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THE BLOSSOM APPROACH TO THE DIMENSION OF THE BIVARIATE SPLINE SPACE*[1)]

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Abstract

The dimension of the bivariate spline space $S_n^r(\Delta)$ may depend on geometric properties of triangulation Δ, in particular if n is not much bigger than r. In the paper, the blossom approach to the dimension count is outlined. It leads to the symbolic algorithm that gives the answer if a triangulation is singular or not. The approach is demonstrated on the case of Morgan-Scott partition and twice differentiable splines.

Key words: Bivariate spline space, Blossom, Dimension.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a closed simply connected polygonal region, and

$$\Delta := \{\Omega_i\}_{i=1}^t, \ \Omega = \bigcup_{i=1}^{t} \Omega_i$$

its regular triangulation, i.e. the triangles

$$\Omega_i, \Omega_j, \ i \neq j,$$

can have in common only a vertex or a whole edge. Let V denote the set of inner vertices, E the set of inner edges, and $\overline{E}$ the set of all edges of Δ. Put

$$m_V := |V|, \quad m_E := |E|.$$

The planar graph $G := (V, \overline{E})$ clearly describes Δ. However, it's sometimes useful to consider also the dual planar graph $\mathcal{G} := (\mathcal{V},\mathcal{E})$, where vertices $i \in \mathcal{V}$ correspond to triangles Ω_i, and $e = (i,j) \in \mathcal{E}$ iff Ω_i,Ω_j share a common edge. Note $|\mathcal{V}| = t$, $|\mathcal{E}| = m_E$, and there is one-to-one correspondance between E and $\mathcal{E}$. So we shall not make any difference between $e = (i,j) \in \mathcal{E}$, and the common edge of Ω_i, Ω_j if not neccessary. In particular, $||e||$ will denote the length of the common edge of the corresponding

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triangles, direction of e will be the direction of this common edge etc. There is another simple relation between G and $\mathcal{G}$, a one to one correspondence between the vertices $v \in V$, and elementary cycles in $\mathcal{G}$,

$$Y_v = ((i_1, i_2), (i_2, i_3), \ldots, (i_d, i_{d+1})),\ i_{d+1} = i_1,\ (i_j, i_{j+1}) \in \mathcal{E},$$

the boundaries of facets. Here, d denotes the degree of v. The cycle Y_v describes the connection between the triangles that meet at inner vertex v. $\mathcal{G}$ is a planar cubic graph with m_V elementary cycles. By Euler's equation,

$$m_E - m_V = t - 1. \tag{1.1}$$

Let $\pi_n(\mathbb{R}^2)$ denote the space of polynomial functions of total degree $\leq n$, and let

$$S_n^r(\Delta) := \{f | f|_{\Omega_i} \in \pi_n(\mathbb{R}^2)\} \cap C^r(\Omega)$$

denote the spline space over a regular triangulation Δ. Quite clearly

$$\dim \pi_n(\mathbb{R}^2) = \binom{n+2}{2}, \tag{1.2}$$

but the dimension of $S_n^r(\Delta)$ may be hard to determine since it might depend on the geometric properties of the triangulation. One can find a lower bound ([9], [10]) as

$$\begin{aligned} \dim S_n^r(\Delta) \quad \geq \quad \Phi_n^r(\Delta) &:= \binom{n+2}{2} + \binom{n-r+1}{2} m_E \\ &- \left(\binom{n+2}{2} - \binom{r+2}{2}\right) m_V + \sum_{i=1}^{m_V} \sigma_i, \\ \sigma_i &:= \sum_{j=1}^{n-r} (r + j + 1 - jn_i)_+,\ i = 1, 2, \ldots, m_V. \end{aligned} \tag{1.3}$$

Here n_i denotes the number of edges with different slopes at inner vertex $v_i \in V$. A similar expression for the upper bound can be established also. Particular partitions show that the lower bound is often very close to actual dimension of the spline space. As an example, in [3] they can differ only by 1. Also, if n is large enough, i.e. $n \geq 3r+2$, and (1.3) actually gives the required dimension ([4]).

In this paper we will tackle the spline space dimension problem by relations, derived from the blossoming formulation of the continuity conditions. In order to proceed let us recall the multiindex notation. Let $\mathbb{Z}_+$ denote the set of nonnegative integers, and let small Greek letters denote the multiindex vectors i.e. vectors with nonnegative integer components. For any multiindices

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{Z}_+^m, \quad \beta = (\beta_1, \beta_2, \cdots, \beta_m) \in \mathbb{Z}_+^m,$$

and a vector $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$, let

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_m, \quad \alpha! := \alpha_1! \alpha_2! \cdots \alpha_m!\ , \quad x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m},$$

$$\binom{n}{\alpha} := \begin{cases} \frac{n!}{\alpha!(n-|\alpha|)!}, & 0 \leq \alpha, |\alpha| \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\alpha \leq \beta$ denotes the relation $\leq$ componentwise i.e. $\alpha_i \leq \beta_i$, all i, and further let $\alpha < \beta$ be $\alpha \leq \beta$ with at least one $\alpha_i < \beta_i$. The generalised binomial coefficient is given by

$$\binom{\alpha}{\beta} := \begin{cases} \prod_{j=1}^{m} \binom{\alpha_j}{\beta_j}, & 0 \leq \beta \leq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

2. Smoothness Conditions in Blossoming Form

Based on the blossoming formulation of the continuity conditions [5] the following conclusion was derived in [3]. Let $L : s \to L(s) := sv + u$ be a straight line that divides $\mathbb{R}^2$ into two halfplanes, the supports of polynomials $p, q \in \pi_n(\mathbb{R}^2)$, and

$$p(x) = \sum_{|\alpha|\le n} a_\alpha \binom{n}{\alpha} x^\alpha, \qquad q(x) = \sum_{|\alpha|\le n} b_\alpha \binom{n}{\alpha} x^\alpha. \tag{2.1}$$

Here, $u = (u_1, u_2)$ denotes a point on L, and $v = (v_1, v_2)$ denotes its unit length direction.

Theorem 2.1. *Let f be a piecewise polynomial function composed of p and q. Then $f \in C^r(\mathbb{R}^2)$ if and only if*

$$\sum_{|\beta|\le n-r} (a_{\alpha+\beta} - b_{\alpha+\beta}) \binom{n-r}{\beta} (sv+u)^\beta = 0 \tag{2.2}$$

for any $s \in \mathbb{R}$, $|\alpha| \le r$.

The relation (2.2) was studied further in [3] for a particular $r = 1$, and dependent relations removed. Here we solve the general case. There can be only

$$(n-r)(r+1) + \binom{r+2}{2}$$

independent conditions. If one expands the lefthand side of (2.2) as a polynomial of s, the relations (2.2) are equivalent to the fact that all the coefficients of

$$s^\ell, \ \ell \ge 0, \ |\alpha| \le r$$

vanish. This gives the total count of conditions in (2.2) as

$$\binom{r+2}{2}(n-r+1) = (n-r)(r+1) + \binom{r+2}{2} + (n-r)\binom{r+1}{2},$$

so the set of the conditions (2.2) must be linearly dependent. We proceed to show that the conditions for $s^\ell, \ell \ge 1$, $|\alpha| < r$ can be expressed as a linear combination of the remaining conditions. We have the following theorem.

Theorem 2.2. *The conditions (2.2) for s^ℓ, $\ell \ge 1$, $|\alpha| < r$ can be expressed by the conditions for s^ℓ, $\ell = 0$, $|\alpha| \le r$, and s^ℓ, $\ell > 0$, $|\alpha| = r$.*

Proof. The multinomial expansion (with $x_\alpha := a_\alpha - b_\alpha$) yields

$$\begin{aligned}
&\sum_{|\beta|\le n-r} x_{\alpha+\beta} \binom{n-r}{\beta} (sv+u)^\beta = \\
&= \sum_{|\beta|\le n-r} \binom{n-r}{\beta} x_{\alpha+\beta} \sum_{0\le\gamma\le\beta} \binom{\beta}{\gamma} (sv)^\gamma u^{\beta-\gamma} \\
&= \sum_{|\gamma|\le n-r} s^{|\gamma|} v^\gamma \sum_{\beta\ge\gamma, |\beta|\le n-r} \binom{n-r}{\beta}\binom{\beta}{\gamma} u^{\beta-\gamma}\, x_{\alpha+\beta} \\
&= \sum_{\ell=0}^{n-r} s^\ell \sum_{|\gamma|=\ell} v^\gamma \sum_{\beta\ge\gamma, |\beta|\le n-r} \binom{n-r}{\beta}\binom{\beta}{\gamma} u^{\beta-\gamma}\, x_{\alpha+\beta}.
\end{aligned} \tag{2.3}$$

Since, for fixed $|\gamma| = \ell$,

$$\binom{n-r}{\beta}\binom{\beta}{\gamma} = \binom{n-r}{\ell}\binom{\ell}{\gamma}\binom{n-r-\ell}{\beta-\gamma}$$

the coefficient by the power s^ℓ, divided by $\binom{n-r}{\ell}$, is given as

$$\sum_{|\gamma|=\ell}\binom{\ell}{\gamma}v^\gamma\sum_{\beta\geq\gamma,|\beta|\leq n-r}\binom{n-r-\ell}{\beta-\gamma}u^{\beta-\gamma}\;x_{\alpha+\beta}=$$

$$=\sum_{|\gamma|=\ell}\binom{\ell}{\gamma}v^\gamma\sum_{\beta\geq 0,|\beta|\leq n-r-\ell}\binom{n-r-\ell}{\beta}u^{\beta}\;x_{\alpha+\beta+\gamma}. \tag{2.4}$$

Let us denote $\epsilon_i := (\delta_{i,j})$. Then it is straightforward to see

$$\binom{|\gamma|}{\gamma}=\sum_i\binom{|\gamma|-1}{\gamma-\epsilon_i} \tag{2.5}$$

for $|\gamma|>0$. Thus for any w_ψ one concludes

$$\sum_{|\gamma|=\ell}\binom{\ell}{\gamma}v^\gamma w_{\gamma+\psi}=$$

$$=\sum_i\sum_{|\gamma|=\ell}\binom{\ell-1}{\gamma-\epsilon_i}v^\gamma w_{\gamma+\psi}=$$

$$=\sum_i\sum_{|\gamma|=\ell-1}\binom{\ell-1}{\gamma}v^{\gamma+\epsilon_i}w_{\gamma+\epsilon_i+\psi}=$$

$$=\sum_i v_i\sum_{|\gamma|=\ell-1}\binom{\ell-1}{\gamma}v^\gamma w_{\gamma+\epsilon_i+\psi}. \tag{2.6}$$

Here we have assumed $\ell>0$. Let us apply (2.6) to (2.4). Then

$$\sum_{|\gamma|=\ell}\binom{\ell}{\gamma}v^\gamma\sum_{\beta\geq 0,|\beta|\leq n-r-\ell}\binom{n-r-\ell}{\beta}u^{\beta}\;x_{\alpha+\beta+\gamma}=$$

$$=\sum_i v_i\sum_{|\gamma|=\ell-1}\binom{\ell-1}{\gamma}v^\gamma\sum_{\beta\geq 0,|\beta|\leq n-r-\ell}\binom{n-r-\ell}{\beta}u^{\beta}\;x_{\alpha+\beta+\gamma+\epsilon_i}=$$

$$=\sum_i v_i\sum_{|\gamma|=\ell-1}\binom{\ell-1}{\gamma}v^\gamma\sum_{\beta\geq 0,|\beta|\leq n-r-(\ell-1)}\binom{n-r-(\ell-1)}{\beta}u^{\beta}\;x_{\alpha+\beta+\gamma+\epsilon_i}+$$

$$+\sum_i v_i\sum_{|\gamma|=\ell-1}\binom{\ell-1}{\gamma}v^\gamma(\sum_{\beta\geq 0,|\beta|\leq n-r-\ell}\binom{n-r-\ell}{\beta}u^{\beta}\;x_{\alpha+\beta+\gamma+\epsilon_i}-$$

$$-\sum_{\beta\geq 0,|\beta|\leq n-r-(\ell-1)}\binom{n-r-(\ell-1)}{\beta}u^{\beta}\;x_{\alpha+\beta+\gamma+\epsilon_i}).$$

The first sum is obviously a linear combination of the continuity conditions for $\ell\;\to\;\ell-1$, $\alpha\;\to\;\alpha+\epsilon_i$. The second, again by (2.6), reduces to

$$\sum_{|\gamma|=\ell}\binom{\ell}{\gamma}v^\gamma(\sum_{\beta\geq 0,|\beta|\leq n-r-\ell}\binom{n-r-\ell}{\beta}u^{\beta}\;x_{\alpha+\beta+\gamma}-$$

$$-\sum_{\beta\geq 0,|\beta|\leq n-r-(\ell-1)}\binom{n-r-(\ell-1)}{\beta}u^{\beta}\;x_{\alpha+\beta+\gamma})=$$

$$= \sum_{|\gamma|=\ell} \binom{\ell}{\gamma} v^\gamma (\sum_{\beta\geq 0, |\beta|\leq n-r-(\ell-1)} (\binom{n-r-\ell}{\beta} - \binom{n-r-(\ell-1)}{\beta}) u^\beta \, x_{\alpha+\beta+\gamma}. \quad (2.7)$$

Here we have taken into account

$$\binom{n-r-\ell}{\beta} = 0, \; |\beta| = n-r-(\ell-1).$$

Further,

$$\binom{n-r-\ell}{\beta} - \binom{n-r-(\ell-1)}{\beta} =$$

$$= \binom{n-r-\ell}{\beta} (1 - \frac{n-r-\ell+1}{n-r-\ell+1-|\beta|}) =$$

$$= -\sum_i \binom{n-r-\ell}{\beta-\epsilon_i}.$$

Thus (2.7) can be written as

$$-\sum_i \sum_{|\gamma|=\ell} \binom{\ell}{\gamma} v^\gamma (\sum_{\beta\geq 0, |\beta|\leq n-r-(\ell-1)} \binom{n-r-\ell}{\beta-\epsilon_i} u^\beta \, x_{\alpha+\beta+\gamma} =$$

$$-\sum_i u_i \sum_{|\gamma|=\ell} \binom{\ell}{\gamma} v^\gamma \sum_{\beta\geq 0, |\beta|\leq n-r-\ell} \binom{n-r-\ell}{\beta} u^\beta \, x_{\alpha+\beta+\gamma+\epsilon_i}.$$

clearly as a linear combination of the continuity conditions for given ℓ, $\alpha \to \alpha+\epsilon_i$. Thus the continuity conditions for given ℓ, α can be expressed as linear combinations of the conditions for $\ell \to \ell, \ell-1$, $\alpha \to \alpha+\epsilon_i$.

Note that now there are only

$$\binom{r+2}{2} + (n-r)(r+1)$$

conditions left, so they must be linearly independent.

3. The Dimension of $S_n^r(\Delta)$

Let Δ be a regular triangulation, and G, $\mathcal{G}$ as defined in the introduction. The continuity conditions given in the previous section have to be written for each

$$e_\ell = (i,j) \in \mathcal{E}, \; \ell = 1, 2, \dots, m_E.$$

Thus for given $e_\ell = (i,j) \in \mathcal{E}$ let u_ℓ denote a point on the edge $\Omega_i \cap \Omega_j$, and v_ℓ the unit length direction of this edge. Further, let c_β^i, $|\beta| \leq n$, denote the coefficients that correspond to the triangle Ω_i, $a_\beta^i := c_\beta^i - c_\beta^1$, and

$$\mathcal{C}_\ell : f \mapsto \mathcal{C}_\ell f := \frac{1}{\ell!} f^{(\ell)}(0).$$

Without loosing generality we can assume that Ω_1 is a boundary triangle. The continuity conditions now read

$$\sum_{|\beta|\leq n-r} (a_{\alpha+\beta}^i - a_{\alpha+\beta}^j) \binom{n-r}{\beta} u_\ell^\beta = 0, \quad (3.1)$$

$$e_\ell = (i,j) \in \mathcal{E}, \; \ell = 1, 2, \dots, m_E, \; |\alpha| \leq r.$$

and

$$\mathcal{C}_k \sum_{|\beta|\le n-r} (a^i_{\alpha+\beta} - a^j_{\alpha+\beta}) \binom{n-r}{\beta} (\bullet\ v_\ell + u_\ell)^\beta = 0, \tag{3.2}$$

$$e_\ell = (i,j) \in \mathcal{E},\ \ell = 1,2,\ldots,m_E,\ |\alpha| = r,$$
$$k = 1,2,\ldots,n-r.$$

This gives the dimension of the spline space $S^r_n(\Delta)$ as the difference between

$$t\ \dim \pi_n(\mathbb{R}^2) \quad = \quad t\ \binom{n+2}{2},$$

and the number of independent conditions, imposed by the relations (3.1) and (3.2). For a particular choice of u_ℓ these conditions can be elaborated further, i.e., the part of conditions that doesn't depend on the geometry will be removed. Let u_ℓ be the inner vertices. We shall show that the conditions (3.1), restricted to the edges of a spanning tree $\mathcal{T}_\mathcal{G}$ of the dual graph $\mathcal{G}$ are linearly independent, and imply the other part of (3.1).

Theorem 3.1. *Let u_ℓ be chosen as the inner vertices of the triangulation Δ, i.e. $u_\ell \in V$. Then*

$$\dim S^r_n(\Delta) = t \binom{n+2}{2} - \binom{r+2}{2} (m_E - m_V) - N,$$

where N denotes the number of independent conditions imposed by

$$\mathcal{C}_k \sum_{0<|\beta|\le n-r} (a^i_{\alpha+\beta} - a^j_{\alpha+\beta}) \binom{n-r}{\beta} (\bullet\ v_\ell + u_\ell)^\beta = 0, \tag{3.3}$$

$$e_\ell = (i,j) \in \mathcal{E}, \ell = 1,2,\ldots,m_E,\ |\alpha| = r,$$
$$k = 1,2,\ldots,n-r.$$

Proof. Consider a given $e_\ell = (i,j)$. If the edge $\Omega_i \cap \Omega_j$, has only one inner vertex u_ℓ, (3.1) is written for that vertex. If both endpoints belong to V, u_ℓ is one of them. But conditions for the other, $\tilde{u}_\ell = \tilde{s}\ v_\ell + u_\ell \in V$,

$$\sum_{|\beta|\le n-r} (a^i_{\alpha+\beta} - a^j_{\alpha+\beta}) \binom{n-r}{\beta} \tilde{u}^\beta_\ell = 0,\ |\alpha| \le r, \tag{3.4}$$

are by theorem 2.2 linearly dependent, and hold automatically if (3.1), (3.2) are satisfied. This implies that it makes no difference which particular interior endpoint is inserted in (3.1). Note also that

$$\sum_{Y_v} \sum_{|\beta|\le n-r} (a^{i_j}_{\alpha+\beta} - a^{i_{j+1}}_{\alpha+\beta}) \binom{n-r}{\beta} v^\beta = \tag{3.5}$$

$$\sum_{|\beta|\le n-r} \binom{n-r}{\beta} v^\beta \sum_{Y_v} (a^{i_j}_{\alpha+\beta} - a^{i_{j+1}}_{\alpha+\beta}) = 0,$$

$$v \in V,\ Y_v = ((i_j, i_{j+1}))$$

since

$$\sum_{Y_v} (a^{i_j}_{\alpha+\beta} - a^{i_{j+1}}_{\alpha+\beta}) = 0.$$

Let now $v \in V$ be any interior point. Let us sum the relations in (3.1) along the cycle Y_v. By (3.4) one can assume that all the edges along the cycle Y_v use the same constant

$u_\ell = v$. But (3.5) shows then that $\binom{r+2}{2}$ equations are annulated. Since there are m_V interior knots, one is finally left with

$$\binom{r+2}{2}(m_E - m_V) = \binom{r+2}{2}(t-1) \tag{3.6}$$

equations corresponding to the edges of a spanning tree $\mathcal{T}_{\mathcal{G}}$. In order to finish the proof we have to show that (3.1) uniquely determines

$$a_\alpha^i, i = 2, 3, \ldots, t-1, \ |\alpha| \leq r,$$

since $a_\alpha^1 = 0$. The equations can be looked for each particular α separately, and it's enough to look at the $(t-1) \times t$ matrix, given rowwise as

$$\delta_{\ell,i} - \delta_{\ell,j}, \ e_\ell = (i,j) \in \mathcal{T}_{\mathcal{G}}$$

with first column omitted. Since $\mathcal{T}_{\mathcal{G}}$ is a tree, is easy to see that the relevant determinant is $\pm$ 1. This completes the proof of the theorem.

Let us write (3.3) in a matrix form. Put

$$a_i^j := (a_{(i,0)}^j, a_{(i-1,1)}^j, \cdots, a_{(0,i)}^j)^T \ \in \mathbb{R}^{i+1}, \tag{3.7}$$

and order the coefficients a_i^j as

$$a := (a_{r+1}^2, a_{r+1}^3, \cdots, a_{r+1}^t, a_{r+2}^2, a_{r+2}^3, \cdots, a_{r+2}^t, \cdots\cdots, a_n^2, a_n^3, \cdots, a_n^t)^T. \tag{3.8}$$

The continuity conditions (3.3) now read

$$M_n a = 0$$

where

$$M_n := (M_{km})_{k=1;m=1}^{n-r;n-r} := (M_{r,km})_{k=1;m=1}^{n-r;n-r} \tag{3.9}$$

is upper triangular block matrix, and the blocks M_{km} of order $(r+1)m_E \times (m+r+1)(t-1)$ correspond to the part of conditions for $\mathcal{C}_k$ with $|\alpha| = r$, $|\beta| = m$, and consequently to the coefficients a_{r+m}^j, $j = 2, 3, \cdots, t$. Note $M_{km} = O$, $k > m$. M_{km} itself is also a block matrix with m_E block rows, and $t-1$ block columns. ℓ-th block row corresponds to conditions for $e_\ell = (i,j) \in \mathcal{E}$, and has at most two nonzero blocks $Q_{\ell i} := Q_{km,\ell i}$, $Q_{\ell j} := Q_{km,\ell j}$, $Q_{km,\ell i} + Q_{km,\ell j} = 0$. Blocks $Q_{km,\ell z}$ are circular matrices of order $(r+1) \times (m+r+1)$. The definition (3.7) implies that that the columns of the matrix $Q_{km,\ell z}$ should be ordered by decreasing first coordinate of β. From

$$\mathcal{C}_k(\bullet\ v + u)^\beta = \mathcal{C}_k \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} (\bullet\ v)^\gamma u^{\beta-\gamma} = \sum_{|\gamma|=k} \binom{\beta}{\gamma} v^\gamma u^{\beta-\gamma},$$

and (3.3) it is straightforward to compute the elements of the matrices. The identity

$$\binom{n-r}{\beta}\binom{\beta}{\gamma} = \binom{n-r}{k}\binom{k}{\gamma}\binom{n-r-k}{\beta-\gamma}$$

gives the first row of a nonzero block $Q_{km,\ell i}$ as

$$\binom{n-r}{k}\sum_{|\gamma|=k}\binom{k}{\gamma}\binom{n-r-k}{\beta-\gamma} v_\ell^\gamma u_\ell^{\beta-\gamma} = f \sum_{|\gamma|=k}\binom{k}{\gamma}\binom{m-k}{\beta-\gamma} v_\ell^\gamma u_\ell^{\beta-\gamma},$$

$$\beta := (\beta_1, m-\beta_1), \ \beta_1 = m, m-1, \cdots, 0, \tag{3.10}$$

with

$$f := f_{n,r,k,m} := \binom{n-r}{k}\binom{n-r-k}{m-k}. \tag{3.11}$$

The rest of the row, i.e. r elements are zero. Note that the factors $\binom{n-r}{k}$ can be removed by a diagonal transformations on M_n. Since this does not affect the rank of M_n we shall assume that M_n is given by (3.10), with $f := f/\binom{n-r}{k}$. This justifies the notation $M_{r,km}$ in (3.9), as well as shows that M_n is obtained from M_{n-1} by introducing additional row and column, but leaving the rest of the matrix unchanged.

As an example to (3.10), take $n = 5, r = 2, k = 1, m = 2, v_\ell = (v_{1,\ell}, v_{2,\ell}), u_\ell = (u_{1,\ell}, u_{2,\ell})$. Then the block $Q_{12,\ell i}$ implied by (3.10) is equal to

$$\pm 2 \begin{pmatrix} v_{1,\ell}u_{1,\ell} & v_{1,\ell}u_{2,\ell} + v_{2,\ell}u_{1,\ell} & v_{2,\ell}u_{2,\ell} & 0 & 0 \\ 0 & v_{1,\ell}u_{1,\ell} & v_{1,\ell}u_{2,\ell} + v_{2,\ell}u_{1,\ell} & v_{2,\ell}u_{2,\ell} & 0 \\ 0 & 0 & v_{1,\ell}u_{1,\ell} & v_{1,\ell}u_{2,\ell} + v_{2,\ell}u_{1,\ell} & v_{2,\ell}u_{2,\ell} \end{pmatrix}.$$

We have finally established the working form of the theorem 3.1

$$\dim S_n^r(\Delta) = t\binom{n+2}{2} - \binom{r+2}{2}(m_E - m_V) - \operatorname{rank} M_n. \tag{3.12}$$

It seems difficult to determine the exact rank of M_n, but some partial answers can be obtained in general. Some simplification of the offdiagonal part of the matrix is provided by the following observation.

Lemma 3.1. *Choose fixed $i \in \mathcal{V}$, $z := (z_1, z_2)$ and $k < m \leq n - r$. If all nonzero blocks $Q_{km,\ell i}$ of M_{km} in block column i are changed to $\tilde{Q}_{km,\ell i}$, with the generating row (3.10) replaced by*

$$f \sum_{|\gamma|=k} \binom{k}{\gamma}\binom{m-k}{\delta-\gamma} v_\ell^\gamma (u_\ell - z)^{\delta-\gamma},$$
$$\delta := (\delta_1, m - \delta_1), \ \ \delta_1 = m, m-1, \cdots, 0, \tag{3.13}$$

the rank of M_n does not change.

Proof. It is enough to proove that $\frac{1}{f}(Q_{km,\ell i} - \tilde{Q}_{km,\ell i})$ can be expressed as linear combination of blocks $Q_{kj,\ell i}$, $j = k, k+1, \cdots, m-1$ with the coefficients depending only on z. To observe that recall (3.10). The s-th row of $\frac{1}{f}\tilde{Q}_{km,\ell i}$, $s = 1, 2, \cdots, r+1$, reads

$$\Big(\underbrace{0, 0, \cdots, 0}_{s-1}, (\sum_{|\gamma|=k} \binom{k}{\gamma}\binom{m-k}{\delta-\gamma} v_\ell^\gamma (u_\ell - z)^{(m-j,j)-\gamma})_{j=0}^m, \underbrace{0, 0, \cdots, 0}_{r+1-s}\Big),$$

and for the nonzero part one has

$$\left(\sum_{|\gamma|=k} \binom{k}{\gamma}\binom{m-k}{(m-j,j)-\gamma} v_\ell^\gamma (u_\ell - z)^{(m-j,j)-\gamma}\right)_{j=0}^m = \tag{3.14}$$

$$(u_\ell - z)^{\epsilon_1}\left((\sum_{|\gamma|=k} \binom{k}{\gamma}\binom{m-1-k}{(m-1-j,j)-\gamma} v_\ell^\gamma (u_\ell - z)^{(m-1-j,j)-\gamma})_{j=0}^{m-1}, 0\right) +$$

$$(u_\ell - z)^{\epsilon_2}\left(0, (\sum_{|\gamma|=k} \binom{k}{\gamma}\binom{m-1-k}{(m-1-j,j)-\gamma} v_\ell^\gamma (u_\ell - z)^{(m-1-j,j)-\gamma})_{j=0}^{m-1}\right).$$

For $m = k+1$ (3.14) gives

$$\left(\sum_{|\gamma|=k} \binom{k}{\gamma}\binom{1}{(k+1-j,j)-\gamma} v_\ell^\gamma (u_\ell - z)^{(k+1-j,j)-\gamma}\right)_{j=0}^{k+1} =$$

$$(u_\ell - z)^{\epsilon_1}\left(\left(\binom{k}{(k-j,j)} v_\ell^{(k-j,j)}\right)_{j=0}^{k}, 0\right) +$$

$$+(u_\ell - z)^{\epsilon_2}\left(0, \left(\binom{k}{(k-j,j)} v_\ell^{(k-j,j)}\right)_{j=0}^{k}\right) =$$

$$\left(\sum_{|\gamma|=k}\binom{k}{\gamma}\binom{1}{(k+1-j,j)-\gamma} v_\ell^{\gamma} u_\ell^{(k+1-j,j)-\gamma}\right)_{j=0}^{k+1} -$$

$$-z_1\left(\left(\binom{k}{(k-j,j)} v_\ell^{(k-j,j)}\right)_{j=0}^{k}, 0\right) -$$

$$-z_2\left(0, \left(\binom{k}{(k-j,j)} v_\ell^{(k-j,j)}\right)_{j=0}^{k}\right),$$

i.e. all the rows of block $m = k+1$ of $\tilde{Q}_{k\ k+1,\ell i}$ can be written as the same linear combination of rows in blocks $Q_{kk,\ell i}$, $Q_{k\ k+1,\ell i}$ what confirms the lemma for this case. The general case follows from (3.14) by induction.

Clearly the choise of $z = u_\ell$ annulls the corresponding offdiagonal blocks. In order to proceed we will make some assumptions. First of all, we shall restrict ourselves to the general case triangulation: no edges in E that share a common vertex have the same direction. Also, since in the most useful cases r is small compared to n, and in some cases $n < 2r$ admits no real splines, we will restrict our discussion to the case $n \geq 2r$.

Theorem 3.2. *Let Δ be a general regular triangulation, and $n \geq n_0 \geq 2r$. Then*

$$\operatorname{rank} M_n \geq \operatorname{rank} M_{n_0} + (r+1)m_E(n-n_0). \tag{3.15}$$

Proof. Consider M_n,

$$M_n = \begin{pmatrix} M_{n-1} & X \\ 0 & M_{n-k\ n-k} \end{pmatrix}.$$

Then $\operatorname{rank} M_n \geq \operatorname{rank} M_{n-1} + \operatorname{rank} M_{n-k\ n-k}$. So, one has to prove

$$\operatorname{rank} M_{n-k\ n-k} = (r+1)m_E,$$

i.e. the rank of $M_{n-k\ n-k}$ is equal to the number of rows. The number of columns in $M_{n-k\ n-k}$ is

$$(n+1)(t-1) \geq 2(r+1)(t-1) \geq \frac{4}{3}(r+1)m_E > (r+1)m_E$$

since $3t \geq 2m_E + 3$, and we only have to establish that rows are linearly independent. Assume they are not. Then there exists a vector $x \neq 0, x \in \mathbb{R}^{(r+1)m_E}$ such that $x^T M_{n-k\ n-k} = 0$. Since the rows corresponding to the boundary triangles with only one interior edge are obviously independent, the corresponding components of x should be zero, and we may assume that all the boundary triangles have only one exterior edge. Let Ω_i be any such boundary triangle of Δ. Then $e_{\ell_1} = (j_1, i)$, $e_{\ell_2} = (i, j_2) \in \mathcal{E}$ for some j_1, j_2, and the block matrices $Q_{\ell_1 i}, Q_{\ell_2 i}$ are the only nonzero blocks in the block column i. Let $x_{|_{\ell_j}}$ denote ℓ_j-th block row of x. Then

$$(x_{|_{\ell_1}}, x_{|_{\ell_2}})^T \begin{pmatrix} Q_{\ell_1 i} \\ Q_{\ell_2 i} \end{pmatrix} = 0.$$

It is easy to see that the matrix

$$\begin{pmatrix} Q_{\ell_1 i} \\ Q_{\ell_2 i} \end{pmatrix} \tag{3.16}$$

of order

$$2(r+1) \times (n+1),\ \ n+1 \geq 2r+1+1 = 2(r+1)$$

is of full rank. Indeed, the rank of (3.16) is not changed by adding $n-2r-1$ zero columns. Further, by adding $n-2r-1$ rows by cyclically continuing $Q_{\ell_1 i}$, and $n-2r-1$ rows by cyclically continuing $Q_{\ell_2 i}$ the rank of (3.16) is increased by at most $2(n-2r-1)$. The obtained matrix is a $2(n-r) \times 2(n-r)$ matrix. Its determinant is equal to the resultant of polynomials

$$p_{\ell_1}(x) := (v_{1,\ell_1}x + v_{2,\ell_1})^{n-r},\ p_{\ell_2}(x) := (v_{1,\ell_2}x + v_{2,\ell_2})^{n-r},$$

with $v_{\ell_i} := (v_{1,\ell_i}, v_{2,\ell_i})$ being the direction that corresponds to e_{ℓ_i}. Since the directions v_{ℓ_1}, v_{ℓ_2} are different, p_{ℓ_1}, p_{ℓ_2} can't have a common zero. This implies the resultant to be nonzero (in fact is equal to $(v_{1,\ell_1}v_{2,\ell_2} - v_{2,\ell_1}v_{1,\ell_2})^{(n-r)^2}$), and consequently (3.16) to be of full rank. Thus $x_{|_{\ell_1}} = 0,\ x_{|_{\ell_2}} = 0$. But this implies that one can consider only $\Delta \setminus \{\Omega_i\}$, and apply the previous argument to the reduced triangulation until only one triangle is left. Thus the theorem is confirmed.

The following theorem reveals the way how to tackle particular triangulations.

Theorem 3.3. *Let Δ be a general regular triangulation, and $n \geq n_0 \geq 2r$. The function*

$$\Xi^r_{n,n_0}(\Delta) := \dim S^r_n(\Delta) - t\left(\binom{n+2}{2} - \binom{n_0+2}{2}\right) + (r+1)m_E(n-n_0)$$

is nonincreasing function of n, and

$$\Phi^r_{n_0}(\Delta) \leq \Xi^r_{n,n_0}(\Delta) \leq \dim S^r_{n_0}(\Delta). \tag{3.17}$$

Proof. It is enough to consider $n > n_0$. The first claim is implied by (3.12), and the theorem 3.2. Recall (1.3). Since $n > n_0 \geq 2r$, and the degree of an inner vertex is at least 3, one has for $j \geq r$

$$(r+j+1-jn_i) \leq (r+j+1-3j) \leq (r+1-2r) = 1-r \leq 0,$$

and σ_i stay unchanged for any $n > n_0$. It is now straightforward to apply (3.12) and (3.15) to obtain (3.17).

Note that for some $n \geq 3r+2$ the function $\Xi^r_{n,n_0}(\Delta)$ always reaches its lower bound. If the right inequality reduces to equality, then so does the left. Thus, for a particular triangulation, one has to determine $\operatorname{rank} M_n,\ n = 2r, 2r+1, .., $ as along as for some $n = n_0$ equality in (3.17) holds, i.e. triangulation becomes nonsingular.

4. An $S^2_n(\Delta)$ Example: Morgan–Scott Partition

Consider the triangulation given on Fig. 1. It was introduced in [6] in order to emphasize the fact that the geometry of triangulation might influence $\dim S^r_n(\Delta)$.

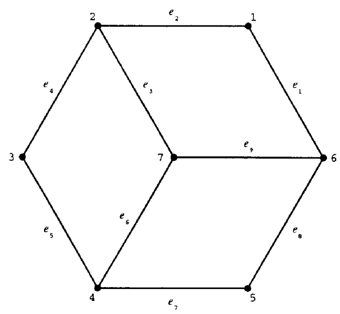

Fig.1. Morgan–Scott triangulation

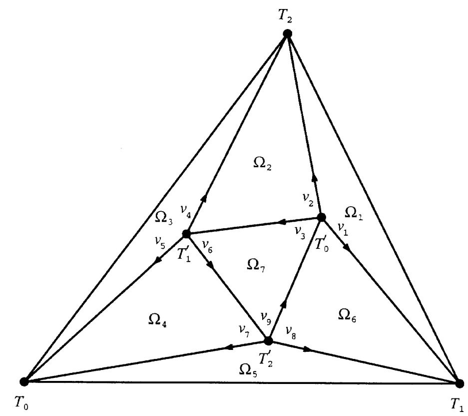

Fig.2. The dual graph for Morgan–Scott triangulation

The case $\dim S_n^1(\Delta)$ was studied further in [11], [3]. Let us demonstrate how to compute the dimension of $S_n^2(\Delta)$. In the view of previous discussion and Fig.2 one has

$$t = 7,\ m_E = 9,\ m_V = 3,\ n = 4, 5, \cdots .$$

Perhaps the simpliest way to look at M_n in this case is to omitt the linearly dependent columns that correspond to $7 \in \mathcal{V}$. We shall choose the value $n_0 = 2r = 4$ as n_0 referenced in the theorems 3.2, 3.3. Thus one has to look at M_4, M_5,

$$M_4 = \begin{pmatrix} M_{11} & M_{12} \\ & M_{22} \end{pmatrix}, \quad M_5 = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ & M_{22} & M_{23} \\ & & M_{33} \end{pmatrix}$$

that will be studied at the same time. The order of M_4 is

$$2(r+1)m_E \times ((r+2)(t-1) + (r+3)(t-1)) = 54 \times 54,$$

and of M_5 similary 81×90. The structure of nonzero blocks in $M_{kk},\ k = 1, 2, 3$ is given as

$$M_{kk} = \begin{pmatrix} Q_{kk,11} & & & & & Q_{kk,16} \\ Q_{kk,21} & Q_{kk,22} & & & & \\ & Q_{kk,32} & & & & \\ & Q_{kk,42} & Q_{kk,43} & & & \\ & & Q_{kk,53} & Q_{kk,54} & & \\ & & & Q_{kk,64} & & \\ & & & Q_{kk,74} & Q_{kk,75} & \\ & & & & Q_{kk,85} & Q_{kk,86} \\ & & & & & Q_{kk,96} \end{pmatrix}$$

with blocks being the matrices of order $3 \times (3+k)$. Further, let us choose the origin as $T_0' = (0,0)$, the points u_ℓ as

$$u_1 = u_2 = u_3 = u_9 = T_0',\ u_4 = u_5 = u_6 = T_1',\ u_7 = u_8 = T_2',$$

and vectors $z_i,\ i \in \mathcal{V}$ that simplify the offdiagonal blocks for both $m = 2, 3$ as

$$z_1 = z_2 = z_6 = u_1,\ z_3 = z_4 = u_4,\ z_5 = u_7.$$

Then by the lemma 3.1 the structure of the nonzero offdiagonal blocks $M_{km}, k < m$, is as follows

$$M_{km} = \begin{pmatrix} & & & \\ & \tilde{Q}_{km,42} & & \\ & & & \\ & & \tilde{Q}_{km,74} & \\ & & & \tilde{Q}_{km,86} \end{pmatrix},$$

and the vectors inserted in (3.13) are

$$\begin{array}{lcccc} \tilde{Q}_{km,42}: & u_4 - z_2 & = & T_1' - T_0' & = & \|e_3\|\ v_3, \\ \tilde{Q}_{km,74}: & u_7 - z_4 & = & T_2' - T_1' & = & \|e_6\|\ v_6, \\ \tilde{Q}_{km,86}: & u_8 - z_6 & = & T_2' - T_0' & = & -\|e_9\|\ v_9. \end{array} \tag{4.1}$$

M_5 is now determined by directions v_ℓ, and lengths of the sides of the inner triangle only. Any rotation of the coordinate system does not change the dimension of the spline space. So we can assume

$$v_3 = (1, 0). \tag{4.2}$$

The rows of M_{33} are by the argument of theorem 3.2 linearly independent. A difference between the number of columns in M_{k3}, and the number of rows in M_{33} is $36 - 27 = 9$. However, the assumption (4.2) simplifies the discussion, and only one out of 9 has to be considered. Take the column 65 of M_5. For $k = 1, 2$ one only has to compute the 5-th columns of blocks $\tilde{Q}_{13,42}, \tilde{Q}_{23,42}$,

$$\tilde{Q}_{13,42}: \quad \|e_3\|^2 \begin{pmatrix} 0 \\ v_4^{\epsilon_2} v_3^{2\epsilon_2} \\ v_4^{\epsilon_1} v_3^{2\epsilon_2} + 2 v_4^{\epsilon_2} v_3^{\epsilon_1+\epsilon_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{Q}_{23,42}: \quad \|e_3\| \begin{pmatrix} 0 \\ v_4^{2\epsilon_2} v_3^{\epsilon_2} \\ v_4^{2\epsilon_2} v_3^{\epsilon_1} + 2 v_4^{\epsilon_1+\epsilon_2} v_3^{\epsilon_2} \end{pmatrix} = \|e_3\| v_4^{2\epsilon_2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

since $v_3^{\epsilon_i} = \delta_{i,1}$. Again by the argument used in the proof of theorem 3.2

$$\operatorname{rank} \begin{pmatrix} Q_{33,\ell i} \\ Q_{33,\ell+1\ i} \end{pmatrix} = 6,\ (\ell, i) = (1,1), (4,3), (7,5). \tag{4.3}$$

But the 5-th column of $Q_{33,32}$ is $(0,0,0)$. This shows that the rows of M_{33} are linearly independent without the column 65 too. So

$$\operatorname{rank} M_5 \geq 27 + \operatorname{rank} M_5',$$

with

$$M_5' = \begin{pmatrix} M_4 & z \end{pmatrix}_{54\times 55}, \; z := (\delta_{i,39})_{i=1}^{54}. \tag{4.4}$$

The constant in the last column was omitted since

$$v_4^{\epsilon_2} = \epsilon_1 \times v_4 = v_3 \times v_4 \neq 0.$$

It is clear what kind of symbolic transformations should be applied to M_5': they should preserve $\det M_4$. If any row or column (of M_4) that has only one nonzero element, say $w_{ij} \neq 0$, the row i and column j can be ommitted, both rank counts increased by 1, and w_{ij} kept as a factor to the final value of $\det M_4$. First of all, note that

$$\det \begin{pmatrix} Q_{11,11} & & Q_{11,16} \\ Q_{11,21} & Q_{11,22} & \\ & Q_{11,32} & \\ & & Q_{11,96} \end{pmatrix} = d,$$

$$d := (v_1 \times v_2)(v_1 \times v_3)(v_1 \times v_9)(v_2 \times v_3)(v_2 \times v_9)(v_3 \times v_9) \neq 0,$$

so the rows $\{1, \cdots, 9, 25, \cdots, 27\}$, and columns $\{1, \cdots, 8, 21, \cdots, 24\}$ may be omitted, with $-d$ kept as a factor for $\det M_4$. The rank count is increased by 12. It would be nice if one could proceed simply by applying some package for symbolic computation but the computations seem to complex. However, a particular structure of M_4, and $Q_{km,\ell i}$ suggest a straightforward sequence of eliminations that can be worked out by hand. A Mathematica program that follows this procedure was written only to doublecheck all the steps. As an example, consider the last block column, i.e. columns $\{38, \cdots, 42\}$. There are 4 nonzero blocks, $\tilde{Q}_{12,86}, \tilde{Q}_{22,16}, Q_{22,86}, Q_{22,96}$, and they all have the same symbolic structure

$$\begin{pmatrix} \times & \times & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times \end{pmatrix}.$$

Nonzero elements are simbolically denoted by $\times$. So one can eliminate all nonzero elements in column 38 by the pivot row 40, then all nonzero elements in column 39 by the pivot row 41, and all nonzero elements in column 40 by the pivot row 42. This reduces the blocks $\tilde{Q}_{12,86}, \tilde{Q}_{22,16}, Q_{22,86}$ to the form

$$\begin{pmatrix} 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{pmatrix}.$$

Thus one can omitt the columns 38,39,40, and rows 40,41,42, increase rank count by 3, but keep the determinant of pivot 3×3 block, i.e. $(v_9^{2\epsilon_1})^3$ as a factor for $\det M_4$. Also, common factors in the columns that are left, are cancled out (and kept) in order to keep complexity as low as possible. These steps can be followed block column by block column from right to left. For $m = 2$ there will be two columns left for each block column, and for $m = 1$ only one. Since $3 * 1 + 6 * 2 = 15$, one is left with the 15×16

matrix symbolicly denoted as

$$\begin{pmatrix}
\times & \times & 0 & 0 & 0 & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\times & \times & 0 & 0 & 0 & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\times & \times & 0 & 0 & 0 & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & 0 & 0 & \times & \times & 0\\
0 & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & 0 & 0 & \times & \times & 0\\
0 & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & 0 & 0 & \times & \times & 0\\
0 & 0 & 0 & \times & \times & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & 0\\
0 & 0 & 0 & \times & \times & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & 0\\
0 & 0 & 0 & \times & \times & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & 0\\
0 & 0 & 0 & 0 & 0 & \times & \times & \times & \times & \times & \times & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \times & \times & \times & \times & \times & \times & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \times & \times & \times & \times & \times & \times & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & \times & \times & \times & \times & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & \times & \times & \times & \times & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & \times & \times & \times & \times & 0
\end{pmatrix}_{15\times 16}.$$

Now a sequence of determinant preserving column eliminations and simplifications with pivotal elements

$$\times_{14,13},\ \times_{13,12},\ \times_{11,9},\ \times_{10,8},\ \times_{8,5},\ \times_{7,4},\ \times_{4,3},\ \times_{1,1},\ \times_{2,2},\ \times_{3,6},\ \times_{5,11},\ \times_{6,15}$$

gives the 3×4 matrix

$$M_5'' := \begin{pmatrix} & 0\\ M_4' & 1\\ & 0\end{pmatrix} := \tag{4.5}$$

$$:= \begin{pmatrix} (v_1\times v_3)^2(v_2\times v_3)^2 & 0 & -(v_1\times v_9)^2(v_2\times v_9)^2 & 0\\ (v_3\times v_4)^2(v_3\times v_5)^2 & -(v_4\times v_6)^2(v_5\times v_6)^2 & 0 & 1\\ 0 & (v_6\times v_7)^2(v_6\times v_8)^2 & -(v_7\times v_9)^2(v_8\times v_9)^2 & 0\end{pmatrix},$$

with determinant of the 3×3 minor that is left of M_4 as

$$\begin{aligned}\det M_4' \;=\;& (v_1\times v_3)^2(v_2\times v_3)^2(v_4\times v_6)^2(v_5\times v_6)^2(v_7\times v_9)^2(v_8\times v_9)^2 -\\ &-(v_1\times v_9)^2(v_2\times v_9)^2(v_3\times v_4)^2(v_3\times v_5)^2(v_6\times v_7)^2(v_6\times v_8)^2.\end{aligned}\tag{4.6}$$

Also,

$$\text{rank}M_4 = 51 + \text{rank}M_4',\ \ \text{rank}M_5 \geq 78 + \text{rank}M_5''.$$

Quite clearly, $\text{rank}M_5'' = 3$, $\text{rank}M_5 \geq 81$, and by (3.12)

$$\Phi_5^2(\Delta) = 30 \leq \dim S_5^2(\Delta) = 111 - \text{rank}M_5 \leq 111 - 81 = 30.$$

Thus the equality must hold, and the theorem 3.3 gives the dimension of $S_n^2(\Delta),\ n\geq 5$. But $\text{rank}M_4' = 2$ or 3 depending on the fact if $\det M_4'$ vanishes or not. By collecting together factors that were saved during elimination one finds out that

$$\begin{aligned}\det M_4 \;=\;& 8\|e_3\|\,\|e_6\|\,\|e_9\|\,(v_1\times v_2)^5(v_1\times v_3)(v_1\times v_9)(v_2\times v_3)(v_2\times v_9)\cdot\\ &(v_3\times v_4)(v_3\times v_5)(v_3\times v_6)(v_3\times v_9)(v_4\times v_5)^5(v_4\times v_6)\cdot\\ &(v_5\times v_6)(v_6\times v_7)(v_6\times v_8)(v_6\times v_9)\cdot\\ &(v_7\times v_8)^5(v_7\times v_9)(v_8\times v_9)\det M_4'. \qquad (4.7)\end{aligned}$$

Since we have restricted to the general triangulations, $\det M_4$ vanishes iff

$$f(\Delta)^2 = 1, \quad f(\Delta) := \frac{v_1\times v_3}{v_1\times v_9}\,\frac{v_2\times v_3}{v_2\times v_9}\,\frac{v_4\times v_6}{v_3\times v_4}\,\frac{v_5\times v_6}{v_3\times v_5}\,\frac{v_7\times v_9}{v_6\times v_7}\,\frac{v_8\times v_9}{v_6\times v_8}. \qquad (4.8)$$

This concludes the proof of the the following results.

Theorem 4.1. *Let Δ be a general Morgan-Scott triangulation. Then*

$$\dim S_n^2(\Delta) = \frac{n(7n-33)}{2} + 25 + \delta$$

with $\delta = 1$ only iff

$$n = 4,\ f(\Delta)^2 = 1,$$

and $\delta = 0$ otherwise.

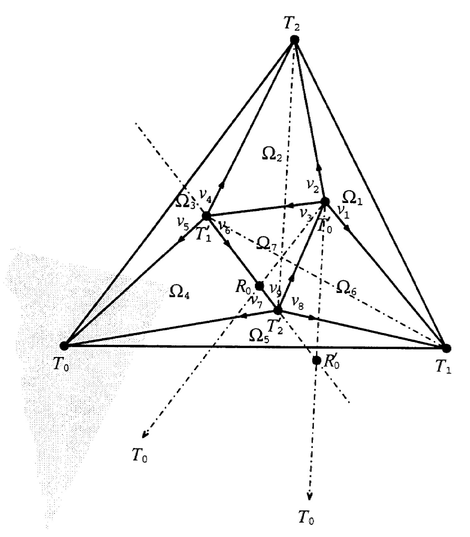

Fig.3. Singular choices of T_0 for the Morgan–Scott triangulation

The Fig.3 shows for which T_0 would $f(\Delta)^2 - 1$ vanish if all the other points of V are left unchanged. The possibility $f(\Delta) = +1$ is given by a line through points T_0', R_0, and the other through the points T_0', R_0'. No T_0 on the second line would produce the Morgan-Scott partition since intersection of this line and the shaded area is empty. However, in general one has to verify both posibilities. This fact was already observed in [2]. Note that $f(\Delta) = +1$ iff the all lines T_iT_i' are concurrent lines ([11], [3]).

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