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METHOD OF NONCONFORMING MIXED FINITE ELEMENT FOR SECOND ORDER ELLIPTIC PROBLEMS*[1)]

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Abstract

In this paper, the method of non–conforming mixed finite element for second order elliptic problems is discussed and a format of real optimal order for the lowest order error estimate.

Key words: Non–conforming mixed finite element, Error estimate, Second order elliptic problems.

1. Introduction

Recently Hiptmair (see[1]) and Farhloul & Fortin (see[2]) have constructed and analyzed some non–conforming finite element mixed methods for second order elliptic problems:

$$\begin{cases} -\mathrm{div}(a\nabla u) = f, & x \in \Omega, \\ u = 0, & x \in \partial, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n$ $(n = 2, 3)$ is a bounded open field with Lipshitz continuous boundary $\partial\Omega$, f is a given function of the space $L^2(\Omega)$ and $a \in L^\infty(\Omega)$ is assumed to be uniformly positive and bounded:

$$0 < a_1 \le a(x) \le a_2, \ \ x \in \bar{\Omega}. \tag{1.2}$$

Introducing the auxiliary variable $p = a\nabla u$, the problems (1.1) may be written as the system:

$$\begin{cases} p - a\nabla u = 0, & x \in \Omega, \\ \mathrm{div} p = -f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Then the mixed variational formulation of (1.3) is:

Find $(p, u) \in H \times M$ such that

$$\begin{cases} a(p, q) + b(q, u) = 0, & \forall q \in H, \\ b(p, v) = -(f, v), & \forall v \in M. \end{cases} \tag{1.4}$$

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where $H = H(\mathrm{div};\Omega) = \{q \in L^2(\Omega)^n;\ \mathrm{div}q \in L^2(\Omega),\ M = L^2(\Omega),\ a(p,q) = (a^{-1}p,q)$, $b(q,v) = (\mathrm{div}q, v)$ and $(.,.)$ is the inner product in $L^2(\Omega)$ or $L^2(\Omega)^n$.

Let $\Im_h$ be a regular triangulation of $\bar{\Omega}$ (cf.[3]) and P_l be the space of polynomials of degrees less or equal to l (where $l \geq 0$ is an integer). The non–conforming discretization of the problem (1.4), constructed in [1] and [2], is to consider two finite–dimensional spaces H_h and M_h such that

1) There is an integer $k \geq 0$ such that $RT^k(\Im_h) \subset H_h$, where $RT^k(\Im_h)$ is the space of vector field arising from kth order Raviart –Thomas elements (see[4]).

2) The moments up to order $l(l \leq k)$ of the discrete flux are continuous across inter elements boundaries, i.e.

$$\int_e (q_{h|K_i}.n_i + q_{h|K_j} n_j) p_l \mathrm{d}s = 0, \ \ \forall p_l \in P_l.$$

for all internal faces $e = \partial K_i \bigcap \partial K_j (i \neq j)$ and all $q_h \in q_h \in H_h$ (where n_i denotes the unite outward normal on ∂K_i).

3) M_h has to satisfy the following condition: if $\forall q_h \in H_h$ and

$$\sum_{K \in \Im_h} \int_K \mathrm{div} q_h v_h \mathrm{d}x = 0, \ \ \forall v_h \in M_h,$$

then $\mathrm{div} q_{h|K} = 0, \ \ \forall K \in \Im_h$.

The non– conformity of this discretization is due to the fact that the discrete flux is not necessarily continuous across inter element boundaries. Hiptemair (see[1]) has proved the convergence and given error estimates for this non–conforming mixed finite elements for $k \geq l \geq 1$. His analysis is based so–called "Generalized Patch Test" (cf.[5]). Farhloul & Fortin have derived a non–conforming approximation of the lowest order in the two–dimensional case (see[2]). We have found that Farhloul & Fortin's format is not optimal as the approximation of the flux $p_{h|K} \in P_1(K)^2, \ \forall K \in \Im_h$, but its accuracy on L^2 norm is only $O(h)$. One knows that if $H_h \subset L^2(\Omega)^n, \ \forall q_h, p_h \in H_h, \ a(p_h, q_h)$ is continuous. Therefore, the error estimates of non–conforming mixed finite element are due to the estimates causing by bilinear forms $b(.,.)$. But in [1], the estimates of non–conforming element causing by $a(.,.)$ and $b(.,.)$ are all discussed. Thus, much work is in vain because $a(.,.)$ cannot cause the error estimates of non–conforming element.

In this paper, C denotes a positive constant independent of h, but may be inequality in different positions.

2. The Non–Conforming Element Analysis

Let $H_h \not\subset H, M_h$ be satisfied 1)–2) in the section 1. Then the discrete problem of (1.4) reads as follows:

Find $(p_h, u_h) \in H_h \times M_h$ such that

$$\begin{cases} a(p_h, q_h) + b_h(q_h, u_h) = 0, & \forall q_h \in H_h, \\ b_h(p_h, v_h) = -(f, v_h), & \forall v_h \in M_h, \end{cases} \tag{2.1}$$

where

$$b_h(q,v) = \sum_{K\in\Im_h}\int_K \mathrm{div}qv dx,\ \ \forall v\in M, \forall q\in H_h\bigcup H. \tag{2.2}$$

To make sure that problem (2.1) has a unique solution and to estimate its error, we need the following hypotheses.

Hypothesis (N_1). There exists a constant α independent of h such that

$$a(q_h,q_h)\geq \alpha\|q_h\|_h^2, \quad \forall q_h\in V_h, \tag{2.3}$$

where

$$V_h=\{q_h\in H_h;\ \ b_h(q_h,v_h)=0, \forall v_h\in M_h\}=\{q_h\in H_h;\ \ \mathrm{div}q_h=0\}. \tag{2.4}$$

and

$$\|q_h\|_h^2=\sum_{K\in\Im_h}(\|q_h\|_{0,K}^2+\|\mathrm{div}q_h\|_{0,K}^2). \tag{2.5}$$

Hypothesis (N_2). There exists an operator $\pi_h: H\to H_h$ such that $\forall q\in H$,

$$b_h(q-\pi_h q, v_h)=0, \quad \forall v_h\in M_h. \tag{2.6}$$

Hypothesis (N_3). When $\forall q\in H^{k+1}(\Omega)^n, k\geq 1$,

$$\|q-\pi_h q\|_{0,\Omega}\leq Ch^{k+1}\|q\|_{k+1,\Omega}. \tag{2.7}$$

And $\forall v\in H^{l+1}(\Omega)\cap H_0^1(\Omega)$, $l\geq 0$

$$\inf_{v_h\in M_h}\|v-v_h\|_{0,\Omega}\leq Ch^{l+1}\|v\|_{l+1,\Omega}. \tag{2.8}$$

Hypothesis (N_4). When $\forall q_h\in V_h$ and $\forall v\in H^{k+2}(\Omega)\cap H_0^1(\Omega)$,

$$\sum_{K\in\Im_h}\int_{\partial K} q_h n v \mathrm{d}s\leq Ch^{k+1}\|q_h\|_{0,\Omega}|v|_{k+2,\Omega}. \tag{2.9}$$

Theorem 2.1. *If (N_1) and (N_2) all hold, then the problem (2.1) has a unique solution $(p_h,u_h)\in H_h\times M_h$. And if ($N_3$) and ($N_4$) all hold and the solution $(p,u)\in H^{k+1}(\Omega)^n\times H^{k+2}(\Omega)$ of the problem (1.4), the following error estimates hold*

$$\|p-p_h\|_{0,\Omega}\leq Ch^{k+1}(\|p\|_{k+1,\Omega}+\|u\|_{k+2,\Omega}), \tag{2.10}$$

$$\|u-u_h\|_{0,\Omega}\leq Ch^{m}(\|p\|_{k+1,\Omega}+\|u\|_{k+2,\Omega}), \tag{2.11}$$

where $m=\min\{l+1,k+1\}$.

Proof. From [1] and (N_2), we obtain the discrete inf–sup condition:

$$\sup_{q_h\in H_h}\frac{b_h(q_h,v_h)}{\|q_h\|_h}\geq \beta\|v_h\|_{0,\Omega}, \quad \forall v_h\in M_h. \tag{2.12}$$

Then, from (2.3), (2.12) and [6], the problem (2.1) has a unique solution. And, from (1.4) and (2.1), we have

$$b_h(p-p_h,v_h)=0, \quad \forall v_h\in M_h. \tag{2.13}$$

Therefore, from (2.5), we have

$$b_h(\pi_h p - p_h, v_h) = 0, \quad \forall v_h \in M_h. \tag{2.14}$$

Thus, $\pi_h p - p_h \in V_h$. And because $p = a\nabla u$, from (2.3), (2.6)–(2.9), we have

$$\begin{aligned}
\alpha\|\pi_h p - p_h\|_{0,\Omega}^2 &= \alpha\|\pi_h p - p_h\|_h^2 \\
&\leq a(\pi_h p - p_h, \pi_h p - p_h) \\
&= a(\pi_h p - p, \pi_h p - p_h) + a(p, \pi_h p - p_h) \\
&= a(\pi_h p - p, \pi_h p - p_h) + \sum_{K\in\Im_h}\int_{\partial K}(\pi_h p - p_h)nu\,\mathrm{d}s \\
&\leq Ch^{k+1}(\|p\|_{k+1,\Omega} + \|u\|_{k+2,\Omega})\|\pi_h p - p_h\|_{0,\Omega}.
\end{aligned} \tag{2.15}$$

By (2.15) and (N_3), we may get (2.10).

Let $P_h : H^m(\Omega) \to M_h$ be L^2–projection, then we have

$$\|u - P_h u\|_{0,\Omega} \leq Ch^m\|u\|_{m,\Omega}, \quad m = \min\{l+1, k+1\}. \tag{2.16}$$

Let

$$H_h^* = \{q_h \in H;\ \ q_{h|K} \in RT^k(\Im_h)\}. \tag{2.17}$$

then, $H_h^* \subset H \bigcup H_h$. Therefore, by (2.12) with the H_h replaced by H_h^*, and from (2.1) and (1.4),we have

$$\begin{aligned}
\|P_h u - u_h\|_{0,\Omega} &\leq \sup_{q_h\in H_h^*}\frac{b_h(q_h, P_h u - u_h)}{\|q_h\|_h} \\
&= \sup_{q_h\in H_h^*}\frac{b_h(q_h, P_h u - u + u - u_h)}{\|q_h\|_h} \\
&\leq \|P_h u - u\|_{0,\Omega} + \sup_{q_h\in H_h^*}\frac{a(p_h - p, q_h)}{\|q_h\|_{0,\Omega}} \\
&\leq C(\|u - P_h u\|_{0,\Omega} + \|p - p_h\|_{0,\Omega}) \\
&\leq Ch^m(\|u\|_{k+2,\Omega} + \|p\|_{k+1,\Omega}), \ \ m = \min\{l+1, k+1\}.
\end{aligned} \tag{2.18}$$

From (2.16) and (2.18), we may obtain (2.11), which completes the proof of Theorem 2.1.

3. A Lowest Order Non–Conforming Element

Let $\Omega \subset R^2$ and $\Im_h$ be a regular triangulation of $\bar{\Omega}$ (cf.[3]). For a vertex M, let $Q(M)$ denote the polygon formed by the triangles adjacent to M and let ψ_M be the pyramid function, linear on every triangle of $Q(M)$ such that $\psi_M(M) = 1$, $\psi_M = 0$ outside $Q(M)$. Then non–conforming mixed finite element spaces M_h and H_h of the lowest order element are taken:

$$M_h = \{v_h \in L^2(\Omega);\ \ v_{h|K} \in P_0(K), \forall K \in \Im_h\}, \tag{3.1}$$

H_h is Crozei–Raviart's space (see[7]), i.e.,

$$H_h = \{q_h \in L^2(\Omega)^2;\ q_{h|K} \in P_1(K), \forall K \in \Im_h, \text{ for all internal vertices } M, \sum_{K\in Q(M)} \int_{\partial K} q_h n_K \psi_M \mathrm{d}s = 0\}, \tag{3.2}$$

where the degrees–freedom of $P_1(K)$ are taken the meddle point $B_1,\ B_2,\ B_3$ of each side $e_1,\ e_2,\ e_3$ on $\partial K = e_1 \bigcup e_2 \bigcup e_3$, see Fig.1.

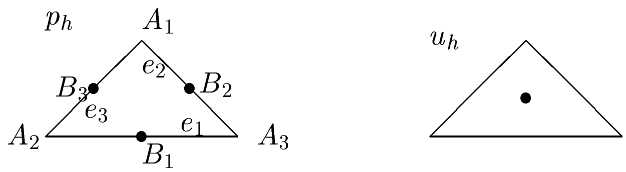

Fig.1

Then, the condition 2) in the section one is satisfied:

$$\int_e (q_{h|K_i} n_i + q_{h|K_j} n_j) p_0 ds = 0,\ \ \forall p_0 \in P_0(e), \tag{3.3}$$

where $e = \partial K_i \bigcap K_j,\ \ i \neq j$.

Let $\pi_h : H \to H_h$ such that $\forall q \in H$

$$b_h(q - \pi_h q, v_h) = 0, \quad \forall v_h \in M_h. \tag{3.4}$$

Then, from the definition of space M_h, we only need

$$\int_{\partial K} (q - \pi_h q) n ds = 0, \quad \forall K \in \Im_h. \tag{3.5}$$

Therefore, we may obtain

$$\pi_h q_{|K} = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3, \tag{3.6}$$

where $\lambda_1,\ \lambda_2,\ \lambda_3$ are the barycentirc coordinates associated with by $e_1,\ e_2,\ e_3$, and

$$a_1 = \frac{1}{mes(e_3)} \int_{e_3} qds + \frac{1}{mes(e_2)} \int_{e_2} qds - \frac{1}{mes(e_1)} \int_{e_1} qds, \tag{3.7}$$

$$a_2 = \frac{1}{mes(e_1)} \int_{e_1} qds + \frac{1}{mes(e_3)} \int_{e_3} qds - \frac{1}{mes(e_2)} \int_{e_2} qds, \tag{3.8}$$

$$a_3 = \frac{1}{mes(e_2)} \int_{e_2} qds + \frac{1}{mes(e_1)} \int_{e_1} qds - \frac{1}{mes(e_3)} \int_{e_3} qds. \tag{3.9}$$

Thus, (3.4) holds. Let $\pi_K q = \pi_h q|_K$, $\forall K \in \Im_h$. Note that $\forall q_1 \in P_1(K)^2$, q_1 may be denoted by

$$q_1 = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3.$$

Thanks to the properties of the barycentric coordinates (see [3] or [6]), we have

$$\int_{e_3} q_1 \mathrm{d}s = \int_0^1 [b_1\lambda_1 + b_2(1-\lambda_1)] \mathrm{mes}(e_3) \mathrm{d}\lambda_1 = (b_1 + b_2)/2, \tag{3.10}$$

$$\int_{e_2} q_1 \mathrm{d}s = \int_0^1 [b_1\lambda_1 + b_3(1-\lambda_1)]\mathrm{mes}(e_2)\mathrm{d}\lambda_1 = (b_1+b_3)/2, \tag{3.11}$$

$$\int_{e_1} q_1 \mathrm{d}s = \int_0^1 [b_2\lambda_2 + b_3(1-\lambda_2)]\mathrm{mes}(e_1)\mathrm{d}\lambda_1 = (b_2+b_3)/2. \tag{3.12}$$

Taking $q = q_1$ in (3.7)–(3.9), by (3.10)–(3.12) we get $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$, in other words,

$$\pi_K q_1 = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = q_1, \quad \forall q_1 \in P_1(K)^2. \tag{3.13}$$

And since $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$, by interpolation theory (see [3] or [6]) we have

$$\|q - \pi_h q\|_{0,K} = \|q - \pi_K q\|_{0,K} \le Ch^2\|q\|_{2,K}, \text{ if } q \in H^2(K)^2. \tag{3.14}$$

Therefore, we get

$$\|q - \pi_h q\|_{0,\Omega} \le Ch^2\|q\|_{2,\Omega}, \text{ if } q \in H^2(\Omega)^2. \tag{3.15}$$

If $q_h \in H_h$, then $\mathrm{div} q_{h|K} \in P_0(K)$. When

$$b_h(q_h, v_h) = 0, \quad \forall v_h \in M_h, \tag{3.16}$$

taking $v_{h|K} = \mathrm{div} q_{h|K}$, one easily gets $\mathrm{div} q_h = 0$, i.e.,

$$\begin{aligned} V_h &= \{q_h \in H_h;\ b_h(q_h, v_h) = 0,\ \forall v_h \in M_h\} \\ &= \{q_h \in H_h;\ \mathrm{div} q_h = 0\}. \end{aligned} \tag{3.17}$$

(2.8) is obvious. Thus, (N_1)–(N_3) are satisfied when $l = 0$ and $k = 1$. In the following we prove that (N_4) (when $k = 1$) is satisfied, too. We denote by b_K the "non–conforming bubble function" defined by (see[8])

$$b_K = 2 - 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2). \tag{3.18}$$

Let

$$W_h = \{v_h \in C^0(\bar{\Omega});\ v_{h|K} \in P_2(K), \forall K \in \Im_h\}, \tag{3.19}$$

$$\Phi_h = \{\phi_h;\ \phi_{h|K} = \alpha_K b_K, \alpha_K \in R, \forall K \in \Im_h\}. \tag{3.20}$$

Then $W_h + \Phi_h$ is nothing else than the non–conforming piece wise quadratic approximation (cf.[8]). Define the following spaces

$$\begin{aligned} X_h = \ & \{q_h \in L^2(\Omega)^2;\ q_{h|K} \in P_1(K),\ \forall K \in \Im_h, \\ & \text{for all internal edges } e = \partial K_i \cap \partial K_j,\ (i \ne j), \\ & \int_e (q_{h|K_i} n_i + q_{h|K_j} n_j) p_0 \mathrm{d}s = 0,\ \forall p_0 \in P_0(e), \text{ and} \\ & \text{for all internal vertices } M, \sum_{K \in Q(M)} \int_{\partial K} q_h n_K \psi_M \mathrm{d}s = 0\}, \end{aligned} \tag{3.21}$$

$$X_h^2 = \{q_h;\ q_{h|K} = \alpha_K \mathrm{curl} b_K,\ \forall K \in \Im_h\}, \tag{3.22}$$

$$X_h^1 = \{q_h \in H;\ q_{h|K} \in BDM_1(K),\ \forall K \in \Im_h\}, \tag{3.23}$$

where BDM_1 denotes the lowest degree finite element of Brezzi–Douglas–Marini [9]. Then, thanks to (3.3) and Lemma 2.1 in [2], one easily gets

$$H_h \subset X_h = X_h^1 + X_h^2. \tag{3.24}$$

Let $q_h \in V_h$, then, by (3.24) and the fact that any $q \in X_h^1$ satisfying $\mathrm{div} q = 0$ in Ω is the curl of a stream function $v_h \in W_h$, there exists $w_h \in W_h$ and $\phi_h \in \Phi_h$ such that

$$q_h = \mathrm{curl} w_h + \mathrm{curl}\phi_h. \tag{3.25}$$

where

$$\begin{cases} \mathrm{curl} w_h &= (\partial w_h/\partial x_2, -\partial w_h/\partial x_1), \\ \mathrm{curl}\phi_{h|K} &= \alpha_K \mathrm{curl} b_K, \ \ \forall K \in \Im_h. \end{cases} \tag{3.26}$$

Thus, $\forall q \in H^2(\Omega)^2$,

$$\begin{aligned} \sum_{K\in\Im_h} \int_{\partial K} q_h n_K v \mathrm{d}s &= \sum_{K\in\Im_h} \alpha_K \int_{\partial K} (\mathrm{curl} b_K) n_K v \mathrm{d}s \\ &= \sum_{K\in\Im_h} \alpha_K \int_{\partial K} (\partial b_K/\partial t) v \mathrm{d}s, \end{aligned} \tag{3.27}$$

where t denotes the unit tangent to the boundary of K. Note

$$\int_{\partial K} (\partial b_K/\partial t) p_2 \mathrm{d}s = 0, \ \ \forall p_2 \in P_2(K). \tag{3.28}$$

If let $P_{2h}v$is the interpolate of v in the W_h, we have

$$\begin{aligned} \sum_{K\in\Im_h} \int_{\partial K} q_h n_K v \mathrm{d}s &= \sum_{K\in\Im_h} \alpha_K \int_{\partial K} (\partial b_K/\partial t)(v - P_{2h}v) \mathrm{d}s \\ &= \sum_{K\in\Im_h} \int_{\partial K} q_h n_K (v - P_{2h}v) \mathrm{d}s \\ &= \sum_{K\in\Im_h} \int_K q_h \nabla (v - P_{2h}v) \mathrm{d}x \\ &\leq Ch^2 \|q_h\|_{0,\Omega} |v|_{3,\Omega}. \end{aligned} \tag{3.29}$$

From the above discussion, we see (N_1)–(N_4) all hold. Therefore, we may obtain the following main result.

Theorem 3.1. *Let* $(p, u) \in H^2(\Omega)^2 \times H^3(\Omega)$ *be the solution of the problem* (1.4) *and* (p_h, u_h) *the solution of the problem* (2.1), *then*

$$\|p - p_h\|_{0,\Omega} \leq Ch^2(\|p\|_{2,\Omega} + \|u\|_{3,\Omega}). \tag{3.30}$$

$$\|u - u_h\|_{0,\Omega} \leq Ch(\|p\|_{2,\Omega} + \|u\|_{3,\Omega}). \tag{3.31}$$

Remark. One may prove that the formats in [1] are suitable to Theorem 2.1. Using Theorem 2.1, one may simplify the procedure of proofs in [1]. In comparison with the result in [2], the freedom degrees of our format is the same as those in [2], but our discrete approximation of flux function is one order higher than that in [2].

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