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CONVERGENCE OF VORTEX METHODS FOR 3-D EULER EQUATIONS*

Jia-fu Lin
(*Institute of System Science, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, 100080, China; School of Mathematical Sciences, Peking University, Beijing 100871, China*)

Abstract

In this paper we apply an approach introduced in [6] [7], where continuous norms and hign order estimates and extension are used, to study the convergence of vortex methods for the 3-D Euler equations in bounded domains as the initial vorticity ω_0 and the curl of the body force f are non-compactly supported functions. Convergence results are proved.

Key words: Euler equations, Vortex methods, Convergence, Initial-boundary value problem.

1. Introduction

The convergence problem of vortex methods for the Euler equations has been studied by many authors. Hald and DelPrete proved the convergence for two-dimensional initial value problems [3]. Three-dimensional initial value problems were studied by Beale and Majda [2] and Beale [1]. Ying [4] and Ying and Zhang [5], [6] proved the convergence of vortex methods for two-dimensional initial-boundary value problems of the Euler equations. Ying [7] proved the convergence of vortex methods for three-dimensional initial-boundary value problems of the Euler equations under the assumption that the initial vorticity ω_0 and the curl of the body force f are compactly supported.

In this paper, we will prove the convergence of the vortex method for three-dimensional initial-boundary value problems without assuming that the ω_0 and $\nabla\times f$ are compactly supported. In contrast to [7], there are two new difficulties. One is how to extend the physical quantities such as velocity and the force function outside Ω. The other one is that the approximate velocity g^ϵ (ie. eqn. (20)) is no longer divergence-free outside Ω. We will use the approach in [4] [5] and [7] to perform the extension of the physical quantites outside Ω. Although $\nabla\cdot g^\epsilon\neq 0$ outside Ω, we will show that $\nabla\cdot g^\epsilon$ is small (see eqn. (48) (49)). This is sufficient for our convergence analysis.

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2. Formulation of Vortex Methods

We consider the initial-boundary value problems of inviscid incompressible flow as follows:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \frac{\nabla p}{\rho} = f, \tag{1}$$

$$\nabla\cdot u = 0, \tag{2}$$

$$u\cdot n|_{x\in\partial\Omega} = 0, \tag{3}$$

$$u|_{t=0} = u_0(x), \tag{4}$$

where $u = (u_1, u_2, u_3)\in\mathbf{R}^3$ is velocity, $p\in\mathbf{R}$ is pressure, ρ is a constant standing for density, f is body force, u_0 is the initial distribution of the velocity satisfying $\nabla\cdot u_0 = 0$ and $u_0\cdot n|_{\partial\Omega} = 0$ and ∇ is the gradient operator $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$. The domain Ω is assumed bounded with a sufficiently smooth boundary $\partial\Omega$ and n is the unit outward normal vector along the boundary. For simplicity we assume Ω is simply connected and convex. Let $\omega = \nabla\times u$ be the vorticity and F be the curl of the body force f, then applying the operator curl to the equation (1) and the initial condition (4), we obtain

$$\frac{\partial \omega}{\partial t} + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = F, \tag{5}$$

$$\omega|_{t=0} = \omega_0 \equiv \nabla\times u_0. \tag{6}$$

To get velocity u from the vorticity ω, we need the stream function, which is not unique for three-dimensional problems. We accept the formulation in [7]

$$-\triangle\psi + \nabla z = \omega,\ u = \nabla\times\psi, \tag{7}$$

$$\nabla\cdot\psi = 0, \tag{8}$$

$$\psi\times n\ |_{x\in\partial\Omega} = 0,\ z\ |_{x\in\partial\Omega} = 0, \tag{9}$$

where ψ is the stream function. Problems (7)-(9) admit a unique smooth solution for any smooth ω (see [7], §4).

We assume that the problem (1)-(4) admits a sufficiently smooth solution (u,p) on $\overline{\Omega}\times[0,T]$. Under this assumption, we consider the vortex method formulation.

Let positive constants h and ϵ be mesh sizes. $j = (j_1, j_2, j_3) \in \mathbf{Z}^3$, $B_j = \{x; j_i h < x_i < (j_i+1)h, i = 1,2,3\}$, $X_j = ((j_1+\frac{1}{2})h, (j_2+\frac{1}{2})h, (j_3+\frac{1}{2})h)$. We define a "vortex blob" function $\zeta(x)$ with a support in ball $\{x; |x| \le 1\}$, which satisfies

$$\int \zeta(x)dx = 1. \tag{10}$$

Consider the following scheme: Set $\Omega_d = \{x; \mathrm{dist}(x,\overline{\Omega}) < d\}$, where $d > 0$ is a parameter. We solve the problem in $\overline{\Omega}_d\times[0,T]$. u_0, F are not defined outside Ω and $\Omega\times[0,T]$ and we extend u_0, F in the following way. By (7)-(9), we get $\psi(x,0)$, then extend it smoothly to $\mathbf{R}^3$. $\psi(x,0)$ needn't satisfy the divergence free condition outside Ω and we assume $\psi(x,0)$ is compactly supported. Set $u_0 = \nabla\times\psi(x,0)$, then u_0 is extended too. u_0 is compactly supported and $\nabla \cdot u_0 = 0$ in $\mathbf{R}^3$. F can be easily extended to $\mathbf{R}^3 \times [0,T]$ with a compact support.

The approximation of ω_0 is

$$\omega_0 \approx \sum_{j\in J} \alpha_j \zeta_\epsilon(x - X_j), \tag{11}$$

where $\alpha_j = \omega_0(X_j)h^3$, $\zeta_\epsilon(x) = \frac{1}{\epsilon^3}\zeta(\frac{x}{\epsilon})$, and $J = \{j; X_j \in \Omega_d\}$.

The approximate solution of (5)-(9) is solved as follows:

$$\omega^\epsilon(x,t) = \sum_{j\in J} \alpha_j^\epsilon(t)\zeta_\epsilon(x - X_j^\epsilon(t)), \tag{12}$$

$$\frac{d\alpha_j^\epsilon}{dt} = (\alpha_j^\epsilon(t)\cdot\nabla)g^\epsilon(X_j^\epsilon(t),t) + h^3F(X_j^\epsilon(t),t), \tag{13}$$

$$\alpha_j^\epsilon(0) = \alpha_j, \tag{14}$$

$$\frac{dX_j^\epsilon}{dt} = g^\epsilon(X_j^\epsilon(t),t), \tag{15}$$

$$X_j^\epsilon(0) = X_j, \tag{16}$$

$$-\triangle\psi^\epsilon + \nabla z^\epsilon = \omega^\epsilon,\ u^\epsilon = \nabla\times\psi^\epsilon, \tag{17}$$

$$\nabla\cdot\psi^\epsilon = 0, \tag{18}$$

$$\psi^\epsilon \times n\ |_{x\in\partial\Omega} = 0,\ z^\epsilon\ |_{x\in\partial\Omega} = 0, \tag{19}$$

$$g^\epsilon(x,t) = \sum_{i=1}^M a_i u^\epsilon(x^{(i)},t), \tag{20}$$

where a_i satisfies the following algebraic system

$$\sum_{i=1}^M (-i)^j a_i = 1,\ (j = 0,1,\cdots,M-1.). \tag{21}$$

And the definition of $x^{(i)}$ is as follows:

$$\text{if } x\in\overline{\Omega},\ \text{then } x^{(i)} = x;\ \text{otherwise } x^{(i)} = (i+1)y - ix, \tag{22}$$

where y is the nearest point on $\partial\Omega$ to x. To solve (12)-(22), further discretization is needed. For instance (17)-(19) may be solved by a finite element scheme [8].

To prove the convergence of the scheme (12)-(22). We extend (u,p) in the following way. Let (u,p) be the solution of (1)-(4). By (7)-(9) we can determine ψ, then extend ψ smoothly to $\mathbf{R}^3\times[0,T]$ and let ψ be compactly supported. Similarly, p is extended to $\mathbf{R}^3\times[0,T]$. By (7) we can determine u, then by (1), we get an extension of f. It is different from the above extension. We write it as $\tilde{f}$, and $\tilde{F} = \nabla\times\tilde{f}$. We require the extension of ψ at $t=0$ is corresponding with the extension of u_0 above. According to this extension, (u,p) is the solution of initial value problems. Convergence of vortex method for three-dimensional initial value problems has been proved in [7]. We can use these results for initial value problems.

3. Convergence for Initial-Boundary Value Problems

First, we define the characteristic curves $\xi(t;\eta,l)$:

$$\frac{d}{dt}\xi(t;\eta,l) = u(\xi(t;\eta,l),t), \tag{23}$$

$$\xi(l;\eta,l) = \eta. \tag{24}$$

Similarly we can define $\xi^\epsilon(t;\eta,l)$ if the function u in (23) is replaced by g^ϵ. Set $x(\eta,t) = \xi(t;\eta,0)$, $x^\epsilon(\eta,t) = \xi^\epsilon(t;\eta,0)$, and $e(t) = x(\cdot,t) - x^\epsilon(\cdot,t)$ in $L^p(\Omega_d)$. The Sobolev norm $|e(t)|_{l,p,\Omega_d}$ is defined in the usual way. Furthermore, let $\alpha(\eta,t)$ and $\alpha^\epsilon(\eta,t)$ be the solutions of the following:

$$\frac{d\alpha}{dt} = (\alpha\cdot\nabla)u(x(\eta,t),t) + h^3\tilde{F}(x(\eta,t),t), \tag{25}$$

$$\alpha(\eta,0) = \omega_0(\eta)h^3, \tag{26}$$

$$\frac{d\alpha^\epsilon}{dt} = (\alpha^\epsilon\cdot\nabla)g^\epsilon(x^\epsilon(\eta,t),t) + h^3F(x^\epsilon(\eta,t),t), \tag{27}$$

$$\alpha^\epsilon(\eta,0) = \omega_0(\eta)h^3. \tag{28}$$

By (5)-(6), (13)-(16), (23)-(28) it is easy to see that $x^\epsilon(X_j,t) = X_j^\epsilon(t)$, $\alpha^\epsilon(X_j,t) = \alpha_j^\epsilon(t)$ and $\alpha(\eta,t) = \omega(x(\eta,t),t)h^3$. We define $\overline{\omega}(t) = (\alpha(\cdot,t) - \alpha^\epsilon(\cdot,t))/h^3$ in $L^p(\Omega_d)$. In the sequel, we will always assume that C is a generic constant which may not be the same in the different expressions and M_1, C_0, C_1, C_2 are some special constants.

Lemma 3.1. *If $p > 3$, and if there is a constant $C_1 > 0$, such that $\|u^\epsilon\|_{2,p,\Omega} \le C_1$, then there exists a constant $C_0 > 0$, such that*

$$|\xi^\epsilon(t;\eta_1,l) - \xi^\epsilon(t;\eta_2,l)| \le C_0|\eta_1-\eta_2|,\ \forall\eta_1,\eta_2\in\Omega_d\ \ t,l\in[0,T]. \tag{29}$$

Proof. Since Ω is convex, by (22), we can get $|\eta_1^{(i)} - \eta_2^{(i)}| \le (i+2)|\eta_1-\eta_2|$. By the assumption of the lemma $\|u^\epsilon\|_{2,p,\Omega} \le C_1$. Using the embedding theorem, we have $\|u^\epsilon\|_{1,\infty,\Omega} \le C$. By (23)-(24) (20), we have

$$\begin{aligned} |\tfrac{d}{dt}(\xi^\epsilon(t;\eta_1,l) - \xi^\epsilon(t;\eta_2,l))| &= |g^\epsilon(\xi^\epsilon(t;\eta_1,l),t) - g^\epsilon(\xi^\epsilon(t;\eta_2,l),t)| \\ &\le C|\xi^\epsilon(t;\eta_1,l) - \xi^\epsilon(t;\eta_2,l)|, \\ \xi^\epsilon(l;\eta_1,l) - \xi^\epsilon(l;\eta_2,l) &= \eta_1-\eta_2. \end{aligned}$$

Using the Gronwall inequality, we obtain (29).

Lemma 3.2. *Under the assumption of Lemma 3.1, we have*

$$\|\overline{\omega}(t)\|_{0,p,\Omega_d} \le C\int_0^t (\|e(t)\|_{0,p,\Omega_d} + |u-u^\epsilon|_{1,p,\Omega} + \|\overline{\omega}(t)\|_{0,p,\Omega_d} + d^{M-1})dt, \tag{30}$$

$$|\overline{\omega}(t)|_{1,p,\Omega_d} \le C \int_0^t (\|e(t)\|_{1,p,\Omega_d} + \|u - u^\epsilon\|_{2,p,\Omega} + \|\overline{\omega}(t)\|_{1,p,\Omega_d} + d^{M-2})dt, \tag{31}$$

where the constant C is independent of d.

Proof. The equations (25) and (27) give

$$\begin{aligned}\frac{d(\alpha - \alpha^\epsilon)}{dt} &= (\alpha \cdot \nabla)(u(x(t),t) - u(x^\epsilon(t),t)) \\ &+(\alpha \cdot \nabla)(u(x^\epsilon(t),t) - g^\epsilon(x^\epsilon(t),t)) \\ &+((\alpha - \alpha^\epsilon)\cdot \nabla) g^\epsilon(x^\epsilon(t),t) \\ &+h^3\tilde{F}(x(t),t) - h^3 F(x^\epsilon(t),t),\end{aligned} \tag{32}$$

where for simplicity we omit the independent variable η. Obviously $|\alpha/h^3| \le C$. By assumption and the embedding theorem, $|\nabla u^\epsilon|_{0,\infty,\Omega} \le C$. By (20)-(21), we have

$$\begin{aligned}&|\nabla \cdot (u(x^\epsilon(t),t) - g^\epsilon(x^\epsilon(t),t))| = \left|\sum_{i=1}^M a_i \nabla \cdot (u(x^\epsilon(t),t) - u^\epsilon(x^\epsilon(t)^{(i)},t))\right| \\ &\le \left|\sum_{i=1}^M a_i \nabla \cdot (u(x^\epsilon(t),t) - u(x^\epsilon(t)^{(i)},t))\right| + \left|\sum_{i=1}^M a_i \nabla \cdot (u(x^\epsilon(t)^{(i)},t) - u^\epsilon(x^\epsilon(t)^{(i)},t))\right| \\ &\le Cd^{M-1} + C\sum_{i=1}^M |D(u-u^\epsilon)(x^\epsilon(t)^{(i)},t)|,\end{aligned}$$

where we have used Taylor expansion and (21) to get the last inequality. We define a mapping $\Phi^{(i)} : x \longmapsto x^{(i)}$, then the second term of equation (32) can be estimated as the following:

$$|(\alpha\cdot\nabla)(u(x^\epsilon(t),t) - g^\epsilon(x^\epsilon(t),t)| \le Ch^3 \left(d^{M-1} + \sum_{i=1}^M |D(u-u^\epsilon)\circ \Phi^{(i)}(x^\epsilon(t))|\right).$$

The forth term also can be estimated as follows.

$$\begin{aligned}&|h^3\tilde{F}(x(t),t) - h^3F(x^\epsilon(t),t)| \\ &\le |h^3\tilde{F}(x(t),t) - h^3F(x(t),t)| + |h^3F(x(t),t) - h^3F(x^\epsilon(t),t)| \\ &\le Ch^3d^M + Ch^3|(x-x^\epsilon)(t)|,\end{aligned}$$

where we also have used Taylor expansion and $F \equiv \tilde{F}$ in $\overline{\Omega} \times [0,T]$. Thus, by (32), we obtain

$$\begin{aligned}\|\overline{\omega}(t)\|_{0,p,\Omega_d} \le C\int_0^t (&\|e(t)\|_{0,p,\Omega_d} + \sum_{i=1}^M \|D(u-u^\epsilon)\circ\Phi^{(i)}(x^\epsilon(t))\|_{0,p,\Omega_d} \\ &+\|\overline{\omega}(t)\|_{0,p,\Omega_d} + \|e(t)\|_{0,p,\Omega_d} + d^{M-1})dt.\end{aligned} \tag{33}$$

The equation (17) implies $\nabla \cdot u^\epsilon = 0$, hence the mapping $\eta \longmapsto x^\epsilon(\eta, t)$ in Ω is measure preserving and by Lemma 3.1 the Jacobi matrix of $\eta \longmapsto x^\epsilon(\eta, t)$ in $\Omega_d \backslash \Omega$ is bounded by C; consequently;

$$\begin{aligned} &\|D(u - u^\epsilon) \circ \Phi^{(i)}(x^\epsilon(t))\|_{0,p,\Omega_d} \\ &\leq C\|D(u - u^\epsilon) \circ \Phi^{(i)}(x^\epsilon(t))\|_{0,p,\Omega} + C\|D(u - u^\epsilon) \circ \Phi^{(i)}(x^\epsilon(t))\|_{0,p,\Omega_d\backslash\Omega} \\ &\leq C\left(|u - u^\epsilon|_{1,p,\Omega} + |D(u - u^\epsilon)|_{0,p,\Omega}\right) = C|u - u^\epsilon|_{1,p,\Omega}. \end{aligned} \tag{34}$$

By substituting (34) into (33), we get (30).

Next, we apply the operator D_η to the equation (32). For notational convenience, we consider one component of α, u, x, and α^ϵ, u^ϵ, x^ϵ and denote by D the derivative with respect to one special variable.

$$\begin{gathered} \frac{d}{dt}D(\alpha - \alpha^\epsilon) = D\alpha D(u(x(t), t) - u(x^\epsilon(t), t)) \\ +\alpha(D^2u(x(t), t)Dx(t) - D^2u(x^\epsilon(t), t)Dx^\epsilon(t)) \\ +D\alpha D(u(x^\epsilon(t), t) - g^\epsilon(x^\epsilon(t), t)) \\ +\alpha(D^2u(x^\epsilon(t), t)Dx^\epsilon(t) - D^2g^\epsilon(x^\epsilon(t), t)Dx^\epsilon(t)) \\ +D(\alpha - \alpha^\epsilon)Dg^\epsilon(x^\epsilon(t), t) \\ +(\alpha - \alpha^\epsilon)D^2g^\epsilon(x^\epsilon(t), t)Dx^\epsilon(t)) \\ +h^3D\tilde{F}(x(t), t)Dx(t) - h^3DF(x^\epsilon(t), t)Dx^\epsilon(t). \end{gathered} \tag{35}$$

To estimate the L^p- norm of the right-hand side, we need to prove $|Dx^\epsilon(t)| \leq C$. Indeed $Dx^\epsilon(t)$ satisfies

$$\frac{d}{dt}Dx^\epsilon(t) = Dg^\epsilon(x^\epsilon(t), t)Dx^\epsilon(t).$$

Using the Gronwall inequality, we can estimate $Dx^\epsilon(t)$. To estimate the sixth term of (35) we notice that

$$\begin{aligned} &\|(\alpha - \alpha^\epsilon)D^2g^\epsilon(x^\epsilon(t), t)Dx^\epsilon(t)\|_{0,p,\Omega_d} \\ &\leq C\max\left(|\alpha - \alpha^\epsilon|\right)|g^\epsilon|_{2,p,\Omega_d} \leq C\|\alpha - \alpha^\epsilon\|_{1,p,\Omega_d}|u^\epsilon|_{2,p,\Omega}, \end{aligned} \tag{36}$$

where we have used the embedding theorem and (20). By the assumption of the lemma, the right-hand side of (36) is bounded by $C\|\alpha - \alpha^\epsilon\|_{1,p,\Omega_d}$. The other terms of (35) are estimated in a straightforward way, then we obtain (31).

Lemma 3.3. *Under the assumption of Lemma* 3.1, *we have*

$$\|e(t)\|_{1,p,\Omega_d} \leq C\int_0^t (\|u - u^\epsilon\|_{1,p,\Omega} + \|e(t)\|_{1,p,\Omega_d} + d^{M-1})dt, \tag{37}$$

$$|e(t)|_{2,p,\Omega_d} \leq C\int_0^t (\|u - u^\epsilon\|_{2,p,\Omega} + \|e(t)\|_{2,p,\Omega_d} + d^{M-2})dt. \tag{38}$$

Proof. From equation (23), we obtain

$$\frac{d}{dt}(x(t)-x^\epsilon(t)) = u(x(t),t) - g^\epsilon(x^\epsilon(t),t)$$

$$= u(x(t),t) - u(x^\epsilon(t),t) + \sum_{i=1}^{M} a_i(u(x^\epsilon(t),t) - u(x^\epsilon(t)^{(i)},t))$$

$$+\sum_{i=1}^{M} a_i(u(x^\epsilon(t)^{(i)},t) - u^\epsilon(x^\epsilon(t)^{(i)},t)),$$

thus

$$\|e(t)\|_{0,p,\Omega_d} \le C\int_0^t (\|e(t)\|_{0,p,\Omega_d} + d^M + \|u-u^\epsilon\|_{0,p,\Omega})dt, \tag{39}$$

where we have also used the abbreviations in Lemma 3.2. By (23) we obtain

$$\frac{d}{dt}D(x(t)-x^\epsilon(t)) = D(u-g^\epsilon)(x^\epsilon(t),t)Dx^\epsilon(t)$$

$$+Du(x(t),t)Dx(t) - Du(x^\epsilon(t),t)Dx^\epsilon(t),$$

$$\frac{d}{dt}D^2(x(t)-x^\epsilon(t)) = D^2u(x(t),t)((Dx(t))^2 - (Dx^\epsilon(t))^2)$$

$$+(D^2u(x(t),t) - D^2g^\epsilon(x^\epsilon(t),t))(Dx^\epsilon(t))^2 + Dg^\epsilon(x^\epsilon(t),t)(D^2x(t) - D^2x^\epsilon(t))$$

$$+(D(u(x(t),t) - Dg^\epsilon(x^\epsilon(t),t)D^2x(t).$$

Using the same deduction as (39), we obtain (37) and (38).

Set $C_2 > 0$, such that $u \equiv 0$ as $|x| \ge C_2$. We consider the following set: $J_1 = \{j; |jh| \le C_2\}$, then J_1 is a finite set. We introduce an operator G as the following: For given $\omega(x)$, there is a unique $u(x)$ satisfies (7)-(9), then we write $u = G\omega$, then

$$\|G\omega\|_{m+1,p,\Omega} \le C\|\omega\|_{m,p,\Omega} \quad m \ge 0. \text{ (see[7], §4 Lemma 4.1)} \tag{40}$$

Now we start to estimate $u - u^\epsilon$. As in [7], we make the following decomposition:

$$\begin{aligned}
&u - u^\epsilon = v_1 + v_2 + v_3,\\
&v_1 = u - G\left(\omega(\cdot,t) * \zeta_\epsilon\right),\\
&v_2 = G\left(\omega(\cdot,t) * \zeta_\epsilon - \sum_{j\in J_1}\alpha_j(t)\zeta_\epsilon(\cdot - X_j(t))\right),\\
&v_3 = G\left(\sum_{j\in J_1}\alpha_j(t)\zeta_\epsilon(\cdot - X_j(t)) - \sum_{j\in J}\alpha_j^\epsilon(t)\zeta_\epsilon(\cdot - X_j^\epsilon(t))\right),
\end{aligned}$$

where $\alpha_j(t) = \alpha(\eta, X_j, 0)$, $X_j(t) = x(X_j, t)$.

Lemma 3.4. *If there is an integer $k \ge 1$ such that*

$$\int_{\mathbf{R}^3} x^r\zeta(x)dx = 0, 1 \le |r| \le k-1, \tag{41}$$

then

$$|v_1(\cdot,t)|_{l,p,\Omega} \le C\epsilon^k, 1 \le p \le +\infty,$$

for any integer $l \ge 0$. *(see*[7], §4 *Lemma* 4.2)

Lemma 3.5. *If* $\zeta \in W^{m+l-1,\infty}(\mathbf{R}^3)$, $m \ge 1$, $l \ge 1$, *then for any* $r \in [1, 3/2]$

$$\|v_2(\cdot,t)\|_{l,p,\Omega} \le C\left(\left(1+\frac{h}{\epsilon}\right)^{3/r}\frac{h^m}{\epsilon^{m+l-1}}\right), \quad 1 \le p < +\infty, \tag{42}$$

as $t \in [0,T]$. *(see* [7], §4 *Lemma* 4.3)

We are now in a position to estimate v_3. Set $J_2 = \{j; X_j \in \Omega_{C_0\epsilon} \bigcap \Omega_d\}$, where C_0 is the constant in Lemma 3.1.

Lemma 3.6. *Under the assumption of Lemma* 3.1, *if* $p > 3$, $l \ge 1$, $\zeta \in W^{l+2,\infty}(\mathbf{R}^3)$, $\|e(t)\|_{0,\infty,\Omega_{C_0\epsilon}} \le M_1\epsilon$, $h \le C_2\epsilon$, *then*

$$\|v_3\|_{l,p,\Omega} \le \frac{C}{\epsilon^{l-1}}\left\{\left(1+\frac{h+\|e(t)\|_{0,\infty,\Omega_{C_0\epsilon}}}{\epsilon^2}\right)\|e(t)\|_{0,p,\Omega_{C_0\epsilon}} + \frac{h}{\epsilon}\|e(t)\|_{2,p,\Omega_{C_0\epsilon}}\right.$$
$$\left. +\|\overline{\omega}(t)\|_{0,p,\Omega_{C_0\epsilon}} + h|\overline{\omega}(t)|_{1,p,\Omega_{C_0\epsilon}} + \int_0^t |u-u^\epsilon|_{1,p,\Omega}dt + \epsilon^{M-1}\right\}. \tag{43}$$

Proof. First, we prove, if $C_0\epsilon < d$, then

$$v_3 = G\left(\sum_{j\in J_2}\alpha_j(t)\zeta_\epsilon(\cdot - X_j(t)) - \sum_{j\in J_2}\alpha_j^\epsilon(t)\zeta_\epsilon(\cdot - X_j^\epsilon(t))\right). \tag{44}$$

Indeed, if $\text{supp}\zeta_\epsilon(\cdot - X_j^\epsilon(t))\bigcap\Omega \ne \emptyset$, then there exists a point $x \in \overline{\Omega}$, such that $|x - X_j^\epsilon(t)| < \epsilon$. By Lemma 3.1 $|\xi^\epsilon(0;x,t) - X_j| < C_0\epsilon$ and $\xi^\epsilon(0;x,t) \in \overline{\Omega}$, thus $X_j \in \Omega_{C_0\epsilon}$. The same reason can be stated for $\zeta_\epsilon(\cdot - X_j(t))$. So we known if $X_j \notin \Omega_{C_0\epsilon}$, then $\zeta_\epsilon(x - X_j(t)) = 0$, $\zeta_\epsilon(x - X_j^\epsilon(t)) = 0, \forall x \in \Omega$, which proves (44).

Let $\omega_3 = \sum_{j\in J_2}(\alpha_j(t)\zeta_\epsilon(x - X_j(t)) - \alpha_j^\epsilon(t)\zeta_\epsilon(x - X_j^\epsilon(t)))$, then $v_3 = G\omega_3$. Let us estimate $\|\partial^\gamma\omega_3\|_{0,p,\Omega}$, where $|\gamma| = l-1$. The function ω_3 can be further decomposed into

$$\omega^{(1)} = \sum_{j\in J_2}(\zeta_\epsilon(x - X_j(t)) - \zeta_\epsilon(x - X_j^\epsilon(t)))\alpha_j(t),$$
$$\omega^{(2)} = \sum_{j\in J_2}\zeta_\epsilon(x - X_j(t))(\alpha_j(t) - \alpha_j^\epsilon(t)),$$
$$\omega^{(3)} = \sum_{j\in J_2}(\zeta_\epsilon(x - X_j(t)) - \zeta_\epsilon(x - X_j^\epsilon(t)))(\alpha_j^\epsilon(t) - \alpha_j(t)).$$

Let us first estimate $\omega^{(1)}$. $\omega^{(1)}$ also can be written as

$$\omega^{(1)} = \int_{\Omega_{C_0\epsilon}}(\zeta_\epsilon(x - \xi(t;X_j,0)) - \zeta_\epsilon(x - \xi^\epsilon(t;X_j,0)))\omega(\xi(t;X_j,0),t)d\eta = I_1 + I_2 + I_3,$$
$$I_1 = \int_{\Omega_{C_0\epsilon}}(\zeta_\epsilon(x - \xi(t;X_j,0)) - \zeta_\epsilon(x - \xi^\epsilon(t;X_j,0)))\omega(\xi(t;X_j,0),t)$$

$$-(\zeta_\epsilon(x-\xi(t;\eta,0))-\zeta_\epsilon(x-\xi^\epsilon(t;\eta,0)))\omega(\xi(t;X_j,0),t)d\eta,$$

$$I_2=\int_{\Omega_{C_0\epsilon}}(\zeta_\epsilon(x-\xi(t;\eta,0))-\zeta_\epsilon(x-\xi^\epsilon(t;\eta,0)))(\omega(\xi(t;X_j,0),t)-\omega(\xi(t;\eta,0),t))d\eta,$$

$$I_3=\int_{\Omega_{C_0\epsilon}}(\zeta_\epsilon(x-\xi(t;\eta,0))-\zeta_\epsilon(x-\xi^\epsilon(t;\eta,0)))\omega(\xi(t;\eta,0),t)d\eta.$$

The following is the same as the estimate of $\omega^{(1)}$ in Lemma 3.6 in [7],

$$\|\partial^\gamma I_1\|_{0,p,\Omega}\le\frac{C}{\epsilon^{l-1}}\left\{\left(\frac{h+\|e(t)\|_{0,\infty,\Omega_{C_0\epsilon}}}{\epsilon^2}\right)\|e(t)\|_{0,p,\Omega_{C_0\epsilon}}+\frac{h}{\epsilon}\|e(t)\|_{2,p,\Omega_{C_0\epsilon}}\right\},\tag{45}$$

$$|\partial^\gamma I_2\|_{0,p,\Omega}\le\frac{C}{\epsilon^{l-1}}\|e(t)\|_{0,p,\Omega_{C_0\epsilon}}.\tag{46}$$

As for the third term, since the mapping $\eta\longmapsto\xi(t;\eta,0)$ is measure preserving, we have

$$\begin{aligned}I_3&=\int_{\Omega_{C_0\epsilon}}\zeta_\epsilon(x-\xi(t;\eta,0))\omega(\xi(t;\eta,0),t)d\eta-\int_{\Omega_{C_0\epsilon}}\zeta_\epsilon(x-\xi^\epsilon(t;\eta,0)))\omega(\xi(t;\eta,0),t)d\eta\\&=\int_{\Omega_{C_0\epsilon}}\zeta_\epsilon(x-\xi)\omega(\xi,t)d\xi-\int_{\Omega_{C_0\epsilon}}\zeta_\epsilon(x-\xi)\omega(\xi(t;\xi^\epsilon(0;\xi,t),0),t)|J|d\xi\\&=A+B,\end{aligned}$$

where

$$A=\int_{\Omega_{C_0\epsilon}}\zeta_\epsilon(x-\xi)(\omega(\xi,t)-\omega(\xi(t;\xi^\epsilon(0;\xi,t),0),t))d\xi,$$

$$B=\int_{\Omega_{C_0\epsilon}}\zeta_\epsilon(x-\xi)\omega(\xi(t;\xi^\epsilon(0;\xi,t),0),t)(|J|-1)d\xi.$$

J is the Jacobi matrix of $\eta\longmapsto\xi^\epsilon(t;\eta,0)$. Analogous to the estimate of $\omega^{(1)}$ in Lemma 3.6 in [7], we have

$$\|\partial^\gamma A\|_{0,p,\Omega}\le\frac{C}{\epsilon^{l-1}}\|e(t)\|_{0,p,\Omega_{C_0\epsilon}}.\tag{47}$$

Applying Lemma 5.1 in [6], Chap2, §5, we obtain

$$|J|=\exp\left(\int_0^t\nabla\cdot g^\epsilon(x^\epsilon(\eta,\tau),\tau)d\tau\right),$$

and

$$\|\nabla g^\epsilon\|_{0,\infty,\Omega_{C_0\epsilon}}\le C\|Du^\epsilon\|_{0,\infty,\Omega}\le C\|u^\epsilon\|_{2,p,\Omega}\le C,$$

where we have used (20), embedding theorem and the assumption of the lemma. Thus

$$\begin{aligned}||J|-1|&=\left|\exp(\int_0^t\nabla\cdot g^\epsilon(x^\epsilon(\eta,\tau),\tau)d\tau)-1\right|\\&\le C\left|\int_0^t\nabla\cdot g^\epsilon(x^\epsilon(\eta,\tau),\tau)d\tau\right|\\&\le C\int_0^t|\nabla\cdot g^\epsilon(x^\epsilon(\eta,\tau),\tau)-\nabla\cdot g(x^\epsilon(\eta,\tau),\tau)|d\tau+C\int_0^t|\nabla\cdot g(x^\epsilon(\eta,\tau),\tau)|d\tau.\end{aligned}\tag{48}$$

where $g(x,t) = \sum_{i=1}^{M} a_i u(x^{(i)},t)$, as $x \in \Omega_{C_0\epsilon}$, so $g(x,t) \equiv u(x,t)$ as $x \in \overline{\Omega}$. Noting that $\nabla \cdot u = 0$, $(x,t) \in \mathbf{R}^3 \times [0,T]$ and applying Taylor expansion, we obtain

$$|\nabla \cdot g(x,t)| = |\nabla \cdot g(x,t) - \nabla \cdot u(x,t)| = \left|\nabla \cdot \sum_{i=1}^{M} a_i(u(x^{(i)},t) - u(x,t))\right| \le C\epsilon^{M-1}, \quad (49)$$

$$\forall x \in \Omega_{C_0\epsilon}.$$

Noting that $|J| = 1$ as $\eta \in \Omega$, by (48)-(49), we have

$$\begin{aligned}
|\partial^\gamma B| &= \left|\int_{\Omega_{C_0\epsilon}} \partial^\gamma \zeta_\epsilon(x-\xi)\omega(\xi(t;\xi^\epsilon(0;\xi,t),0),t)(|J|-1)d\xi\right| \\
&= \left|\int_{\Omega_{C_0\epsilon}\setminus\Omega} \partial^\gamma \zeta_\epsilon(x-\xi)\omega(\xi(t;\xi^\epsilon(0;\xi,t),0),t)\,(|J|-1)\,d\xi\right| \\
&\le C\int_{\Omega_{C_0\epsilon}\setminus\Omega} \left|\partial^\gamma \zeta_\epsilon(x-\xi)\int_0^t (\nabla \cdot g^\epsilon(x^\epsilon(\eta,\tau),\tau) - \nabla \cdot g(x^\epsilon(\eta,\tau),\tau))d\tau\right| d\xi \\
&\quad + C\int_{\Omega_{C_0\epsilon}} |\partial^\gamma \zeta_\epsilon(x-\xi)|\epsilon^{M-1}d\xi.
\end{aligned}$$

Applying Lemma 3.5 in [7], we obtain

$$\|\partial^\gamma B\|_{0,p,\Omega} \le \frac{C}{\epsilon^{l-1}}\left(\int_0^t |u-u^\epsilon|_{1,p,\Omega}dt + \epsilon^{M-1}\right). \quad (50)$$

Combining (45)-(47) and (50), we get

$$\|\partial^\gamma \omega^{(1)}\|_{0,p,\Omega} \le \frac{C}{\epsilon^{l-1}}\left\{\left(1 + \frac{h + \|e(t)\|_{0,\infty,\Omega_{C_0\epsilon}}}{\epsilon^2}\right)\|e(t)\|_{0,p,\Omega_{C_0\epsilon}} + \frac{h}{\epsilon}\|e(t)\|_{2,p,\Omega_{C_0\epsilon}}\right.$$

$$\left. + \int_0^t |u-u^\epsilon|_{1,p,\Omega}dt + \epsilon^{M-1}\right\}.$$

Analogous to the estimate of $\omega^{(2)}$, $\omega^{(3)}$ in Lemma 3.6 in [7], we have

$$\|\partial^\gamma \omega^{(2)}\|_{0,p,\Omega} \le \frac{C}{\epsilon^{l-1}}\left(\|\overline{\omega}(t)\|_{0,p,\Omega_{C_0\epsilon}} + h|\overline{\omega}(t)|_{1,p,\Omega_{C_0\epsilon}}\right),$$

$$\|\partial^\gamma \omega^{(3)}\|_{0,p,\Omega} \le \frac{C}{\epsilon^{l-1}}\left(\|\overline{\omega}(t)\|_{0,p,\Omega_{C_0\epsilon}} + h|\overline{\omega}(t)|_{1,p,\Omega_{C_0\epsilon}}\right).$$

Thus we obtain the estimate for ω_3

$$\|\partial^\gamma \omega_3\|_{0,p,\Omega} \le \frac{C}{\epsilon^{l-1}}\left\{\left(1 + \frac{h + \|e(t)\|_{0,\infty,\Omega_{C_0\epsilon}}}{\epsilon^2}\right)\|e(t)\|_{0,p,\Omega_{C_0\epsilon}} + \frac{h}{\epsilon}\|e(t)\|_{2,p,\Omega_{C_0\epsilon}}\right.$$

$$\left. + \|\overline{\omega}(t)\|_{0,p,\Omega_{C_0\epsilon}} + h|\overline{\omega}(t)|_{1,p,\Omega_{C_0\epsilon}} + \int_0^t |u-u^\epsilon|_{1,p,\Omega}dt + \epsilon^{M-1}\right\}.$$

By (40), we get

$$\|v_3\|_{l,p,\Omega} = \|G\omega_3\|_{l,p,\Omega} \leq C\|\omega_3\|_{l-1,p,\Omega},$$

which proves the inequality (43).

Finally, let us prove the convergence theorem.

Theorem 3.1. *If $p > 3$, (41) holds for $k = M - 1 \geq 3$, $\zeta \in W^{m+1,\infty}(\mathbf{R}^3)$, $m \geq 3$ and $h \leq C_2\epsilon^2$, then there is a constant $\epsilon_0 > 0$, such that if $\epsilon \leq \epsilon_0$, then*

$$\|u - u^\epsilon\|_{1,p,\Omega} + \|e(t)\|_{1,p,\Omega_{C_0\epsilon}} + \|\overline{\omega}(t)\|_{0,p,\Omega_{C_0\epsilon}} \leq C\left(\epsilon^k + \frac{h^m}{\epsilon^m}\right), \tag{51}$$

$$\|u - u^\epsilon\|_{2,p,\Omega} + \|e(t)\|_{2,p,\Omega_{C_0\epsilon}} + \|\overline{\omega}(t)\|_{1,p,\Omega_{C_0\epsilon}} \leq C\left(\epsilon^{k-1} + \frac{h^m}{\epsilon^{m+1}}\right). \tag{52}$$

Proof. Let $t = 0$, then $X_j(0) = X_j^\epsilon(0) = X_j$, $\alpha_j(0) = \alpha_j^\epsilon(0) = \alpha_j$, so $v_3(x,0) = 0$, $u(x,0) - u^\epsilon(x,0) = v_1(x,0) + v_2(x,0)$. By Lemma 3.4 and 3.5, $\|u^\epsilon(\cdot,0)\|_{2,p,\Omega} \leq \|u(\cdot,0)\|_{2,p,\Omega}+C$. Set a constant $C_3>\|u(\cdot,0)\|_{2,p,\Omega}+C$, then, by continuity, $\|u^\epsilon(\cdot,t)\|_{2,p,\Omega} < C_3$ in a neighbourhood of $t = 0$. We again fix a constant $M_2 > 0$. Since $\|e(0)\|_{0,\infty,\Omega_{C_0\epsilon}} = 0$, $\|e(t)\|_{0,\infty,\Omega_{C_0\epsilon}} < M_2\epsilon^2$ in a neighbourhood, too. Let the intersection of these two intervals be $[0, T_*]$. We notice that T_* depends on h and ϵ.

We define the following norms:

$$\begin{aligned}
\|u - u^\epsilon\|_{2,p,\Omega}^* &= \|u - u^\epsilon\|_{1,p,\Omega} + \epsilon|u - u^\epsilon|_{2,p,\Omega}, \\
\|e(t)\|_{2,p,\Omega_{C_0\epsilon}}^* &= \|e(t)\|_{1,p,\Omega_{C_0\epsilon}} + \epsilon|e(t\|_{2,p,\Omega_{C_0\epsilon}}, \\
\|\overline{\omega}(t)\|_{1,p,\Omega_{C_0\epsilon}}^* &= \|\overline{\omega}(t)\|_{0,p,\Omega_{C_0\epsilon}} + \epsilon|\overline{\omega}(t)|_{1,p,\Omega_{C_0\epsilon}}.
\end{aligned}$$

We may assume $\epsilon \leq 1$, then Lemma 3.2 gives

$$\|\overline{\omega}(t)\|_{1,p,\Omega_{C_0\epsilon}}^* \leq C\left(\int_0^t \|e(t)\|_{1,p,\Omega_{C_0\epsilon}} + \|u - u^\epsilon\|_{2,p,\Omega}^* + \|\overline{\omega}(t)\|_{1,p,\Omega_{C_0\epsilon}}^* dt + \epsilon^{M-1}\right). \tag{53}$$

Lemma 3.3 gives

$$\|e(t)\|_{2,p,\Omega_{C_0\epsilon}}^* \leq C\left(\int_0^t \|e(t)\|_{2,p,\Omega_{C_0\epsilon}}^* + \|u - u^\epsilon\|_{2,p,\Omega}^* dt + \epsilon^{M-1}\right). \tag{54}$$

And Lemma 3.4 ,3.5 and 3.6 give

$$\|u - u^\epsilon\|_{2,p,\Omega}^* \leq C\left\{\epsilon^k + \frac{h^m}{\epsilon^m} + \|e(t)\|_{2,p,\Omega_{C_0\epsilon}}^* + \|\overline{\omega}(t)\|_{1,p,\Omega_{C_0\epsilon}}^* + \int_0^t \|u - u^\epsilon\|_{2,p,\Omega}^* dt\right\}, \tag{55}$$

for $t \in [0, T_*]$. Applying Gronwall inequality, we obtain

$$\|u - u^\epsilon\|_{2,p,\Omega}^* \leq C\left\{\epsilon^k + \frac{h^m}{\epsilon^m} + \|e(t)\|_{2,p,\Omega_{C_0\epsilon}}^* + \|\overline{\omega}(t)\|_{1,p,\Omega_{C_0\epsilon}}^*\right\}. \tag{56}$$

By substituting (56) into (53) and (54), we get

$$\|\overline{\omega}(t)\|_{1,p,\Omega_{C_0\epsilon}}^* \leq C\int_0^t (\epsilon^k + \frac{h^m}{\epsilon^m} + \|e(t)\|_{2,p,\Omega_{C_0\epsilon}}^* + \|\overline{\omega}(t)\|_{1,p,\Omega_{C_0\epsilon}}^*)dt,$$

$$\|e(t)\|^*_{2,p,\Omega_{C_0\epsilon}} \le C\int_0^t (\epsilon^k + \frac{h^m}{\epsilon^m} + \|e(t)\|^*_{2,p,\Omega_{C_0\epsilon}} + \|\overline{\omega}(t)\|^*_{1,p,\Omega_{C_0\epsilon}})dt.$$

Applying Gronwall inequality again, we obtain

$$\|\overline{\omega}(t)\|^*_{1,p,\Omega_{C_0\epsilon}} + \|e(t)\|^*_{2,p,\Omega_{C_0\epsilon}} \le C\left(\epsilon^k + \frac{h^m}{\epsilon^m}\right).$$

Equation (56) gives

$$\|u - u^\epsilon\|^*_{2,p,\Omega} \le C\left(\epsilon^k + \frac{h^m}{\epsilon^m}\right),$$

thus (51) and (52) hold for $t \in [0, T_*]$. We notice that the constant C is independent of h and ϵ.

By virtue of the embedding theorem, we have

$$\|e(t)\|_{0,\infty,\Omega_{C_0\epsilon}} \le C\left(\epsilon^k + \frac{h^m}{\epsilon^m}\right) = C\left(\epsilon^{k-2} + \frac{h^m}{\epsilon^{m+2}}\right)\epsilon^2.$$

We take $\epsilon_0 \le 1$ small enough such that $C\left(\epsilon^{k-2} + \frac{h^m}{\epsilon^{m+2}}\right) < M_2$ and $\|u\|_{2,p,\Omega} + C\left(\epsilon^{k-1} + \frac{h^m}{\epsilon^{m+1}}\right) < C_3$ if $\epsilon \le \epsilon_0$, then it is easy to show that T_* is indeed independent of ϵ and h and equal to T. Now the proof is complete.

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