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A POSTERIORI ERROR ESTIMATES IN ADINI FINITE ELEMENT FOR EIGENVALUE PROBLEMS *1)

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Abstract

In this paper, we discuss a posteriori error estimates of the eigenvalue λ_h given by Adini nonconforming finite element. We give an assymptotically exact error estimator of the λ_h. We prove that the order of convergence of the λ_h is just 2 and the λ_h converge from below for sufficiently small h.

Key words: eigenvalue, nonconforming finite element, error estimate

Consider eigenvalue problems: Find pairs (λ, u), $\lambda \in R, u \in H_0^2(G), \|u\|_0 = 1$, such that

$$a(u,v) = \lambda(u,v), \quad \forall v \in H_0^2(G) \tag{1}$$

and their nonconforming finite element approximations:Find pairs $(\lambda_h, u_h), \lambda_h \in R, u_h \in V_h, \|u_h\|_0$=1, such that

$$a_h(u_h,v) = \lambda_h(u_h,v), \quad \forall v \in V_h \tag{2}$$

where $a(u,v) = \sum \int_G (a_{ijkl}\partial_i\partial_j u \partial_k\partial_l v + a_{pq}\partial_p u \partial_q v)$ is the symmetric, continuous, H_0^2-elliptic bilinear form, $(u,v) = \int_G uv; V_h$ is a nonconforming finit element space associated with a regular triangulations

$$T_h = \{T\},\ V_h \not\subset H_0^2(G),\ a_h(u,v) = \sum\sum_T \int_T (a_{ijkl}\partial_i\partial_j u \partial_k\partial_l v \ + \ a_{pq}\partial_p u \partial_q v)$$

are uniformly V_h -elliptic; i,j,k,l=1,2; p,q=0,1,2; $\partial_1 = \frac{\partial}{\partial x}$,$\partial_2 = \frac{\partial}{\partial y}$,$\partial_0 = id$, $\partial_1\partial_2 = \frac{\partial^2}{\partial x \partial y}$.
Let(λ_h, u_h) and (λ, u) be an eigenpair of (2) and of (1), respectivery, and (λ_h, u_h) converge (λ, u). In [3], the abstract error estimates has been presented and the following estimates has been proved for Adini finite element:

$$|\lambda_h - \lambda| \le Ch^2 \tag{3}$$

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In this paper ,we discuss a posteriori error estimates .We prove that the order of convergence is just 2, and give an asymptotically exact estimator for Adini finite element. Consider the steady state problems: Find $w \in H_0^2(G)$, such that

$$a(w,v) = (f,v), \quad \forall v \in H_0^2(G) \tag{4}$$

In the case of $f \equiv u_h$, let $u^\star$ *and* $u_h^\star \in V_h$ denote the exact solution and nonconforming finite element solution, respectively. It is obvious that $u_h^\star \equiv \lambda_h^{-1} u_h$.

Lemma 1. *The following estimates hold*

$$\frac{\lambda_h - \lambda}{\lambda} = \frac{\lambda_h}{(u,u_h)}(u^\star - u_h^\star, u) \tag{5}$$

$$\|u_h - u\|_s \le C\|u^\star - u_h^\star\|_s, \quad s = 0,1 \tag{6}$$

Proof. Let P_λ be the orthogonal projection operator of the $L_2(G)$ onto eigenspace V_λ corresponding to the eigenvalue λ. Taking u=$\frac{P_\lambda u_h}{\|P_\lambda u_h\|_0}$.

$$\begin{aligned}(u^\star - u_h^\star, u) &= (u^\star - \lambda_h^{-1}u_h, u) = \lambda^{-1}(u_h,u) - \lambda_h^{-1}(u_h,u) \\ &= (\lambda^{-1} - \lambda_h^{-1})(u_h,u)\end{aligned}$$

which is just (5). The proof of the (6) is the same as that of [5, (1.4)].
In the case of $f \equiv \lambda u$,it is obvious that the exact solution of the associated (4) is just u and nonconforming finite element solution $u_h^0 \in V_h$ satisfies

$$a_h(u_h^0, v) = \lambda(u,v), \quad \forall v \in V_h \tag{7}$$

Lemma 2. *The following inequality holds*

$$\|u_h - u\|_h \le \|u_h^0 - u\|_h + C\|\lambda_h u_h - \lambda u\|_0 \tag{8}$$

Proof. From (2) and (7) we have

$$a_h(u_h - u_h^0, v) = (\lambda_h u_h - \lambda u, v)$$

Taking v=$u_h - u_h^0$, we get by uniformly elliptic

$$\begin{aligned}\|u_h - u_h^0\|_h^2 &\le Ca_h(u_h - u_h^0, u_h - u_h^0) \\ &\le C\|\lambda_h u_h - \lambda u\|_0 \|u_h - u_h^0\|_0\end{aligned}$$

and hence

$$\|u_h - u_h^0\|_h \le C\|\lambda_h u_h - \lambda u\|_0$$

using the above inequality and the triangle inequality we obtain (8).

Lemma 3. *The following equality holds*

$$\lambda_h - \lambda \ = \ a_h(u-u_h, u-u_h) - \lambda\|u-u_h\|_0^2 + 2D_h \tag{9}$$

where $D_h = a_h(u, u_h) - (\lambda u, u_h)$.

Proof. From (1) and (2) we have

$$\begin{aligned} a_h(u-u_h, u-u_h) &= a_h(u,u) + a_h(u_h,u_h) - 2a_h(u,u_h) = \lambda + \lambda_h - 2(\lambda u, u_h) - 2D_h \\ &= \lambda_h - \lambda + \lambda(2 - 2(u,u_h)) - 2D_h \\ &= \lambda_h - \lambda + \lambda\|u-u_h\|_0^2 - 2D_h \end{aligned}$$

it is jast (9).
The above lemma valid for all nonconforming finite element methods . Let V_h^0 be the piecewise constant functions space given by

$$V_h^0 \ = \ \{v, v|_T \in P_0(T), T \in T_h\}$$

and P be the orthogonal projection of $L_2(G)$ onto V_h^0. From [6] we easily prove that

Lemma 4. *Assume* $w \in W_{1,2}(G)$, *then*

$$\|Pw - w\|_0 \ \leq \ Ch\|w\|_1 \tag{10}$$

In the remainder of this paper, we shall essentially discuss Adini finite element. For simplicity, we consider the biharmonic equation:

$$-\Delta^2 u \ = \ \lambda u \quad in\ G; \quad u = \frac{\partial u}{\partial \gamma} = 0 \quad on\ \partial G \tag{11}$$

where G is a rectangle. In this case, we have $a(u,v) = \int_G[\Delta u \Delta v + (1-\nu)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)]$ and $a_h(u,v) = \sum_T \int_T[\nu\Delta u\Delta v + (1-\nu)(\partial_{11}u\partial_{11}v + \partial_{22}u\partial_{22}v + 2\partial_{12}u\partial_{12}v)]$. $\partial_{ij} = \partial_i\partial_j$, $\partial_{ijk} = \partial_i\partial_j\partial_k$. Let T_h be a triangulations made up of rectangles $T = (x_T - h_T, x_T + h_T) \times (y_T - k_T, y_T + k_T)$ with the center (x_T, y_T) and V_h the Adini finite element space associated with T_h. $h = \max_T \sqrt{h_T^2 + k_T^2}$. Denote that $A_1(u) = \Delta u - (1-\nu)\partial_{22}u, B_1 = \partial_1 u_h - \wedge_T \partial_1 u_h, A_2(u) = \Delta u - (1-\nu)\partial_{11}u, B_2 = \partial_2 u_h - \wedge_T\partial_2 u_h$, $\wedge_T$ denotes the bilinear interpolation operator (see [1] P302–P309).

Theorem 1. *Assume that the eigenfunction* $u \in H^4(G), (\lambda_h, u_h)$ *is the Adini eigenpair, then*

$$\lambda_h - \lambda \ = \ 2\sum_T \int_T \{F\partial_1 A_1(u)\partial_{221}u_h + E\partial_2 A_2(u)\partial_{112}u_h\} + O(h^3) \tag{12}$$

where $E = \frac{1}{2}((x-x_T)^2 - h_T^2), F = \frac{1}{2}((y-y_T)^2 - k_T^2)$.

Proof. It is well known that (see [1],[3])

$$\|u^\star - u_h^\star\|_s \leq Ch^2\|u\|_3, \quad s=0,1 \tag{13}$$

$$\|u - u_h^0\|_h \leq Ch^2\|u\|_4 \tag{14}$$

Substitute (13) into (6) and (14) into (8), we have

$$\|u_h - u\|_s \leq Ch^2\|u\|_3, \quad s=0,1 \tag{15}$$

$$\|u_h - u\|_h \leq Ch^2\|u\|_4 \tag{16}$$

From [1] P298–P309, we have

$$\begin{aligned} D_h &= \sum_T [\int_{T_1'} A_1(u)B_1 - \int_{T_1''} A_1(u)B_1] + \sum_T [\int_{T_2'} A_2(u)B_2 - \int_{T_2''} A_2(u)B_2] \\ &\equiv D_h^1 + D_h^2 \end{aligned} \tag{17}$$

Using the identity argument (see [3]) and $F, \|B_1\|_0 = O(h^2), \|B_1\|_1' = O(h), \partial_{122}B_1 = 0$, we have

$$\begin{aligned} D_h^1 &= \sum_T \int_T F\partial_{122}(A_1(u)B_1) \\ &= \sum_T \int_T F(\partial_1 A_1(u)\partial_{221}u_h + \partial_{122}A_1(u)B_1 + 2\partial_2 A_1(u)\partial_{12}B_1) + O(h^3) \end{aligned} \tag{18}$$

Notice that F=0 on T_2' and T_2'', using the Green formula, we have

$$\sum_T \int_T F\partial_{122}A_1(u)B_1 = \sum_T [-\int_T \partial_{21}A_1(u)\partial_2(FB_1) + 0] = O(h^3) \tag{19}$$

Using the Green formula and (10) we have

$$\begin{aligned} \sum_T \int_T F\partial_2 A_1(u)\partial_{12}B_1 &= \sum_T -\int_T \partial_2(F\partial_2 A_1(u))\partial_1 B_1 \\ &= \sum_T -\int_T \partial_2 F\partial_2 A_1(u)\partial_1 B_1 + O(h^3) \\ &= \sum_T -P\partial_2 A_1(u)\int_T \partial_2 F\partial_1 B_1 + O(h^3) \\ &= 0 + O(h^3) = O(h^3) \end{aligned} \tag{20}$$

Substitute (19) and (20) into (18), we have

$$D_h^1 = \sum_T \int_T F\partial_1 A_1(u)\partial_{221}u_h + O(h^3) \tag{21}$$

Notice that since $(C^\infty(\bar{G}))^- = H^4(G)$, the closure being understood in the sense of the norm $\|\bullet\|_4$, the (21) holds as long as the $u \in H^4(G)$. And similarly we have

$$D_h^2 = \sum_T \int_T E\partial_2 A_2(u)\partial_{112}u_h + O(h^3) \tag{22}$$

Substitute (21) and (22) into (17) and combining relations (15), (16), (17) and (9), we get (12). The theorem is proved.

Corollary 1. *Assume that eigenfunctions* $u \in H^4(G)$ *, then*

$$\lambda_h - \lambda = -\frac{2}{3}\sum_T (h_T^2 \int_T \partial_2 A_2(u)\partial_{112}u + k_T^2 \int_T \partial_1 A_1(u)\partial_{221}u) + O(h^3) \quad (23)$$

$$\lambda_h - \lambda = -\frac{2}{3}\sum_T (h_T^2 \int_T \partial_2 A_2(u_h)\partial_{112}u_h + k_T^2 \int_T \partial_1 A_1(u_h)\partial_{221}u_h) + O(h^3) \quad (24)$$

Proof. Using (16) and (10) we deduce

$$\begin{aligned}\sum_T \int_T F\partial_1 A_1(u)\partial_{221}u_h &= \sum_T \int_T F\partial_1 A_1(u)\partial_{221}u + O(h^3) \\ &= \sum_T P(\partial_1 A_1(u))P(\partial_{221}u)\int_T F + O(h^3) \\ &= \sum_T -\frac{1}{3}k_T^2 \int_T P(\partial_1 A_1(u))P(\partial_{221}u) + O(h^3) \\ &= \sum_T -\frac{1}{3}k_T^2 \int_T \partial_1 A_1(u)\partial_{221}u + O(h^3) \qquad (25)\end{aligned}$$

Similarly, we can prove

$$\sum_T \int_T E\partial_2 A_2(u)\partial_{112}u_h = \sum_T -\frac{1}{3}h_T^2 \int_T \partial_2 A_2(u)\partial_{112}u + O(h^3) \quad (26)$$

Substitute (25) and (26) into (12) we get (23).
From (16) and (23) we get (24).
The (23) shows that the order of convergence is just 2 for the Adini finite element eigenvalue.The (24) shows that $-\frac{2}{3}\sum_T(---)$ is an asymptotically exact estimator of the λ_h and the corrected eigenvalue $\lambda_h + \frac{2}{3}\sum_T(---)$ increases the accuracy from $O(h^2)$ *to* $O(h^3)$.

Corollary 2. *Assume that rectangulay mesh is uniform and* $u \in H^4(G)$ *, then*

$$\lambda_h - \lambda = -\frac{2}{3}h_T^2 \int_G [(\partial_{221}u)^2 + \nu(\partial_{112}u)^2] - \frac{2}{3}k_T^2 \int_G [(\partial_{112}u)^2 + \nu(\partial_{221}u)^2] + O(h^3) \quad (27)$$

Proof. Since T_h is an uniform mesh, from (23) we have

$$\lambda_h - \lambda = -\frac{2}{3}h_T^2 \int_G \partial_2 A_2(u)\partial_{112}u - \frac{2}{3}k_T^2 \int_G \partial_1 A_1(u)\partial_{221}u + O(h^3) \quad (28)$$

from the Green formula [1, P34] and u=$\frac{\partial u}{\partial \gamma}$=0, on ∂G, we deduce

$$\int_G \partial_1 A_1(u)\partial_{221}u = \int_G (\partial_{111}u + \partial_{122}u - (1-\nu)\partial_{122}u)\partial_{221}u$$