma\|x - z_0\| \\ &\le 1 + \phi'(t) < 1 \quad (\frac{\tau}{2} \le t < t^*), \end{aligned}$$

so that $[f'(z_0)^{-1}f'(x)]^{-1}$ exits and

$$\|[f'(z_0)^{-1}f'(x)]^{-1}\| \le \frac{1}{1 - \|I - f'(z_0)^{-1}f'(x)\|} \le -\phi'(t)^{-1} \quad (\frac{\tau}{2} \le t < t^*)$$

holds by Neumann's Theorem.

Lemma 3.4. *If $x, y, u, v \in D$ satisfy*

$$\|x - x_0\| \le t, \|y - y_0\| \le s - \tau,$$
$$\|u - x\| \le p - t, \|v - y\| \le q - s,$$

where $0 < t < s < p < q < t^*$, then

$$\|f(z_0)^{-1}[f'(\frac{x+y}{2}) - f'(\frac{u+v}{2})]\| \le -[\phi'(\frac{s+t}{2}) - \phi'(\frac{p+q}{2})]. \tag{3.4}$$

Proof. Since

$$f'(\frac{x+y}{2}) - f'(\frac{u+v}{2}) = A + B,$$

where

$$A = -\int_0^1 [f''(\frac{x+y}{2} + \theta(\frac{u+v-x-y}{2})) - f''(\frac{x+y}{2})]d\theta(\frac{u+v-x-y}{2}),$$
$$B = -f''(\frac{x+y}{2})(\frac{u+v-x-y}{2}).$$

So,

$$\begin{aligned}\|f'(z_0)^{-1}[f'(\frac{x+y}{2}) - f'(\frac{u+v}{2})]\| &\le \|f'(z_0)^{-1}A\| + \|f'(z_0)^{-1}B\| \\ &\le \int_0^1 K\theta d\theta(\frac{p-t+q-s}{2})^2 + \phi''(\frac{s+t}{2})(\frac{p-t+q-s}{2}) \\ &= -[\phi'(\frac{s+t}{2}) - \phi'(\frac{p+q}{2})]\end{aligned}$$

follows by Lemma 3.3.

For the proof of Theorems, we need the following expansion of $f(u)$: If $x, y \in D, u = \omega(x, \frac{x+y}{2})$, then

$$\begin{aligned}f(u) &= f(u) - f(x) - f'(\frac{x+y}{2})(u-x) \\ &= \int_0^1 f''(y + \theta(u-y))(1-\theta)d\theta(u-y)^2 + \int_0^1 f''(\frac{x+y}{2} + \theta\frac{y-x}{2})d\theta\frac{(y-x)(u-y)}{2} \\ &\quad + \int_0^1 C(x,y)(1-\theta)d\theta(\frac{y-x}{2})^2 + \int_0^1\int_0^1 D(x,y)d\tau(1-\theta)d\theta\frac{(y-x)^2}{2},\end{aligned} \tag{3.5}$$

where

$$C(x,y) = f''(\frac{x+y}{2} + \theta\frac{y-x}{2}) - f''(\frac{x+y}{2}),$$
$$D(x,y) = f''(\frac{x+y}{2}) - f''(x + \frac{\theta - \tau\theta + \tau}{2}(y-x)).$$

Proof of Theorem 1. Let us prove that

$$(\dot{1}) \qquad x_n \in \overline{O(x_0, t_n)} \tag{3.6}$$

$$(\dot{2}) \qquad \|y_n - x_n\| \le s_n - t_n \tag{3.7}$$

$$(\dot{3}) \qquad y_n \in \overline{O(z_0, s_n - \frac{\tau}{2})} \tag{3.8}$$

$$(\dot{4}) \qquad \|[f'(z_0)^{-1} f'(z_n)]^{-1}\| \le -\phi'(r_n)^{-1} \tag{3.9}$$

$$(\dot{5}) \qquad \|x_{n+1} - y_n\| \le t_{n+1} - s_n \tag{3.10}$$

hold for all $x_n, y_n, z_n \in kw(P; x_0, y_0)$, where t_n, s_n, r_n are defined by (2.4). In fact, they hold for $n = 0$. Given they be true for $n = 0, 1, \ldots, k$, then

$$\begin{aligned}
&\|x_{k+1} - x_0\| \le \|x_{k+1} - y_k\| + \|y_k - x_k\| + \|x_k - x_0\| \le t_{k+1}. \\
&\|y_k - x_0\| \le \|y_k - x_k\| + \|x_k - x_0\| \le s_k, \\
&\|y_k + \theta(x_{k+1} - y_k) - z_0\| \le s_k + \theta(t_{k+1} - s_k) - \frac{\tau}{2} \le t^* - \frac{\tau}{2}, \\
&\|z_k - z_0\| \le r_k - \frac{\tau}{2} \le t^* - \frac{\tau}{2}, \\
&\|z_k + \theta\frac{y_k - x_k}{2} - z_0\| \le r_k + \theta\frac{s_k - t_k}{2} - \frac{\tau}{2} \le t^* - \frac{\tau}{2}, \qquad 0 \le \theta \le 1.
\end{aligned} \tag{3.11}$$

By Lemma 3.3 and Lemma 3.4,

$$\begin{aligned}
&\|f'(z_0)^{-1} f(x_{k+1})\| \\
&\quad \le \int_0^1 \|f'(z_0)^{-1} f''(y_k + \theta(x_{k+1} - y_k))\|(1-\theta)d\theta \|x_{n+1} - y_k\|^2 \\
&\quad + \int_0^1 \|f'(z_0)^{-1} f''(z_k + \theta\frac{y_k - x_k}{2})\|d\theta \frac{\|(y_k - x_k)(x_{k+1} - y_k)\|}{2} \\
&\qquad + \int_0^1 \|f'(z_0)^{-1} C(x_k, y_k)\|(1-\theta)d\theta (\frac{\|y_k - x_k\|}{2})^2 \\
&\qquad + \int_0^1\int_0^1 \|f'(z_0)^{-1} D(x_k, y_k)\|d\tau(1-\theta)d\theta \frac{\|y_k - x_k\|^2}{2} \\
&\quad \le \int_0^1 \phi''(s_k + \theta(t_{k+1} - s_k))(1-\theta)d\theta (t_{k+1} - s_n)^2 \\
&\qquad + \int_0^1 \phi''(r_k + \theta\frac{s_k - t_k}{2})d\theta \frac{(s_k - t_k)(t_{k+1} - s_k)}{2} \\
&\qquad + \int_0^1 K\theta(\frac{s_k - t_k}{2})(1-\theta)d\theta (\frac{s_k - t_k}{2})^2 \\
&\qquad + \int_0^1\int_0^1 K\frac{(1-\tau)(1-\theta)}{2}(s_k - t_k)d\tau(1-\theta)d\theta (\frac{s_k - t_k}{2})^2, \\
&\quad = \phi(t_{k+1}),
\end{aligned} \tag{3.12}$$

which yields

$$\begin{aligned}\|y_{k+1}-x_{k+1}\| &\le \|[f'(z_0)^{-1}f'(z_k)]^{-1}\|\|f'(z_0)^{-1}f(x_{k+1})\|\\ &\le -\phi'(r_k)^{-1}\phi(t_{k+1})\\ &= s_{k+1}-t_k. \end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned}\|y_{k+1}-x_0\| &\le \|y_{k+1}-x_n\|+\|x_k-x_0\|\\ &\le s_{k+1}-t_k+t_k-t_0=s_{k+1}\\ &\le t^* \end{aligned} \tag{3.14.}$$

and

$$\|z_{k+1}-z_0\| \le r_{k+1}-\frac{\tau}{2} \le t^*-\frac{\tau}{2}.$$

By Lemma 3.2, we get

$$\|[f'(z_0)^{-1}f'(z_{k+1})]^{-1}\| \le -\phi'(r_{k+1})^{-1}. \tag{3.15}$$

Since

$$\begin{aligned}x_{n+2}-y_{n+1} &= (f'(z_k)^{-1}-f'(z_{k+1})^{-1})f(x_{n+1})\\ &= -[f'(z_0)^{-1}f'(z_k)]^{-1}\cdot f'(z_0)^{-1}[f'(z_k)-f'(z_{k+1})]\\ &\quad \times[-f'(z_{k+1})^{-1}f(x_{k+1})].\end{aligned}$$

By Lemma 3.3, Lemma 3.4, it follows that

$$\begin{aligned}\|x_{n+2}-y_{n+1}\| &\le \|-[f'(z_0)^{-1}f'(z_k)]^{-1}\|\cdot\|f'(z_0)^{-1}[f'(z_k)-f'(z_{k+1})]\|\\ &\quad \times\|-f'(z_{k+1})^{-1}f(x_{k+1})\|\\ &\le -\phi'(r_k)^{-1}\cdot[\phi'(r_k)-\phi'(r_{k+1})]\cdot[-\phi'(r_{k+1})^{-1}\phi(t_{k+1})]\\ &= [\phi'(r_k)^{-1}-\phi'(r_{k+1})^{-1}]\phi(t_{k+1})\\ &= t_{k+2}-s_{k+1}.\end{aligned} \tag{3.16}$$

(3.11),(3.13),(3.14) and (3.15),(3.16) show that (3.6)-(3.10) hold for $n=k+1$. By induction, the proof of (3.6)-(3.10) is completed.

Lemma 3.2 implies that the sequence $kw(\phi;0,\tau)$ is a Cauchy sequence. From (3.6)-(3.10), $kw(f;x_0,y_0)$becomes a Cauchy sequence, too. Then it follows that

$$\exists x^* \in \overline{O(z_0,t^*-\frac{\tau}{2})} \subset \overline{O(x_0,t^*)} \qquad \text{such that} \qquad x_n \to x^*, y_n \to x^*, n\to\infty,$$

and

$$\|x_n-x^*\| \le t^*-t_n, \qquad \|y_n-x^*\| \le t^*-s_n. \tag{3.17}$$

From (3.12), $f(x^*) = 0$. Now to complete the proof of Theorem 1 , it is sufficient to proof that there is an unique solution of (1.1) in $\overline{O(z_0, t^* - \frac{\tau}{2})} \cup O(z_0, t^{**} - \frac{\tau}{2}) \cap D$. Suppose

$$\exists \widetilde{x} \in \overline{O(z_0, t^* - \frac{\tau}{2})} \cup O(z_0, t^{**} - \frac{\tau}{2}) \cap D, \qquad \text{such that} \qquad f(\widetilde{x}) = 0,$$

Let $\xi_\theta := x^* - z_0 + \theta(\widetilde{x} - x^*)$, then for $0 \le \theta \le 1$, holds $\|\xi_\theta\| < t^{**} - \frac{\tau}{2}$. Since

$$\begin{aligned}
&\| \int_0^1 f'(z_0)^{-1} f'(x^* + \theta(\widetilde{x} - x^*)) d\theta - I \| \\
&\le \int_0^1 d\theta \int_0^1 \| f'(z_0)^{-1} f''(z_0 + \kappa \xi_\theta) \| d\kappa \|\xi_\theta\| \\
&< \int_0^1 d\theta \int_0^1 \phi''(\frac{\tau}{2} + \kappa(t^* - \frac{\tau}{2} + \theta(t^{**} - t^*))) d\kappa (t^* - \frac{\tau}{2} + \theta(t^{**} - t^*)) \\
&= \int_0^1 \phi'(t^* + \theta(t^{**} - t^*)) d\theta + 1 = 1.
\end{aligned}$$

So, $[\int_0^1 f'(z_0)^{-1} f'(x^* + \theta(\widetilde{x} - x^*)) d\theta]^{-1}$ exists by Neumann's Theorem, it follows that the operator equation

$$\int_0^1 f'(z_0)^{-1} f'(x^* + \theta(\widetilde{x} - x^*)) d\theta (\widetilde{x} - x^*) = f'(z_0)^{-1}(f(\widetilde{x}) - f(x^*)) = 0$$

has an unique solution $\widetilde{x} - x^* = 0$, that is, $\widetilde{x} = x^*$.

Proof of Theorem 2.3. Using Taylor expansion at x_n, we have

$$f(x_{n+1}) = (E + F) G \cdot [f'(z_0)^{-1} f(x_n)]$$

where

$$\begin{aligned}
E &= \int_0^1 f''(x_n + \theta z_n) d\theta (\frac{y_n - x_n}{2}) \\
F &= \int_0^1 f''(x_n + \theta(x_{n+1} - x_n))(1 - \theta) d\theta (x_{n+1} - x_n) \\
G &= [-f'(z_0)^{-1} f'(z_n)]^{-1}.
\end{aligned}$$

By Lemma 3.2 and the proof of Theorem 1,

$$\begin{aligned}
\|f'(z_0)^{-1} f(x_{n+1})\| &\le \|f'(z_0)^{-1}(E + F) G\| \cdot [f'(z_0)^{-1} f(x_n)] \\
&\le [\int_0^1 \phi''(t_n + \theta r_n) d\theta \frac{s_n - t_n}{2} \\
&\quad + \int_0^1 \phi''(t_n + \theta(t_{n+1} - t_n))(1 - \theta) d\theta (t_{n+1} - t_n)] \\
&\quad \times [-\phi'(r_n)^{-1}] \|f'(z_0)^{-1} f(x_n)\| \\
&= \frac{\phi(t_{n+1})}{\phi(t_n)} \|f'(z_0)^{-1} f(x_n)\|.
\end{aligned}$$

It follows that,

$$\begin{aligned}\frac{\|f'(z_0)^{-1}f(x_{n+1})\|}{\|f'(z_0)^{-1}f(x_n)\|} &\leq \frac{\phi(t_{n+1})}{\phi(t_n)} \qquad (3.18)\\ &= \frac{\phi'(r_{n+1})}{\phi'(r_n)} \cdot \frac{t_{n+2}-t_{n+1}}{t_{n+1}-t_n}\\ &\leq \frac{t_{n+2}-t_{n+1}}{t_{n+1}-t_n}, \qquad (3.19)\end{aligned}$$

which completes the proof by induction.

Example 1. If $\tau \neq 0$, that condition (2.1) is true does not imply that the condition (2.2) is true(It is different from the case $\tau = 0$.).

Let

$$K = 0, \eta = \frac{4-\sqrt{15}}{6}, \gamma = 2, \tau = \sqrt{\frac{5}{3}},$$

we have

$$\gamma\eta + \psi(\tau)\gamma = \frac{1}{2},$$

and

$$\gamma(\frac{\tau}{2} + \eta) - 1 = \frac{1}{3} > 0.$$

That is, the condition (2.1) is true, but the condition (2.2) is not true.

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