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ASYMPTOTIC ANALYSIS OF SHELLS VIA Γ-CONVERGENCE*[1)]

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Abstract

We give a new justification of the linear membrane and flexural shell models. We prove that the sequence of scaled energy functionals associated with the scaled problem Γ-converges to the energy functional associated with a two-dimensional model. This two-dimensional model is a membrane or flexural one, depending on the geometric and kinematic conditions. Then, a classical argument allows to give a new proof of the convergence theorems recently obtained by P.G. Ciarlet, V. Lods and B. Miara.

Key words: Γ-converges, linear elastic shell

Introduction

The deformations of an elastic body submitted to forces are governed by three-dimensional mathematical equations. This means that the unknown, which is the vector formed by the components of the displacement, depends on three variables. however, when the elastic body is "thin" in one dimension, for instance when it is a shell, one can use two-dimensional shell models, such as those of Naghdi, Koiter, Budiansky-Sanders etc. Thus, the important point is to explain which model is the "good" one in a given situation and why. Hence, an important aspect of the mathematical analysis in elasticity consists in studying the validity of the two-dimensional equations to describe the physical behavior of a three-dimensional body. This is what is called the justification of the model.

Deriving lower-dimensional models can be achieved through a formal asymptotic analysis. The method is the following: first, one has to make the "scalings" on the unknown and the "right" assumptions on the forces in order to set the problem over a fixed domain, i.e, a domain independent of the thickness ε. Next, it is assumed that the scaled three-dimensional displacement field obtained in this fashion can be expanded in powers of the small parameter ε. Finally, replacing this formal expansion in the variational equations, one can identify the leadin term by equating to 0 the coefficients of the powers of ε.

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In the study of linearly elastic shells, the first contribution of that kind is due to [1]. Then, [2] pointed out the importance of the geometry of the shell: depending on the geometric and kinematic conditions, the formal asymptotic analysis leads to identify one of two distinct models: the “membrane” model or the “flexural” model. Thus, it is not possible to derive these two models simultaneously for shells, unlike the case of plates. For other works in this spirit, see [3–6].

Essentially, a two-dimensional model is considered justified when one can prove convergence of the three-dimensional unknown to the leading term of the asymptotic expansion, as the thickness ε of the shell goes to zero. In the linear case, the articles of [7, 8], [9] give the complete justification of the membrane and flexural models by using the techniques of asymptotic analysis. For nonlinear membranes, such results were obtained by [10], using Γ-convergence and following the approach of [11].

Here, we give another method to obtain convergence theorems in the linear case using Γ-convergence theory. A similar approach was done for linearly elastic plates by [12].

We study separately the membrane case and the flexural case. First, we recall the main notations about the geometry of the shell, and we make appropriate scalings, in order to define the scaled three-dimensional problem. Next, we prove the Γ-convergence of the energy functionals associated with the scaled three-dimensional problem to a functional corresponding to a variational problem posed over a two-dimensional domain. We then deduce the weak convergence of the displacements, the strong convergence being obtained as in [7], [9].

1. The Three-Dimensional Shell Problem in Linearized Elasticity

We begin with geometric preliminaries. Throughout this work, Greek indices and exponents (except ε) belong to the set $\{1,2\}$, Latin indices and exponents (except when used to index sequences) take their values in the set $\{1, 2, 3\}$, and we use the summation convention on repeated indices and exponents.

Let ω be a bounded, open and connected subset of $\boldsymbol{R}^2$, with a Lipschitz-continuous boundary γ. We note $y = (y_\alpha)$ a generic point of $\bar{\omega}$, and $\partial_\alpha := \partial/\partial y_\alpha$ the partial derivatives. let $\varphi : \bar{\omega} \to \boldsymbol{R}^3$ be an injective mapping, at least of class $\mathcal{C}^3$. We assume that the two vectors

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha \varphi(y)$$

are linearly independent at all points $y \in \bar{\omega}$. They form the *covariant basis* of the tangent plane to the surface $S = \varphi(\bar{\omega})$ at the point $\varphi(y)$; the two vectors $\boldsymbol{a}^\alpha(y))$ defined by

$$\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{\alpha}_\beta(y) = \delta^\alpha_\beta$$

constitute the contravariant basis at this same point $\varphi(y)$. We also define the vector

$$\boldsymbol{a}_3 = \boldsymbol{a}^3 := \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}.$$

The covariant and contravariant components of the *metric tensor* are given by

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta,$$

the covariant and mixed components of the curvature tensor are

$$b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha, \quad b_\alpha^\beta := a^{\beta\sigma} b_{\sigma\alpha},$$

and the Christoffel symbolds of the surface S are defined by

$$\Gamma_{\alpha\beta}^\sigma := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha.$$

Since the metric tensor is symmetric and definite positive at all points of $\bar{\omega}$, there exists a constant a_0 such that:

$$a(y) := \det\left(a_{\alpha\beta}(y)\right) \geq a_0 > 0 \quad \text{for all } y \in \bar{\omega}.$$

The area element along S is $\sqrt{a}dy$.

For each $\varepsilon > 0$, we define the sets

$$\Omega^\sigma := \omega \times [-\varepsilon, \varepsilon] \text{ and } \Gamma^\sigma := \gamma \times [-\varepsilon, \varepsilon].$$

We let $x^\varepsilon = (x_i^\varepsilon)$ denote a generic point in the set Ω^ε, and $\partial_i^\varepsilon = \partial / \partial x_i^\varepsilon$, so that we have $x_\alpha^\varepsilon = y_\alpha$ and $\partial_\alpha^\varepsilon = \partial_\alpha$. Let the mapping $\boldsymbol{\Phi} : \overline{\Omega}^\varepsilon \to \boldsymbol{R}^3$ be defined by

$$\boldsymbol{\Phi}(x^\varepsilon) := \varphi(y) + x_3^\varepsilon \boldsymbol{a}^3(y) \quad \text{for all } x^\varepsilon = (y, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon.$$

The three vectors

$$\boldsymbol{g}_i^\varepsilon(x^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Phi}(x^\varepsilon)$$

form the covariant basis at the point $\boldsymbol{\Phi}(x^\varepsilon)$, with which we associate the contravariant basis formed by the vectors $\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)$ given by

$$\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon) \cdot \boldsymbol{g}_j^\varepsilon(x^\varepsilon) = \delta_j^i.$$

The covariant and contravariant components of the metric tensor, and the Christoffel symbols are

$$g_{ij}^\varepsilon := \boldsymbol{g}_i^\varepsilon \cdot \boldsymbol{g}_j^\varepsilon, \quad g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}, \quad \Gamma_{ij}^{p,\varepsilon} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial_i^\varepsilon \boldsymbol{g}_j^\varepsilon.$$

The volume element in the set $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ is $\sqrt{g^\varepsilon}dx^\varepsilon$, where

$$g^\varepsilon := \det\ (g_{ij}^\varepsilon).$$

For all $\varepsilon > 0$, the set $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ is the *reference configuration* of an elastic shell with middle surface S and thickness $2\varepsilon > 0$. We assume that the elastic material constituting the shell is homogeneous and isotropic, and that the reference configuration is a natural state. The Lamé constants of the material are $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$. We assume that the shell is clamped on a portion $\Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon]$ of its lateral surface, where $\gamma_0 \subset \gamma$

satisfies length $\gamma_0 > 0$, and that it is subjected to volume forces with density $f^{i,\varepsilon}\boldsymbol{g}^{i,\varepsilon}$, where $f^{i,\varepsilon} \in L^2(\Omega)$.

The covariant components $u_i^\varepsilon : \overline{\Omega}^\varepsilon \to \boldsymbol{R}$ of the displacement $u_i^\varepsilon(x^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)$ of the points of the shell are the unknowns of the *three-dimensional shell problem.* In linearized elasticity, this problem can be written as (see e.g. [13]): Find

$$\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) := \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon); \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_0^\varepsilon\}, \tag{1.1}$$

such that

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k\|l}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}dx^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}dx^\varepsilon \quad \text{for all } \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon), \tag{1.2}$$

where the covariant components of the three-dimensional elasticity tensor are defined by

$$A^{ijkl,\varepsilon} := \lambda^\varepsilon g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu^\varepsilon(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}), \tag{1.3}$$

and the covariant components of the *linearized strain tensor* are

$$e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon) := \frac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon + \partial_j^\varepsilon v_i^\varepsilon) - \Gamma_{ij}^{p,\varepsilon} v_p^\varepsilon. \tag{1.4}$$

The definition of the mapping $\boldsymbol{\Phi}$ implies that $\Gamma_{3,3}^{p,\varepsilon}=\Gamma_{\alpha,3}^{3,\varepsilon}=0$ in $\overline{\Omega}^\varepsilon$, $A^{\alpha\beta\sigma 3,\varepsilon}=A^{\alpha 333,\varepsilon}=0$ in $\overline{\Omega}^\varepsilon$. For ease of exposition, we do not consider surface loads. Taking them into account would not add any essential difficulty.

The variational problem (1.1)–(1.2) has one and only one solution for each $\varepsilon > 0$; this follows from the $\boldsymbol{V}(\Omega^\varepsilon)$-ellipticity of the bilinear form appearing in (1.2), see [13].

From now on, we will not work with the variational formulation (1.1)–(1.2) (which is the approach of [7], [9]) but with the formulation in terms of energy. Indeed, the vector field $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ can also defined as the unique solution of the following minimization problem: Find $\boldsymbol{u}^\varepsilon$ such that

$$\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) \quad \text{and } J^\varepsilon(\boldsymbol{u}^\varepsilon) = \inf_{\boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon)} J^\varepsilon(\boldsymbol{v}^\varepsilon), \tag{1.5}$$

where

$$J^\varepsilon(\boldsymbol{v}^\varepsilon) := \frac{1}{2}\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k\|l}^\varepsilon(\boldsymbol{v}^\varepsilon) e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}dx^\varepsilon - \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}dx^\varepsilon. \tag{1.6}$$

Of course, formulations (1.1)–(1.2) and (1.5)–(1.6) are equivalent.

2. Justification of the Two-Dimensional Membrane Shell Model

In this section, we assume that the shell is clamped along its whole lateral face $\boldsymbol{\Phi}(\Gamma^\varepsilon)$, i.e, $\gamma = \gamma_0$, and the displacements vanish on $\boldsymbol{\Phi}(\Gamma^\varepsilon)$. We also assume that the middle surface of the shell is uniformly elliptic in the sense that there exists a constant $b > 0$ such that

$$|b_{\alpha\beta}(y)\xi^\alpha\xi^\beta| \geq b\xi^\alpha\xi^\alpha \tag{2.1}$$

for all $y \in \bar{\omega}$ and $(\xi^\alpha) \in \boldsymbol{R}^2$. For instance, a portion of a sphere or a portion of an ellipsoid are uniformly elliptic surfaces. Inequality (2.1) means that the principal radii of curvature $R_1(y)$ and $R_2(y)$ have the same sign at all points $\varphi(y) \in S$ and that there exists a constant $\rho > 0$ such that

$$\rho^{-1} \leq R_\alpha(y)| \leq \rho, \quad \alpha = 1, 2, \quad \text{for all } y \in \bar{\omega}.$$

2.1. The Scaled Three-Dimensional Problem.

As in [7], we first set the three-dimensional problem (1.1)–(1.2) over a domain indpendent of ε, and we next make appropriate scalings on the components of the displacement and assumptions on the data. We let

$$\Omega := \omega \times [-1, 1], \quad \Gamma := \gamma \times [-1, 1],$$

and with any point $x^\varepsilon = (x_i^\varepsilon) \in \overline{\Omega}^\varepsilon$, we associate the point $x = (x_i) \in \overline{\Omega}$ defined by $x_\alpha = x_\alpha^\varepsilon (= y_\alpha)$ and $x_3 = (1/\varepsilon) x_3^\varepsilon$.

Then, for all $x^\varepsilon \in \overline{\Omega}^\varepsilon$, the scaled displacement $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$ is defined by

$$u_i(\varepsilon)(x) = u_i^\varepsilon(x^\varepsilon), \tag{2.2}$$

and with the vector field $\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{V}(\Omega^\varepsilon)$ we associate the scaled vector field

$$\boldsymbol{v} = (v_i(x)) = (v_i^\varepsilon(x^\varepsilon)) \quad \text{for all } x^\varepsilon \in \overline{\Omega}^\varepsilon.$$

We next assume that there exist constants $\lambda > 0$ and $\mu > 0$ independent of ε, and there exist functions $f^i \in L^2(\Omega)$ independent of ε such that

$$\lambda^\varepsilon = \lambda, \quad \mu^\varepsilon = \mu, \tag{2.3}$$

$$f^{i,\varepsilon}(x^\varepsilon) = f^i(x) \quad \text{for all } x \in \Omega. \tag{2.4}$$

The choice of these scalings is of prime importance in order to find the "right" two-dimensional model; in particular, one has to make different assumptions in the flexural case.

Finally, with the functions $\Gamma_{ij}^{p,\varepsilon}, g^\varepsilon, A^{ijkl,\varepsilon} : \overline{\Omega}^\varepsilon \to \boldsymbol{R}$, we associate the functions $\Gamma_{ij}^p(\varepsilon)$, $g(\varepsilon)$, $A^{ijkl}(\varepsilon) : \overline{\Omega}^\varepsilon \to \boldsymbol{R}$ defined by (cf. [7]), for all $x \in \overline{\Omega}$,

$$\Gamma_{ij}^p(\varepsilon)(x) := \Gamma_{ij}^{p,\varepsilon}(x^\varepsilon), \; g(\varepsilon)(x) := g^\varepsilon(x^\varepsilon), \; A^{ijkl}(\varepsilon)(x) := A^{ijkl,\varepsilon}(x^\varepsilon). \tag{2.5}$$

In addition, for any vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$, we define the symmetric tensor $(e_{i\|j}(\varepsilon)(\boldsymbol{v})) \in \boldsymbol{L}^2(\Omega)$ given by

$$e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) := \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma_{\alpha\beta}^p(\varepsilon) v_p, \tag{2.6}$$

$$e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{2}\Big(\partial_\alpha v_3 + \frac{1}{\varepsilon}\partial_3 v_\alpha\Big) - \Gamma_{\alpha 3}^\sigma(\varepsilon) v_\sigma, \tag{2.7}$$

$$e_{3\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3. \tag{2.8}$$

One can easily check that $e^{\varepsilon}_{i\|j}(\boldsymbol{v}^{\varepsilon}) = e_{i\|j}(\varepsilon)(\boldsymbol{v})$.

We then have the following results:

Theorem 2.1. *The scaled unknown* $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$ *defined in* (2.2) *satisfies*

$$\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in H^1(\Omega); \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\}, \tag{2.9}$$

$$J(\varepsilon)(\boldsymbol{u}(\varepsilon)) = \inf_{\boldsymbol{v} \in \boldsymbol{V}(\Omega)} J(\varepsilon)(\boldsymbol{v}), \tag{2.10}$$

where

$$J(\varepsilon)(\boldsymbol{v}) = \frac{1}{2}\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon)(\boldsymbol{v}) e_{i\|j}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}dx - L(\varepsilon)(\boldsymbol{v}), \tag{2.11}$$

and

$$L(\varepsilon)(\boldsymbol{v}) = \int_\Omega f^i v_i \sqrt{g(\varepsilon)}dx. \tag{2.12}$$

Remark 2.2. We give another expression for the functional $J(\varepsilon)$. Using (1.3) and (2.5) we can write

$$\begin{aligned} J(\varepsilon)(\boldsymbol{v}) =& \frac{1}{2}\int_\Omega \Big\{\frac{2\lambda\mu}{\lambda+2\mu} g^{\alpha\beta}(\varepsilon) g^{\sigma\tau}(\varepsilon)\Big\} e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) e_{\sigma\|\tau}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}dx \\ &+ \frac{1}{2}\int_\Omega \{2\mu g^{\alpha\sigma}(\varepsilon) g^{\beta\tau}(\varepsilon)\} e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) e_{\sigma\|\tau}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}dx - L(\varepsilon)(\boldsymbol{v}) \\ &+ \frac{1}{2}\int_\Omega \Big\{(\lambda+2\mu)\Big[\frac{\lambda}{\lambda+2\mu} g^{\alpha\beta}(\varepsilon) e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) + g^{33}(\varepsilon) e_{3\|3}(\varepsilon)(\boldsymbol{v})\Big]^2\Big\}\sqrt{g(\varepsilon)}dx \\ &+ \frac{1}{2}\int_\Omega \{4\mu g^{\alpha\sigma}(\varepsilon) g^{33}(\varepsilon) e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) e_{\sigma\|3}(\varepsilon)(\boldsymbol{v})\}\sqrt{g(\varepsilon)}dx. \end{aligned} \tag{2.13}$$

Note that a similar expression was used for the scaled functional of the three-dimensional plate problem to study via Γ-convergence theory the asymptotic behavior of the three-dimensional unknown (cf. [12]).

2.2. A Korn's Inequality.

We recall a fundamental result of [7] which plays a crucial role in their proof as well as in the one we give in Section 2.4. It is a Korn's inequality for uniformly elliptic surfaces, which will allow us to prove a priori estimates for $(\boldsymbol{u}(\varepsilon 0)_{\varepsilon>0}$.

From now on, we will suppose that one of the following assumptions holds:

$$\boldsymbol{\varphi} \text{ is analytic in an open set containing } \bar{\omega} \text{ and } \gamma \text{ is of class } \mathcal{C}^3 \tag{2.14}$$

or

$$\boldsymbol{\varphi} \in \mathcal{C}^5(\bar{\omega}; \boldsymbol{R}^3) \text{ and } \gamma \text{ is of class } \mathcal{C}^4. \tag{2.15}$$

These assumptions were stated in [14] and [15], respectively, to prove existence theorems for the two-dimensional linear membrane shell problem. In the first proof, the authors introduce an auxiliary system which is amenable to the theory of [16], and they show the ellipticity of the bilinear form of the variational problem. In the second proof, the existence follows from a lemma of J.L. Lions and the uniqueness for the Cauchy problem for certain elliptic equations is used to obtain uniqueness in that cases.

The following theorm, whose proof is given in [7], Thm. 4.1, states a generalized Korn's inequality. We denote by $\| \cdot \|_{0,\Omega}$, resp. $\| \cdot \|_{1,\Omega}$, the norm in the space $L^2(\Omega)$, resp. $H^1(\Omega)$.

Theorem 2.3. *Assume that S is elliptic, and that either* (2.14) *or* (2.15) *holds. Then, there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_0$ and for all $\boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega)$*

$$\Big\{ \sum_{\alpha} \|v_\alpha\|^2_{1,\Omega} + \|v_3\|^2_{0,\Omega} \Big\}^{1/2} \leq C \Big\{ \sum_{ij} \|e_{i\|j}(\varepsilon)(\boldsymbol{v})\|^2_{0,\Omega} \Big\}^{1/2} \tag{2.16}$$

where the functions $e_{i\|j}(\varepsilon)(\boldsymbol{v})$ and the space $\boldsymbol{V}(\Omega)$ are defined as in (2.6)–(2.9).

Remark 2.4. Inequality (2.16) involves the $H^1(\Omega)$-norms of the horizontal components (v_α), but only the $L^2(\Omega)$ norm of the vertical component v_3.

Remark 2.5. For any $\boldsymbol{\eta} = (\eta_i) \in H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$, let us define the functions

$$\gamma_{\alpha\geq}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3.$$

A sufficient conditon to establish (2.16) is that there exists a constant $c > 0$ such that

$$\Big\{ \sum_{\alpha} \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{0,\omega} \Big\}^{1/2} \leq c \Big\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} \Big\}^{1/2}$$

for all $\boldsymbol{\eta} \in H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$. One can show that hte assumptions of Theorem 2.3 imply this condition (cf. [14] and [15]).

2.3. Review on Γ-convergence

In this section, we recall the definitions and the main properties on Γ-convergence theory that we will use in the proofs of section 2.4. For a general survey, see [17] or [18].

Definition 2.6. *Let $\mathcal{V}$ be a reflexive Banach space, and let $(\mathcal{J}^*(\varepsilon))_{\varepsilon>0}$ be a sequence of functionals $J^*(\varepsilon) : \mathcal{V} \to \boldsymbol{R} \cup \{+\infty\}$, where ε is a parameter approaching zero. We denote by rightharpoonup the weak convergence in $\mathcal{V}$. We say that the functional $J^* : \mathcal{V} \to \boldsymbol{R}$ is the Γ-limit of the functionals $J^*(\varepsilon)$ if the following propertie hold:*

(i) *If $(\boldsymbol{v}(\varepsilon))_{\varepsilon>0}$ is a weakly convergent sequence in $\mathcal{V}$, then*

$$\boldsymbol{v}(\varepsilon) \rightharpoonup \boldsymbol{v} \in \mathcal{V} \Longrightarrow J^*(\boldsymbol{v}) \leq \liminf_{\varepsilon\to 0} J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)). \tag{2.17}$$

(ii) *For any $\boldsymbol{v} \in \mathcal{V}$, there exists a sequence $(\boldsymbol{v}(\varepsilon))$ in $\mathcal{V}$ such that*

$$\boldsymbol{v}(\varepsilon) \rightharpoonup_{\varepsilon\to 0} \boldsymbol{v} \text{ and } J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) \to J^*(\boldsymbol{v}). \tag{2.18}$$

Remark 2.7. When the Γ-limit exists, it is unique. This is an easy consequence of Definition 2.6. One can prove the existence of the Γ-limit by verifying the properties (2.17)–(2.18): then the functional has to be known in advance. This is the approach we use in Section 2.4. For nonlinear membrane shells, [10] proceed by first extracting a Γ-convergent subsequence.

Remark 2.8. We gave the definition using the weak topology of $\mathcal{V}$ as it is best adapted to our analysis.

The following theorem will allow us to prove weak convergence results on the displacements once we have a Γ-convergence theorem for the energies.

Theorem 2.9. *Assume that the sequence $(J^*(\varepsilon))_{\varepsilon>0}$ is Γ-convergent to J^*, and assume that there exists a compact subset $\mathcal{U}$ of $\mathcal{V}$ independent of ε such that, for all $\varepsilon > 0$, there exists $\boldsymbol{u}(\varepsilon)$ satisfying*

$$\boldsymbol{u}(\varepsilon) \in \mathcal{U} \quad \textit{and} \quad J^*(\varepsilon)(\boldsymbol{u}(\varepsilon)) = \inf_{\boldsymbol{v}\in V} J^*(\varepsilon)(\boldsymbol{v}).$$

Then there exists $\boldsymbol{u} \in \mathcal{U}$ such that

$$\boldsymbol{u}(\varepsilon) \rightharpoonup_{\varepsilon\to 0} \boldsymbol{u} \quad \textit{and} \quad J^*(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in \mathcal{V}} J^*(\boldsymbol{v}).$$

In addition, one has

$$J^*(\varepsilon)(\boldsymbol{u}(\varepsilon)) \to_{\varepsilon\to 0} J^*(\boldsymbol{u}).$$

2.4. Convergence Theorems

We first establish the Γ-convergence of a sequence of energy functionals by identifying its Γ-limit. Let $\mathcal{V}$ be the space defined by

$$\mathcal{V} := \{\boldsymbol{v} = (v_i); v_\alpha \in H_0^1(\Omega), v_3 \in L^2(\omega)\}. \tag{2.19}$$

We extend the energies of the three-dimensional problem to $\mathcal{V}$ by letting, for any $\boldsymbol{v} \in \mathcal{V}$,

$$J^*(\varepsilon)(\boldsymbol{v}) := \begin{cases} J(\varepsilon)(\boldsymbol{v}) & \text{if } \boldsymbol{v} \in \boldsymbol{V}(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.20}$$

where the space $\boldsymbol{V}(\Omega)$ and the functional $J(\varepsilon)$ are defined as in (2.9) and (2.14), respectively. Let J^* be the functional given by

$$J^*(\boldsymbol{v}) := \begin{cases} J(\boldsymbol{v}) & \text{if } \boldsymbol{v} \text{ does not depend on } x_3, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.21}$$

where

$$J(\boldsymbol{v}) = \frac{1}{4}\int_\Omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{v})\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx - L(\boldsymbol{v}), \tag{2.22}$$

$$L(\boldsymbol{v}) = \int_\Omega f^i v_i \sqrt{a} dx, \tag{2.23}$$

the functions $a^{\alpha\beta\rho\sigma}$ are the contravariant components of the elasticity tensor of S, defined by

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \tag{2.24}$$

and the functions $\gamma_{\alpha\beta}(\cdot)$ are the covariant components of the linearized change of metric tensor given by

$$\gamma_{\alpha\beta}(\boldsymbol{v}) := \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\geq} v_\sigma - b_{\alpha\beta}v_3 \quad \text{for all } \boldsymbol{v} \in \mathcal{V}. \tag{2.25}$$

The main result is the following:

Theorem 2.10. *Let the space* $\mathcal{V} := (H_0^1(\omega))^2 \times L^2(\Omega)$ *be equipped with the weak topology of* $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$. *Then, for this topology, the functional* J^* *is the* Γ*-limit of the functionals* $J^*(\varepsilon)$.

Proof. It suffices to check that the functional J^* satisfies properties (i) and (ii) of Definition 2.6.

$$\boldsymbol{v}(\varepsilon) \rightharpoonup \boldsymbol{v} \text{ in } \mathcal{V} \Longrightarrow J^*(\boldsymbol{v}) \leq \lim \inf_{\varepsilon\to 0} J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)). \tag{2.26}$$

First case: the weak limit $\boldsymbol{v}$ depends on x_3. By definition of J^*, we have $J^*(\boldsymbol{v}) = +\infty$. It then suffices to show that $\lim \inf_{\varepsilon\to 0} J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) = +\infty$. Assume that this is false, i.e, assume that $\liminf_{\varepsilon\to 0} J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) < +\infty$. Then one can show that there exist a constant $c_1 > 0$ and a subsequence, still indexed by ε, such that $J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) \geq c_1$. Thus $\boldsymbol{v}(\varepsilon) \in \boldsymbol{H}^1(\Omega)$ and $J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) = J(\varepsilon)(\boldsymbol{v}(\varepsilon 0)$. It follows that there exist a constant $c_2 > 0$ such that

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon)(\boldsymbol{v}(\varepsilon)) e_{i\|j}(\varepsilon)(\boldsymbol{v}(\varepsilon))\sqrt{g(\varepsilon)}dx \leq c_2.$$

Using the generalized Korn's inequality (2.16), we deduce that th enorms $\|e_{i\|j}(\varepsilon_k)(\boldsymbol{v}\ (\varepsilon_k))\|_{0,\Omega}$, $\|v_\alpha(\varepsilon_k)\|_{1,\Omega}$ and $\|v_3(\varepsilon_k)\|_{0,\Omega}$ are all bounded independently of ε. Consequently, there exists a subsequence, still denoted $(\boldsymbol{v}(\varepsilon))_{\varepsilon>0}$ for convenience, and there exist functions $e_{i\|j} \in L^2(\Omega)$ such that

$$e_{i\|j}(\varepsilon_k)(\boldsymbol{v}(\varepsilon_k)) \Longrightarrow e_{i\|j} \text{ in } L^2(\Omega), \tag{2.27}$$

$$v_\alpha(\varepsilon_k) \Longrightarrow v_\alpha \text{ in } H^1(\Omega) \text{and } v_\alpha(\varepsilon_k) \to v_\alpha \text{ in } L^2(\Omega), \tag{2.28}$$

$$v_3(\varepsilon_k) \Longrightarrow v_3 \text{ in } L^2(\Omega), \tag{2.29}$$

where the function $\boldsymbol{v} = (v_i)$ is the weak limit in (2.26). It follows from the convergences (2.27)–(2.29) that the functions v_i are independent of x_3 (see Theorem 5.1 of [7]), which is a contradiction. Thus $\liminf_{\varepsilon\to 0} J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) = +\infty$.

Second Case: the weak limit $\boldsymbol{v}$ does not depend on x_3. If $\liminf_{\varepsilon\to 0} J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) = +\infty$, then (2.26) always holds. Assume that $\liminf_{\varepsilon\to 0} J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) < +\infty$. As in the first case, it follows that, for a subsequence still indexed by ε, we have $\boldsymbol{v}(\varepsilon) \in \boldsymbol{H}^1(\Omega)$, and there exist functions $e_{i\|j} \in L^2(\Omega)$ such that the convergencs (2.27)–(2.29) hold. Using the convergences $\Gamma^\sigma_{\alpha\beta} \to \Gamma^\sigma_{\alpha\beta}$, $\Gamma^3_{\alpha\beta}(\varepsilon) \to b_{\alpha\beta}$ in $\mathcal{C}^0(\overline{\Omega})$ (see Lemma 3.1 of [7]), we get

$$e_{\alpha\|\beta} = \gamma_{\alpha\beta}(\boldsymbol{v}). \tag{2.30}$$

The expression (2.13) of $J(\varepsilon)$, combined with the positive definiteness of the tensor $(g^{ij}(\varepsilon))$, implies

$$\begin{aligned} J(\varepsilon)(\boldsymbol{v}) \geq & \frac{1}{2}\int_\Omega \Big\{\frac{2\lambda\mu}{\lambda+2\mu} g^{\alpha\beta}(\varepsilon)\Big\} e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}dx \\ & + \frac{1}{2}\int_\Omega \{2\mu g^{\alpha\sigma}(\varepsilon) g^{\beta\tau}(\varepsilon)\} e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) e_{\sigma\|\tau}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}dx - L(\varepsilon)(\boldsymbol{v}). \end{aligned}$$

With the convergences $g^{\alpha\beta}(\varepsilon) \to a^{\alpha\beta}$ and $g(\varepsilon) \to a$ in $\mathcal{C}^0(\overline{\Omega})$, this inequality becomes

$$\lim_{\varepsilon \to 0} J(\varepsilon)(\boldsymbol{v}(\varepsilon)) \geq \frac{1}{4} \int_\Omega a^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\boldsymbol{v}) \gamma_{\sigma\tau}(\boldsymbol{v}) \sqrt{a} dx - L(\boldsymbol{v}).$$

This means that, for any convergent subsequence $J(\varepsilon)(\boldsymbol{v}(\varepsilon))$, we have

$$\lim_{\varepsilon \to 0} J(\varepsilon)(\boldsymbol{v}(\varepsilon)) \geq J(\boldsymbol{v}).$$

Then relation (2.26) is satisfied.

(ii) We now show that, for any $\boldsymbol{v} \in \mathcal{V}$, there exists a sequence $(\boldsymbol{v}(\varepsilon))$ in $\mathcal{V}$ such that

$$(\boldsymbol{v}(\varepsilon)) \Longrightarrow \text{ in } \mathcal{V} \text{ and } J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) \to J^*(v). \tag{2.31}$$

First case: the weak limit $\boldsymbol{v}$ depends on x_3. Let $(\boldsymbol{v}(\varepsilon))_{\varepsilon>0}$ be the sequence defined by $\boldsymbol{v}(\varepsilon) = \boldsymbol{v}$ for all $\varepsilon > 0$. Then it follows from the first part of the proof that

$$J^*(v) = \lim \inf_{\varepsilon \to 0} J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)).$$

On the other hand, since $\boldsymbol{v}$ depends on x_3, the definitiomn (2.21) of the functional J^* implies $J^*(\boldsymbol{v}) = +\infty$. Thus

$$J^*(\boldsymbol{v}) = \lim \inf_{\varepsilon \to 0} J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)),$$

and property (2.31) is verified in this cae.

Second case: the weak limit $\boldsymbol{v}$ does not depend on x_3. We consider a sequence $(\boldsymbol{v}(\varepsilon))_{\varepsilon>0}$ in the space $\boldsymbol{V}(\Omega)$ where, for all $\varepsilon > 0$, the function $\boldsymbol{v}(\varepsilon)$ is the unique solution of the minimization problem:

$$\boldsymbol{v}(\varepsilon) \in \boldsymbol{V}(\Omega) \quad \text{and} \quad J_{\boldsymbol{v}}(\varepsilon)(\boldsymbol{v}(\varepsilon)) = \inf_{\boldsymbol{w} \in \boldsymbol{V}(\Omega)} J_{\boldsymbol{v}}(\varepsilon)(\boldsymbol{w}),$$

where, for all $\boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{V}(\Omega)$,

$$J_{\boldsymbol{v}}(\varepsilon)(w) := \frac{1}{2} B(\varepsilon)(\boldsymbol{w}, \boldsymbol{w}) - \tilde{L}(\varepsilon)(\boldsymbol{v}, \boldsymbol{w}),$$

the bilinear form $B(\varepsilon)$ being defined on $\boldsymbol{V}(\Omega) \times \boldsymbol{V}(\Omega)$ by

$$B(\varepsilon)(\boldsymbol{u}, \boldsymbol{w}) := \int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon)(\boldsymbol{u}) e_{i\|j}(\varepsilon)(\boldsymbol{w}) \sqrt{g(\omega)} dx,$$

and the linear form $\tilde{L}(\varepsilon)$ being defined on $\boldsymbol{V}(\Omega)$ by

$$\tilde{L}(\varepsilon)(\boldsymbol{v}, \boldsymbol{w}) = \frac{1}{2} \int_\Omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{v}) e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{w}) \sqrt{a} dx.$$

We recall that the function $\boldsymbol{v}$ is fixed.

Equivalently, the function $\boldsymbol{v}(\varepsilon)$ solves the variational problem

$$B(\varepsilon)(\boldsymbol{v}(\varepsilon), \boldsymbol{w}) = \tilde{L}(\varepsilon)(\boldsymbol{v}, \boldsymbol{w}) \quad \text{for all } \boldsymbol{w} \in V(\Omega). \tag{2.32}$$

Since $J_{\boldsymbol{v}}(\varepsilon)(\boldsymbol{v}(\varepsilon 0) \leq J_{\boldsymbol{v}}(\varepsilon)(\mathbf{0}) = 0$, we deduce that there exists a constant $C(\boldsymbol{v})$ depending on $\boldsymbol{v}$, such that

$$\sum_{i,j} \|e_{i\|j}(\varepsilon)(\boldsymbol{v}(\varepsilon))\|_{0,\Omega}^2 \leq C(\boldsymbol{v}).$$

This, together with Korn's inequality (2.16), implies that the norms $\|e_{i\|j}(\varepsilon)(\boldsymbol{v}(\varepsilon))\|_{0,\Omega}$, $\|v_\alpha(\varepsilon)\|_{1,\Omega}$ and $\|v_3(\varepsilon)\|_{0,\Omega}$ are all bounded independently of ε. Consequently, there exists a subsequence, still denoted $(\boldsymbol{v}(\varepsilon))$, and there exist functions $e_{i\|j} \in L^2(\Omega)$, $\tilde{v}_\alpha \in H_0^1(\Omega)$ and $\tilde{v}_3 \in L^2(\Omega)$ such that

$$e_{i\|j}(\varepsilon)(\boldsymbol{v}(\varepsilon)) \Longrightarrow e_{i\|j} \text{ in } L^2(\Omega), \tag{2.33}$$

$$v_\alpha(\varepsilon) \Longrightarrow \tilde{v}_\alpha \text{ in } H^1(\Omega) \text{ and } v_\alpha(\varepsilon) \to \tilde{v}_\alpha \text{ in } L^2(\Omega), \tag{2.34}$$

$$v_3(\varepsilon) \Longrightarrow \tilde{v}_3 \text{ in } L^2(\Omega). \tag{2.35}$$

These convergences imply that (see step (i)) the functions $\tilde{v}_i$ are independent of x_3 and that

$$e_{\alpha\|\beta} = \gamma_{\alpha\beta}(\tilde{\boldsymbol{v}}) \tag{2.36}$$

where $\tilde{\boldsymbol{v}} := (\tilde{v}_i)$.

We fix $\boldsymbol{w} = (w_i)$ in the space $\boldsymbol{V}(\Omega)$, we multiply equation (2.32) by ε, and we let ε tend to 0. Using the weak convergences (2.33)–(2.35), we get

$$\int_\Omega \{2\mu a^{\alpha\sigma} e_{\sigma\|3} \partial_3 w_\alpha + [\lambda a^{\sigma\tau} e_{\sigma\|\tau} + (\lambda + 2\mu) e_{3\|3}] \partial w_3\} \sqrt{a} dx = 0. \tag{2.37}$$

By letting $\boldsymbol{w}$ vary in $\boldsymbol{V}(\Omega)$, we obtain

$$e_{3\|3} = -\frac{\lambda}{\lambda + 2\mu} a^{\alpha\beta} e_{\alpha\|\beta}, \tag{2.38}$$

$$e_{\alpha\|3} = 0. \tag{2.39}$$

Next, in equation (2.32), we choose $\boldsymbol{w} \in \boldsymbol{V}(\Omega)$ such that $\partial_3 \boldsymbol{w} = 0$, and we let ε tend to 0. We find that the function $\tilde{\boldsymbol{v}}$ satisfies

$$\int_\Omega \Big\{\frac{2\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})\Big\} \gamma_{\sigma\tau}(\tilde{\boldsymbol{v}}) \gamma_{\alpha\beta}(\boldsymbol{w}) \sqrt{a} dx$$
$$= \frac{1}{2} \int_\Omega a^{\alpha\sigma\beta\tau} \gamma_{\sigma\tau}(\boldsymbol{v}) \gamma_{\alpha\beta}(\boldsymbol{w}) \sqrt{a} dx.$$

Since the functions $\tilde{\boldsymbol{v}}$ and $\boldsymbol{w}$ do not depend on x_3, this equation can be written as

$$B(\bar{\tilde{\boldsymbol{v}}}, \bar{\boldsymbol{w}}) = B(\bar{\boldsymbol{v}}, \bar{\boldsymbol{w}}) \quad \text{for } \bar{\boldsymbol{w}} \in \boldsymbol{V}_M(\omega), \tag{2.40}$$

where the space $\boldsymbol{V}_M$ is defined by

$$\boldsymbol{V}_M(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \tag{2.41}$$

the bilinear form B is given by

$$B(\bar{\boldsymbol{z}},\bar{\boldsymbol{w}}) = \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\bar{z})\gamma_{\alpha\beta}(\bar{\boldsymbol{w}})\sqrt{a}dy \text{ for } (\bar{\boldsymbol{z}},\bar{\boldsymbol{w}}) \in \boldsymbol{V}_M(\omega)\times\boldsymbol{V}_M(\omega), \tag{2.42}$$

and where, if $\boldsymbol{w}$ is a function defined almost everywhere on Ω, we denote by $\bar{\boldsymbol{\omega}} = (\bar{\omega}_i)$ the function defined almost everywhere on ω by

$$\bar{\boldsymbol{\omega}}(y) := \frac{1}{2}\int_{-1}^{1}\boldsymbol{w}(y,x_3)dx_3. \tag{2.43}$$

In particular, if $\boldsymbol{w}$ does not depend on x_3, then

$$\boldsymbol{w}(y,x_3) = \bar{\boldsymbol{w}}(y).$$

Under either one of assumptions (2.14) and (2.15), it was shown by [14] and [15] that the bilinear form B is coercive on the space $\boldsymbol{V}_M(\omega)$. Thus, equation (2.40) implies

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}.$$

Therefore, since $\boldsymbol{v}$ is unique, the whole sequence $\boldsymbol{v}(\varepsilon)$ weakly converges to $\boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)$.

Finally, we have

$$\begin{aligned} J(\varepsilon)(\boldsymbol{v}(\varepsilon)) =& \frac{1}{2}B(\varepsilon)(\boldsymbol{v}(\varepsilon),\boldsymbol{v}(\varepsilon)) - L(\varepsilon)(\boldsymbol{v}(\varepsilon)) = \frac{1}{2}\tilde{L}(\varepsilon)(\boldsymbol{v},\boldsymbol{v}(\varepsilon)) - L(\varepsilon)(\boldsymbol{v}(\varepsilon)) \\ =& \frac{1}{4}\int_\Omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{v})e_{\alpha\|\beta}(\boldsymbol{v}(\varepsilon))\sqrt{a}dx - L(\varepsilon)(\boldsymbol{v}(\varepsilon)). \end{aligned}$$

Hence

$$J(\varepsilon)(\boldsymbol{v}(\varepsilon)) \to \frac{1}{4}\int_\Omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{v})\sqrt{a}dx - L(\boldsymbol{v}) = J(\boldsymbol{v}),$$

as $\varepsilon \to 0$, which we may write as

$$J^*(\varepsilon)(\boldsymbol{v}(\varepsilon)) \to J^*(\boldsymbol{v}).$$

□

This proof justifies the choice of the function spaces. Indeed, the estimates give bounds for the norms $\|v_\alpha(\varepsilon)\|_{1,\omega}$ and $\|v_3(\varepsilon)\|_{0,\omega}$. In other words, the bilinear forms corresponding to the functionals $J(\varepsilon)$ are equi-coercive on $H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)$. The spaces and the topology are thus "natural" for this problem.

Remark 2.11. The last step of the proof is a bit "artificial" since we use the existence and uniqueness of the two-dimenional linear problem: the coercivity of the bilinear form B is a certral argument. In [19], we give two variants of the proof, which do not require this result, and we show that an approach similar that of [10] in the nonlinear case leads to the result of Theorem 2.10: Consider the function $W : \boldsymbol{M}_{3,3} \to \boldsymbol{R}$ defined by

$$W(F) = \frac{\mu}{4}\|F + F^T - 2I\|^2 + \frac{\lambda}{8}[\mathrm{tr}(F + F^T - 2I)]^2,$$

where $\boldsymbol{M}_{3,3}$ is the space of 3×3 real matrices equipped with the norm $\|F\| = \sqrt{\mathrm{tr}(FF^T)}$, and I is the unit matrix of $\boldsymbol{M}_{3,3}$. Then it is easy to check that W is convex and satisfies the following growth property: There exists a constant $c > 0$ such that $W(F) \leq c(1 + \|F\|^2)$ for all $F \in \boldsymbol{M}_{3,3}$. If z_i, $i = 1,2,3$, are three vectors of $\boldsymbol{R}^3$, we denote by $(z_1|z_2|z_3)$ the 3×3 matrix which i-th column vector is z_i. Let us introduce the function $W_0 : \bar{\omega} \times \boldsymbol{M}_{3,2} \to \boldsymbol{R}$ by letting

$$W_0(y, \bar{F}) = \inf_{z\in\boldsymbol{R}^3} W((\bar{F}|z)A^{-1}(y)),$$

where $A^{-1}(y) = (\boldsymbol{a}^1(y)|\boldsymbol{a}^2(y)\boldsymbol{a}^3(y))^T$ for all $y \in \bar{\omega}$. Then the Γ-limit functional J of the sequence $J(\varepsilon)$ can be written as

$$J(\boldsymbol{v}) = \int_\Omega W_0(y, (\boldsymbol{a}_1 + \partial_1\boldsymbol{v}|\boldsymbol{a}_2 + \partial_2\boldsymbol{v}))\sqrt{a}dy - \int_\Omega f^i v_i\sqrt{a}dy$$

for all $\boldsymbol{v} \in \mathcal{V}$, $\boldsymbol{v}$ independent of x_3. This is another expression for the functional J defined in (2.22). The main difficulty of the proof remains the same as in that of Theorem 2.10, since Korn's inequality is used. Note that in the general nonlinear case studied by [10], the function W is assumed to satisfy a coerciveness property: $W(F) \geq c_1\|F\|^2 = c_2$ for all $F \in \boldsymbol{M}_{3,3}$, with $c_1 > 0$ and $c_2 \geq 0$. The function W that we choose does not satisfy this relation, but it is in fact Korn's inequality that plays that role, as a property of "coercivity after integration".

Note that the functional J defined in (2.22) is that which is associated with the two-dimensional membrane problem for a linearly elastic shell: for $\boldsymbol{v} \in \mathcal{V}$, $\boldsymbol{v}$ independent of x_3, we have

$$J(\boldsymbol{v}) = \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\bar{\boldsymbol{v}})\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\sqrt{a}dy - \int_\omega \Big(\int_{-1}^1 f^i dx_3\Big)v_i\sqrt{a}dy.$$

Now, we deduce a convergence result for the displacements:

Theorem 2.12. *The sequence* $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ *where, for all* $\varepsilon > 0$, $\boldsymbol{u}(\varepsilon)$ *is the solution of the scaled minimization problem* (2.9)–(2.10), *satisfies*

$$u_\alpha(\varepsilon) \to u_\alpha \quad \text{in } H^1(\Omega), \quad u_3(\varepsilon) \to u_3 \quad \text{in } L^2(\Omega),$$

where the function $\boldsymbol{u}$ *is independent of* x_3 *and* $\boldsymbol{u}$ *is the solution of the minimization problem*

$$J^*(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\mathcal{V}} J^*(\boldsymbol{v}),$$

the functional J^* *being defined in* (2.21). *The function* $\bar{\boldsymbol{u}}$ *is the solution of the two-dimensional membrane shell problem in linearized elasticity.*

Proof. We infer from Korn's inequality (2.16) that $\|u_\alpha(\varepsilon)\|_{0,\Omega}$ and $\|u_3(\varepsilon)\|_{0,\Omega}$ are bounded independently of ε. Thus the vector fields $\boldsymbol{u}(\varepsilon)$ belong to a compact set of $\mathcal{V}$ for the weak topology of $H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)$. We can also show that any weak limit of $\boldsymbol{u}(\varepsilon)$ is independent of x_3. Theorem 2.7 implies that there exist a function $\boldsymbol{u} \in \mathcal{V}$ and a subsequence $(\boldsymbol{u}(\varepsilon_k))$ such that $\boldsymbol{u}(\varepsilon_k) \Longrightarrow \boldsymbol{u}$ in $\mathcal{V}$ and $\boldsymbol{u}$ satisfies $J^*(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\mathcal{V}} J^*(\boldsymbol{v})$. Thus, the function $\boldsymbol{u}$ is unique and the whole sequence $(\boldsymbol{u}(\varepsilon))$ converges to $\boldsymbol{u}$ in $\mathcal{V}$.

The strong convergences are then obtained as in [7], Thm 5.1. □

3. Justification of the Two-Dimensional Flexural Model

We proceed as in the membrane case. There is no geometric assumption on the middle surface S and the shell may be only partially clamped on $\Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon]$. The "right" choice of scalings is then (cf. [9]):

$$u_i(\varepsilon)(x) = u_i^\varepsilon(x^\varepsilon) \quad \text{for all } x \in \Omega, \tag{3.1}$$

and the forces must be $O(\varepsilon^2)$ in the sense that:

$$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^i(x) \quad \text{for all } x \in \Omega, \tag{3.2}$$

where $x^\varepsilon \in \Omega^\varepsilon$ and $x \in \Omega$ are in the correspondence defined in section 2.1. The energy functional corresponding to the scaled three-dimensional problem is

$$J(\varepsilon)(\boldsymbol{v}) = \frac{1}{2\varepsilon}\int_\Omega \{A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)(\boldsymbol{v})e_{i\|j}(\varepsilon)(\boldsymbol{v})\}\sqrt{g(\varepsilon)}dx - L(\varepsilon)(\boldsymbol{v}); \tag{3.3}$$

see (2.5)–(2.8) and (2.12) for the definitions of the functions $A^{ijkl}(\varepsilon)$, $e_{i\|j}(\varepsilon)$, and $L(\varepsilon)$.

Let $\boldsymbol{V}_F(\varepsilon)$ be the space defined by:

$$\begin{aligned}\boldsymbol{V}_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in & H^1(\omega)\times H^1(\omega)\times H^2(\omega),\\ & + b_\alpha^\sigma(\partial_\beta\eta_\sigma - \Gamma_{\beta\sigma}^\tau\eta_\tau) + b_\alpha^\sigma|_\beta\eta_\sigma - c_{\alpha\beta}\eta_3,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned} b_\beta^\sigma|_\alpha &:= \partial_\alpha b_\beta^\alpha + \Gamma_{\alpha\tau}^\sigma b_\beta^\tau - \Gamma_{\beta\alpha}^\tau b_\tau^\sigma,\\ c_{\alpha\beta} &:= b_\alpha^\sigma b_{\sigma\beta}.\end{aligned}$$

For any $\boldsymbol{v} \in \boldsymbol{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in H^1(\Omega);\ \boldsymbol{v} = 0 \text{ sur } \Gamma_0\}$, the functional J^*: $\boldsymbol{V}(\Omega) \to \boldsymbol{R}$ is defined in the following way:

$$J^*(v) = \begin{cases} J(\boldsymbol{v}) & \text{if } \boldsymbol{v} \text{ is independent of } x_3 \text{ and } \bar{\boldsymbol{v}} \in \boldsymbol{V}_F(\omega),\\ +\infty & \text{otherwise,}\end{cases} \tag{3.5}$$

where

$$J(\boldsymbol{v}) := \bar{J}(\bar{\boldsymbol{v}}) = \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\bar{\boldsymbol{v}})\rho_{\alpha\beta}(\bar{\boldsymbol{v}})\sqrt{a}dy - L(\bar{\boldsymbol{v}}), \tag{3.6}$$

$$L(\bar{\boldsymbol{v}}) := \int_\omega \Big(\int_{-1}^1 f^i dx_3\Big)\bar{\boldsymbol{v}}_i\sqrt{a}dy, \tag{3.7}$$

and $\bar{\boldsymbol{v}}$ is the average associated with $\boldsymbol{v}$ (see (2.43)).

We use the following Korn's inequality (see [9], Thm. 5.1):

$$\|\boldsymbol{v}\|_{1,\Omega} \leq \frac{C}{\varepsilon}\Big\{\sum_{i,j}\|e_{i\|j}(\varepsilon)(\boldsymbol{v})\|_{0,\Omega}^2\Big\}^{1/2} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega), \tag{3.8}$$

to prove the Γ-convergence of the functionals $J(\varepsilon)$.

Theorem 3.1. *The functional J^* is the Γ-limit of functionals $J(\varepsilon)$ in $\boldsymbol{V}(\Omega)$ equipped with the weak topology of $H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$.*

For a proof, see [19].

As in the membrane case, we deduce from this result a weak convergence theorem for the displacements. The strong convergences are then obtained as in [9], Thm. 5.1. We call $\boldsymbol{u}(\varepsilon)$ the solution of the scaled minimization problem:

$$\boldsymbol{v}(\varepsilon) \in \boldsymbol{V}(\Omega) \text{ and } J(\varepsilon)(\boldsymbol{v}(\varepsilon)) = \inf_{\boldsymbol{w}\in\boldsymbol{V}(\Omega)} J(\varepsilon)(\boldsymbol{w}), \tag{3.9}$$

where $J(\varepsilon)$ is defined as in (3.3).

Theorem 3.2. *The sequence $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ satisfies*

$$u_i(\varepsilon) \to u_i \text{ in } H^1(\Omega), \tag{3.10}$$

where the function $\boldsymbol{u}$ is independent of x_3, $\bar{\boldsymbol{u}} \in \boldsymbol{V}_F(\omega)$ and $\boldsymbol{u}$ is the solution of the minimization problem

$$J^*(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\boldsymbol{V}(\Omega)} J^*(\boldsymbol{v}), \tag{3.11}$$

where J^* is defined by (3.5). The function $\bar{\boldsymbol{u}}$ is the solution of the two-dimensional flexural shell problem in linearized elasticity.

References

[1] A.L. Goldenveizer, Derivation of an approximate theory of shells by means of asymptotic integration of the equations of the theory of elasticity, *Prikl. Mat. Mech.,* **27** (1963), 593–608.

[2] E. Sanchez-Palencia, Passages à la limite de l'élasticité tridimensionnelle à la théorie asymptotique des coques minces, C.R. Acad. Sci. Paris, Sér. II **311** (1990), 909–916.

[3] P. Destuynder, Sur une Justification des Modèles de Plaques et de Coques par les Méthodes Asymptotiques, Doctoral Dissertation, Université Pierre et Marie Curie, Paris, 1980.

[4] P. Destuynder, A classification of thin shell theories, *Acta Applicandæ,* **4** (1985), 15–63.

[5] D. Caillerie, E. Sanchez-Palencia, Elastic thin shells: asymptotic theory in the anisotropic and heterogeneous cases, *Math. Models Methods Appl. Sci.,* **5** (1996), 473–496.

[6] B. Miara, E. Sanchez-Palencia, Asymptotic Analysis of linearly elastic shells, *Asymptotic Anal.,* **12** (1996), 41–54.

[7] P.G. Ciarlet, V. Lods, Asymptotic analysis of linearly elastic shells. I. Justification of membrane shell equations, *Arch. Rational Mech. Anal.,* to appear.

[8] P.G. Ciarlet, V. Lods, Asymptotic analysis of linearly elastic shells: "Generalized membrane shells", *J. Elasticity,* **43** (1996), 147–188.

[9] P.G. Ciarlet, C. Lods, B. Miara, Asymptotic analysis of linearly elastic shells. II. Justification of flexural shell equations, to appear.

[10] H. Le Dret, A. Raoult, The membrane shell model in nonlinear elasticity: A variational asymptotic derivation, *J. Nonlinear Sci.,* **6** (1996), 59–84.

[11] E. Acerbi, G. Buttazzo, D. Percivale, A variational definition of the strain energy for an elastic string, *J. Elasticity,* **25** (1991), 137–148.

[12] F. Bourquin, P.G. Ciarlet, G. geymonat, A. Raoult, Γ-convergence et analyse asymptotique des plaques minces, *C.R. Acad. Sci. Paris, Sér. I,* **315** 1017–1024.

[13] P.G. Ciarlet, Mathematical Elasticity, Vol. III: Theory of Shells, North-Holland, to appear.

[14] P.G. Ciarlet, E. Sanchez-Palencia, An existence and uniqueness theorem for the two-dimensional linear membrane shell equations, *J. Math. Pures Appl.,* **75** (1996), 51–67.

[15] P.G. Ciarlet, V. Lods, On the ellipticity of linear membrane shell equations, *J. Math. Pures Appl.,* **75** (1996), 107–124.

[16] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, *Comm. Pure Appl. Math.,* **17** (1964), 35–92.

[17] G. Dal Maso, An Introduction to Γ-Convergence, Progress in Nonlinear Differential Equations and Their Applications, Birkäuser, Basel, 1993.

[18] H. Attouch, Variational Convergence for Functions and Operators, Pitman, Boston, 1984.

[19] K. Genevey, Sur Quelques Questions Liées à la Théorie mathématique des Coques Minces, Doctoral Dissertation, Université Pierre et Marie Curie.