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THE L^2- NORM ERROR ESTIMATE OF NONCONFORMING FINITE ELEMENT METHOD FOR THE 2ND ORDER ELLIPTIC PROBLEM WITH THE LOWEST REGULARITY [*1)]

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Abstract

The abstract L^2-norm error estimate of nonconforming finite element method is established. The uniformly L^2-norm error estimate is obtained for the nonconforming finite element method for the second order elliptic problem with the lowest regularity, i.e., in the case that the solution $u \in H^1(\Omega)$ only. It is also shown that the L^2-norm error bound we obtained is one order heigher than the energe-norm error bound.

Key words: L^2-norm error estimate, nonconforming f.e.m., lowest regularity

1. Introduction

This paper is concerned with the uniformly L^2- norm error estimate of the nonconforming finite method for the second order elliptic problem with the lowest regularity, i.e., in the case that the solution $u \in H^1(\Omega)$ only, but not in $H^2(\Omega)$.

For the conforming finite element method of the second order elliptic problem, it is well known that using the Aubin-Nitsche lemma obtained the L^2- norm error bound, which is one order of h, the parameter of triangulation, higher than the H^1- norm error bound, in the case that the solution u of the primale problem is smooth enough, i.e., $u \in H^2(\Omega)$ (c.f.[1]). And recently, Schatz and Wang [2] considered the uniformly L^2- norm error bound for the conforming finite element method of second order elliptic problem in the case that the solution u is not smooth enough, i.e., $u \in H^1(\Omega)$ only, but not in $H^2(\Omega)$.

In order to consider the L^2- norm error estimate for the nonconforming finite element method, we need the Aubin-Nitsche lemma for the nonconforming finite element method, which has been considered in [4], and for which we now give a clear expression and a simple proof.

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Firstly, let us state the Aubin-Nitsche lemma for the conforming finite element method.

Consider the variational elliptic problem as follows

$$\begin{cases} \text{Find } \ u \in H_0^1(\Omega) \ \text{ such that} \\ a(u,v) = (f,v) \qquad \forall v \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where

$$a(u,v) \equiv \int_\Omega a_{ij}(x)\partial_i u \partial_j v dx, \tag{1.2}$$

$$(f,v) \equiv \int_\Omega f \cdot v dx \tag{1.3}$$

and $a_{ij}(x) \in L^\infty(\Omega), f \in L^2(\Omega)$,

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{2} \xi_i^2 \quad \forall x \in \Omega, \quad \xi = (\xi_1,\xi_2)^T \in R^2. \tag{1.4}$$

Then the conforming finite element approximation of (1.1) is as follows, let $\tilde{V_h} \subset H_0^1(\Omega)$ be the finite element subspace of $H_0^1(\Omega)$

$$\begin{cases} \text{Find } \ u_h \in \tilde{V_h}, \qquad \text{such that} \\ a(u_h,v_h) = (f,v_h) \qquad \forall \ \ v_h \in \tilde{V_h}. \end{cases} \tag{1.5}$$

Then it is well known that

Theorem 1. (Aubin-Nitsche Lemma)(c.f.[1])

Let u and u_h be the solutions of the problems (1.1) *and* (1.5) *respectively, then there exists $C = Const. > 0$, such that*

$$\|u-u_h\|_0 \leq \|u-u_h\|_1 \sup_{g\in L^2(\Omega)} \{\frac{1}{\|g\|_0} \inf_{\phi_h\in\tilde{V_h}} \|\phi_g - \phi_h\|_1\}, \tag{1.6}$$

where, for any given $g \in L^2(\Omega), \phi_g \in H_0^1(\Omega)$ such that

$$a(v,\phi_g) = (g,v) \qquad v \in H_0^1(\Omega). \tag{1.7}$$

Corollary 2. ([2])

Assume that $f \in L^2(\Omega)$, then given any $\epsilon > 0$, there exists an $h_0 = h_0(\epsilon) > 0$ such that for all $0 < h \leq h_0(\epsilon)$,

$$\|u-u_h\|_0 \leq \epsilon\|u-u_h\|_1. \tag{1.8}$$

The proof can be completed from that $\|\phi_g - (\phi_g)_h\|_1 \leq \epsilon\|g\|_0$ (c.f.[2]) and (1.6).

Note that the Corollary 2 shows that the L^2- norm error bound is one order of ϵ higher than the H^1- norm error bound for the conforming finite element approximation to the second order problem in the case that the solution $u \in H^1(\Omega)$ only, but not in $H^2(\Omega)$.

2. L^2- norm Error Estimate for Nonconforming Finite Element Method

We now turn to consider the L^2- norm error estimate for nonconforming finite element method for second order elliptic problem (1.1) in the case that the solution $u \in H_0^1(\Omega)$ only.

Firstly, we give a clear expression and a simple proof for the Aubin-Nitsche Lemma for the nonconforming finite element approximation to the second order problem. Let $V_h \not\subset H_0^1(\Omega)$ be the nonconformin finite element space, and $u_h \in V_h$ be the solution of the following problem

$$a_h(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h, \tag{2.1}$$

where

$$a_h(u_h, v_h) \equiv \sum_K \int_k a_{ij}(x) \partial_i u_h \partial_j v_h dx, \tag{2.2}$$

and

$$\|v_h\|_h \equiv (\sum_K |v_h|_{1,K}^2)^{\frac{1}{2}} \tag{2.3}$$

Theorem 3. *Let u and u_h be the solutions of the problems* (1.1) *and* (2.1) *respectively, then there exists C= Const.* > 0, *such that*

$$\|u - u_h\|_0 \le C \sup_{g \in L^2(\Omega)} \frac{1}{\|g\|_0} \{ \|u - u_h\|_h \|\phi_g - (\phi_g)_h\|_h + E_h(u, (\phi_g)_h - \phi_g) + E_h^*(u_h - u, \phi_g) \}, \tag{2.4}$$

where $\phi_g \in H_0^1(\Omega)$ is the solution of (1.7), and $(\phi_g)_h$ is the nonconforming finite element approximation of ϕ_g : $(\phi_g)_h \in V_h$, such that

$$a_h(v_h, (\phi_g)_h) = (g, v_h) \qquad \forall \ \ v_h \in V_h, \tag{2.5}$$

and

$$E_h(u, w_h) = a_h(u, w_h) - a_h(u_h, w_h) = a_h(u, w_h) - (f, w_h), \tag{2.6}$$

$$E_h^*(w_h, \phi_g) = a_h(w_h, \phi_g) - a_h(w_h, (\phi_g)_h) = a_h(w_h, \phi_g) - (g, w_h). \tag{2.7}$$

Before proving the theorem, it should be noted that, when $V_h \subset H_0^1(\Omega)$, i.e., the conforming finite element method, the abstract error estimate (2.4) reduced the estimste (1.6). In fact, for the conforming finite element method, the expressions (2.6) and (2.7) vanish and that

$$\|\phi_g - (\phi_g)_h\|_h = \|\phi_g - (\phi_g)_h\|_1 \le C \inf_{\phi_h \in V_h} \|\phi_g - \phi_h\|_1. \tag{2.8}$$

Proof of Theorem 3.

Noting that

$$\|u - u_h\|_0 = \sup_{g \in L^2(\Omega)} \frac{|(g, u - u_h)|}{\|g\|_0}, \tag{2.9}$$

and

$$(g, u-u_h) = a(u,\phi_g) - a_h(u_h,(\phi_g)_h) = a_h(u-u_h, \phi_g - (\phi_g)_h)$$
$$+ \{a_h(u,(\phi_g)_h) - a_h(u_h,(\phi_g)_h)\} + \{a_h(u_h,\phi_g) - a_h(u_h,(\phi_g)_h)\}$$
$$= a_h(u-u_h, \phi_g - (\phi_g)_h) + E_h(u,(\phi_g)_h) + E_h^*(u_h,\phi_g). \tag{2.10}$$

We have

$$|a_h(u-u_h, \phi_g - (\phi_g)_h)| \le C\|u-u_h\|_h \cdot \|\phi_g - (\phi_g)_h\|_h. \tag{2.11}$$

Taking into account that u and ϕ_g are the solutions of the problems (1.1) and (1.7) respectively, we have

$$E_h(u,\phi_g) = E_h^*(u,\phi_g) = 0, \tag{2.12}$$

from which, it can be seen that

$$E_h(u,(\phi_g)_h) = E_h(u,(\phi_g)_h - \phi_g), \tag{2.13}$$

$$E_h^*(u_h,\phi_g) = E_h^*(u_h - u,\phi_g). \tag{2.14}$$

Summarizing (2.9)–(2.14), the proof is completed.

We now give the uniformly L^2- norm error estimate of nonconforming finite element approximation of the problem (1.1) with the solution $u \in H_0^1(\Omega)$ only.

Theorem 4. *Assume that the solution of the problem* (1.1) $u \in H_0^1(\Omega)$, *and* $f \in L^2(\Omega)$, *the triangulation* $\mathcal{T}_h$ *of the polygonal* Ω *is quasi-uniform and satisfies the inverse hypothesis* (*c.f.*[1]), *and the nonconforming finite element space* $V_h \not\subset H_0^1(\Omega)$ *possesses the following property, for any given* $\phi \in C_0^\infty(\Omega)$, *there exists C= Const.* > 0 *independent of h, such that*

$$|\sum_K \int_{\partial K} \partial_\nu \phi \cdot w_h ds| \le Ch\|\phi\|_2 \cdot \|w_h\|_h, \qquad \forall \ \ w_h \in V_h, \tag{2.15}$$

where $K \in \mathcal{T}_h$ *is the element with the edge* $\partial K, \partial_\nu$ *denotes the conormal derivative operator associated with the bilinear form* $a(\cdot,\cdot)$ *in* (1.2) *on* ∂K. *Then for any given* $\epsilon > 0$, *there exists* $h_1 = h_1(\epsilon) > 0$, *such that*

$$\|u-u_h\|_0 \le \epsilon\{\|u-u_h\|_h + \epsilon\|f\|_0\}, \qquad as \ \ 0 < h \le h_1(\epsilon). \tag{2.16}$$

Proof. By the Lemma 1 in [2], let $D = \{f : f \in L^2(\Omega), \|f\|_0 = 1\}, W = \{u : u = Tf, \forall f \in D\}$, and $W_* = \{u_* : u_* = T^*g, \forall g \in D\}$ where $u = Tf \in H_0^1(\Omega)$, and $u_* = T^*g \in H_0^1(\Omega)$ are the solutions of (1.1) and (1.7) respectively, i.e.,

$$a(Tf,v) = (f,v) \qquad \forall \ \ v \in H_0^1(\Omega),$$

and

$$a(v,T^*g) = (g,v) \qquad \forall \ \ v \in H_0^1(\Omega),$$

then W and W_* are precompact in $H_0^1(\Omega)$. And due to that the space $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, then there exists a finite open cover $\{S(\phi_i;\epsilon)\}_{i=1}^n, \bar{W} \subset \cup_{i=1}^n S(\phi_i;\epsilon), \bar{W}_* \subset \cup_{i=1}^n S(\phi_i;\epsilon)$ where $\phi_i \in C_0^\infty(\Omega), S(\phi_i;\epsilon) = \{v \in H_0^1(\Omega) : \|v - \phi_i\|_1 \leq \epsilon\}, 1 \leq i \leq n$.

For any given $f, g \in L^2(\Omega), f, g \neq 0$, set

$$\bar{f} = \frac{f}{\|f\|_0}, \quad \bar{u} = \frac{u}{\|f\|_0} \quad \bar{u}_h = \frac{u_h}{\|f\|_0}, \tag{2.17}$$

and

$$\bar{g} = \frac{g}{\|g\|_0}, \quad \bar{\phi}_g = \frac{\phi_g}{\|g\|_0}, \quad (\bar{\phi}_g)_h = \frac{(\phi_g)_h}{\|g\|_0}, \tag{2.18}$$

then

$$a(\bar{u}, v) = (\bar{f}, v) \qquad \forall \ v \in H_0^1(\Omega), \tag{2.19}$$

$$a_h(\bar{u}_h, v_h) = (\bar{f}, v_h) \qquad \forall \ v_h \in V_h; \tag{2.20}$$

and

$$a(v, \bar{\phi}_g) = (\bar{g}, v) \qquad \forall \ v \in H_0^1(\Omega), \tag{2.21}$$

$$a_h(v_h, (\bar{\phi}_g)_h) = (\bar{g}, v_h) \qquad \forall \ v_h \in V_h. \tag{2.22}$$

By the Theorem 3, we have

$$\begin{aligned} \|\bar{u} - \bar{u}_h\|_0 \leq C \sup_{g \in L^2(\Omega)} \{ & \|\bar{u} - \bar{u}_h\|_h \cdot \|\bar{\phi}_g - (\bar{\phi}_g)_h\|_h \\ & + E_h(\bar{u}, (\bar{\phi}_g)_h - \bar{\phi}_g) + E_h^*(\bar{u}_h - \bar{u}, \bar{\phi}_g)\}. \end{aligned} \tag{2.23}$$

From the Theorem 3.1 in [3], it can be seen that there exists $h_1' = h_1'(\epsilon) > 0$, such that

$$\|\bar{\phi}_g - (\bar{\phi}_g)_h\|_h \leq \alpha_1 \epsilon, \qquad \text{as} \ \ 0 < h \leq h_1'. \tag{2.24}$$

And by the similar way as in the proof of Theorem 3.1 in [3], we have, there exists $h"_1 = h"_1(\epsilon, \bar{W}, \bar{W}_*)$ such that

$$\begin{cases} |E_h(\bar{u}, (\bar{\phi}_g)_h - \bar{\phi}_g)| \leq \alpha_2 \epsilon \|(\bar{\phi}_g)_h - \bar{\phi}_g\|_h, \\ |E_h^*(\bar{u} - \bar{u}_h, \bar{\phi}_g)| \leq \alpha_2 \epsilon \|\bar{u}_h - \bar{u}\|_h, \qquad \text{as} \ \ 0 < h \leq h"_1, \end{cases} \tag{2.25}$$

where the parameter $\alpha_1, \alpha_2 > 0$ will be determined in the following. Thus we have

$$\begin{aligned} \|\bar{u} - \bar{u}_h\|_0 &\leq C\epsilon\{(\alpha_1 + \alpha_2)\|\bar{u} - \bar{u}_h\|_h + \alpha_2 \|\bar{\phi}_g - (\bar{\phi}_g)_h\|_h\} \\ &\leq C\epsilon\{(\alpha_1 + \alpha_2)\|\bar{u} - \bar{u}_h\|_h + \alpha_1 \alpha_2 \epsilon\} \\ &\leq \epsilon\{\|\bar{u} - \bar{u}_h\|_h + \epsilon\}, \qquad \text{as} \ \ 0 < h \leq h_1(\epsilon) = \min(h_1', h"_1), \end{aligned} \tag{2.26}$$

when the parameter α_1, α_2 have been chosen as follows

$$\alpha_1 = \alpha_2 = \frac{1}{2C},$$

and it is not losing the generality to assume that the C= Const. > 1 in (2.4).

Finally from $\|\bar{u} - \bar{u}_h\|_0 = \|u - u_h\|_0 \cdot \|f\|_0$ and $\|\bar{u} - \bar{u}_h\|_h = \|u - u_h\|_h \cdot \|f\|_0$, the proof is completed.

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