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# AN EXPLICIT PSEUDO–SPECTRAL SCHEME WHIT ALMOST UNCONDITIONAL STABILITY FOR THE CAHN–HILLIARD EQUATION*1)

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**Abstract**

In this paper, an explicit fully discrete three-level pseudo-spectral scheme with almost unconditional stability for the Cahn–Hilliard equation is proposed. Stability and convergence of the scheme are proved by Sobolev's inequalities and the bounded extensive method of the nonlinear function (B.N. Lu[4] (1995)). The scheme possesses the almost same stable condition and convergent accuracy as the Creak-Nicloson scheme but it is an explicit scheme. Thus the iterative method to solve nonlinear algebraic system is avoided. Moreover, the linear stability of the critical point $u_0$ is investigated and the linear dispersive relation is obtained. Finally, the numerical results are supplied, which checks the theoretical results.

*Key words*: Cahn-Hilliard equation, Pseudo-spectral scheme, Almost unconditional stability, Linear stability for criticlal points, Numerical experiments.

## 1. Introduction

In this paper we consider a class of the nonlinear Cahn–Hilliard equation with periodic initial-value problem:

$$\begin{cases} u_t = M\Delta(\phi(u) - \gamma\Delta u), & (x,t) \in R \times J & (1.1)\\ u(x,0) = u_0(x), & x \in R & (1.2)\\ u(x+2\pi,t) = u(x,t). & (x,t) \in R \times J & (1.3)\end{cases}$$

where $M > 0$ is the mobility (assumed to be a constant) and $\gamma > 0$ is a phenomena logical constant modeling the effect of interfacial energy. The Laplace operator is denoted by $\Delta$, $\phi(u) = \psi'(u)$, $\psi(u) = \frac{1}{4}(u^2 - \beta^2)^2$ is called the homogeneous free energy. $\phi(\cdot)$ is the real function; $u_0(x)$ and $u(x,t)$ are the given and unknown real functions defined on $R$ and $R \times J$, $2\pi$–periodic with respect to $x$, respectively. $J = [0,T](T > 0)$. $R$ is the real line.

Theoretical results about the existence uniqueness and regularity for (1) can be found in [1]. Numerical approximations of (1) based on the finite element method[2], the finite difference method[3] have also been considered.

In this paper, we devote a three-level explicit pseudo–spectral with almost same stable condition and same accuracy as Creak-Nicloson implicit scheme. We prove its

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convergence and stability by using the bounded extensive method of the nonlinear function[4]. Therefore we avoid quite difficult a priori estimates. We don't need to solve nonlinear algebraic system.

Throughout this paper, the $c$ will be used to indicate generic constants, dependent of constant $M$, $\gamma$, $T$, function $\phi$, $u_0$, and so on.

## 2. The Pseudo–Spectral Scheme

We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm of $L^2(I)$ defined by

$$(u,v)=\int_I u\cdot\bar{v}dx, \quad \|u\|^2=(u,u)$$

where $I=[0,2\pi)$. Moreover, we define the Sobolev norm and seminorm:

$$\|u\|_s^2=\sum_{j=0}^{s}\Big\|\frac{\partial^j u}{\partial x^j}\Big\|^2, \quad |u|_j^2=\Big\|\frac{\partial^j u}{\partial x^j}\Big\|^2.$$

The definition of periodical Sobolev space $H_p^s(I)$ may be found in [4, 6–7].

The Fourier modes $\chi_j(x)=\dfrac{1}{\sqrt{2\pi}}e^{ijx}$, $j=0,\pm1,\pm2,\cdots$ are an orthogonal Hilbert basis of $L_p^2(I)$. For any positive even integer $N$ we set

$$S_N=\mathrm{Span}\Big\{\chi_j(x):-\frac{N}{2}\le j\le\frac{N}{2}-1\Big\}$$

and we denote by $P_N$ the orthogonal project of $H_p^s(I)$ upon $S_N$.

Let $K$ be a positive integer and $k=T/K$ be the time–step length. The notation $u_N^n$ is used to denote the approximation of $u_N$ at $t=nk$.

We define the following difference quotients:

$$u^n_{N\hat{t}}=\frac{u_N^{n+1}-u_N^{n-1}}{2k}; \qquad u_N^{n+\frac12}=\frac12(u_N^{n+1}+u_N^{n-1}).$$

Let $h=2\pi/N$ be the space–step length and $x_j=jh$ $(0<j\le N)$. The discrete inner product in the interval $I$ is define by

$$(u,v)_h=h\sum_{j=1}^{N}u(x_j)\bar{v}(x_j), \quad \|u\|_h^2=(u,u)_h.$$

The approximation $u_N^n$ to $u_N$ at $t=nk$ given by the pseudo–spectral method is defined by the equations:

$$\begin{cases}(u^n_{N\hat{t}},\chi)_h=M(\phi(u_N^n),\Delta\chi)_h-M\gamma(\Delta u_N^{n+\frac12},\Delta\chi)_h, & \forall\chi\in S_N, & (2.1)\\ \dfrac1k(u_N^1-u_N^0,\chi)_h=M(\phi(u_N^0),\Delta\chi)_h-M\gamma(\Delta u_N^0,\Delta\chi)_h, & \forall\chi\in S_N, & (2.2)\\ (u_N^0,\chi)_h=(u_0,\chi)_h, & \forall\chi\in S_N. & (2.3)\end{cases}$$

**Lemma 1.**[5] *For any $f,g\in C(\bar{I})$.*

$$(I_Nf,I_Ng)_h=(I_Nf,I_Ng)=(f,g)_h,$$

*where $I_N$ is the interpolative operator defined by $I_Nf\in S_N$ and $I_Nf(x_j)=f(x_j)$.*

**Lemma 2.**[5] *Assume that $v\in H_p^s(I)$, for any $s\ge\mu\ge0$, then there exists a positive constant c, independent of $v$ and $N$, such that $\|v-I_Nv\|_\mu\le cN^{\mu-s}|v|_s$.*

**Lemma 3.**[5] *Let $\sigma\ge\mu\ge0$, for any $v\in S_N$, then $|v|_\sigma\le(N/2)^{\sigma-\mu}|v|_\mu$, $\sigma\ge1/2$.*

By lemma 1, we can obtain the following result:

**Lemma 4.** *The equation* (2) *is equivalent to the following equation:*

$$\begin{cases} (u^n_{N\hat{t}},\chi) = M(I_N\phi(u^n_N),\Delta\chi) - M\gamma(\Delta u_N^{n+\frac{1}{2}},\Delta\chi) & \forall\chi\in S_N \qquad (3.1)\\ \frac{1}{k}(u^1_N - u^0_N,\chi) = M(I_N\phi(u^0_N),\Delta\chi) - M\gamma(\Delta u^0_N,\Delta\chi), & \forall\chi\in S_N \qquad (3.2)\\ u^0_N = I_N u_0. & \forall\chi\in S_N \qquad (3.3)\end{cases}$$

The equation (3) always can be solved by an inductive formulation (See below in the section 4) because the equation (3) can be written as an explicit scheme. Thus there exists a unique solution of the equation (3).

**Theorem 1.** *Assume that* $u \in C^0(J, H^s_p(I)) \bigcap C^3(J, L^2_p(I))$ *and* $u^n_N \in S_N$ *solve equations* (1) *and* (3) *respectively. If* $\phi(\cdot)$ *is in* $C^{s+1}(R)$*, then there exist positive constants* $c_1, M_1$*, such that for* $k \le c_1$*,* $N \ge M_1$*, and for any positive constant* $B$*, if* $k^{2-\delta}N^\sigma \le B$*:*

$$\sup_{0\le n\le K} \|u^n_N - u(t_n)\| \le c(N^{-s} + k^2) \qquad (4)$$

*where* $s > \sigma \ge 1$, $2 > \delta > 0$.

*Proof.* Set $\widetilde{u}\,(t_n) = P_N u(t_n)$, $e^n = u(t_n) - u^n_N$, $\xi_n = u(t_n) - \widetilde{u}\,(t_n)$ and $\eta_n = \widetilde{u}\,(t_n) - u^n_N$, then $e^n = \xi_n + \eta_n$, by lemma 2,

$$\|e^n\| \le cN^{-s}|u|_s + \|\eta_n\| \qquad (5)$$

The Taylor formula yields the identity

$$\widetilde{u_t}\,(t_n) = \widetilde{u}^{\,n}_{\hat{t}} - \frac{k^2}{6}\,\widetilde{u}_{ttt}\,(t_n + k\theta), \quad (|\theta| < 1)$$

where $\widetilde{u_t}\,(t_n) = \dfrac{\partial\,\widetilde{u}\,(t_n)}{\partial t}$, $\widetilde{u}^{\,n}_{\hat{t}} = \dfrac{\widetilde{u}\,(t_{n+1}) - \widetilde{u}\,(t_{n-1})}{2k}$.

Note that $(\widetilde{u_t},\chi) = (u_t,\chi)$, $(\widetilde{u}_{ttt},\chi) = (u_{ttt},\chi)$. We have

$$(\widetilde{u}^{\,n}_{\hat{t}},\chi) = M(\phi(u(t_n)),\Delta\chi) - M\gamma(\Delta\,\widetilde{u}\,(t_{n+\frac{1}{2}}),\Delta\chi) + \frac{k^2}{6}(\widetilde{u}_{ttt}\,(t_n + k\theta),\chi). \qquad (6)$$

Subtracting (3.1) from (6) with the choice $\chi = \eta_{n+1} + \eta_{n-1}$ yields

$$\frac{1}{2k}(\|\eta_{n+1}\|^2 - \|\eta_{n-1}\|^2) = M(\phi(u(t_n)) - I_N\phi(u^n_N), \Delta(\eta_{n+1} + \eta_{n-1})) - \frac{M\gamma}{2}\|\Delta(\eta_{n+1} + \eta_{n-1})\|^2 + \frac{k^2}{6}(u_{ttt}(t_n + k\theta), \eta_{n+1} + \eta_{n-1}) \qquad (7)$$

Following [3,4,6,7] we make a temporary assumption that $\phi(\cdot) \in C^{s+1}_b(R)$, by lemmas 1 and 2, we get

$$\begin{aligned}&(\phi(u(t_n)) - I_N(\phi(u^n_N)), \Delta(\eta_{n+1} + \eta_{n-1}))\\ &= (\phi(u(t_n)) - \phi(u^n_N) + \phi(u^n_N) - I_N(\phi(u^n_N)), \Delta(\eta_{n+1} + \eta_{n-1}))\\ &\le c(N^{-s} + \|\eta_n\|)\|\Delta(\eta_{n+1} + \eta_{n-1})\| \le \frac{c^2}{2\varepsilon}(N^{-2s} + \|\eta_n\|^2) + \varepsilon\|\Delta(\eta_{n+1} + \eta_{n-1})\|^2\end{aligned} \qquad (8)$$

Taking $\varepsilon = \frac{\gamma}{2}$ in (8), and because $\|u_{ttt}\| \le c$, from (5), (7) and (8), we have

$$\|\eta_{n+1}\|^2 - \|\eta_{n-1}\|^2 \le kc_2(\|\eta_{n+1}\|^2 + \|\eta_{n-1}\|^2 + \|\eta_n\|^2 + N^{-2s} + k^4) \qquad (9)$$

where $c_2 = \max\Big\{\dfrac{3Mc + c^2}{2\gamma}, \dfrac{\gamma}{18}, \dfrac{6Mc^2}{\gamma}\Big\}$.

Summation of (9) with respect to $n$ from 1 to $m(< K)$ yields

$$\|\eta_{m+1}\|^2 + \|\eta_m\|^2 \le \|\eta_1\|^2 + \|\eta_0\|^2 + c_2T(N^{-2s} + k^4) + 2kc_2\sum_{n=0}^{m}(\|\eta_{n+1}\|^2 + \|\eta_n\|^2).$$

by Gronwall's inequality and $\|\eta_0\| = 0$, we get

$$\|\eta_{m+1}\|^2 + \|\eta_m\|^2 \le c_3(N^{-2s} + k^4 + \|\eta_1\|^2).$$

It is not difficult to prove that $\|\eta_1\| \le c(N^{-s} + k^2)$.

Therefore

$$\|\eta_n\| \le c(N^{-s} + k^2). \tag{10}$$

Now the desired estimate (4) follows (5) and (10) for $\phi \in C_b^{s+1}(R)$.

Finally, similarly to the standard argument in [3,4,6,7], we can remove the hypothesis that $\phi$ and its first derivative are bounded. In fact that

$$\|e^n\|_\infty \le c\|e^n\|_1 \le c(N^{1-s} + N\|\eta_n\|),$$

where $\|\cdot\|_\infty$ denotes the uniform norm. If $s > 1$, then there exists $\sigma$, such that $s > \sigma \ge 1$. By lemma 2 and 3,

$$\|e^n\|_\sigma \le \|\eta_n\|_\sigma + \|\xi_n\|_\sigma \le c(N^{\sigma-s} + k^2N^\sigma),$$

by Sobolev's inequality and the hypothesis of the theorem, we get that there exits an $M_1$ and $c_1$, when $N \ge M_1$ and $k < c_1$,

$$\|e^n\|_{L^\infty(J;L^\infty(I))} \le cN^{\sigma-s} + ck^\delta(k^{2-\delta}N^\sigma) \le cM_1^{\sigma-s} + cBk^\delta.$$

Thus we have $\|e^n\|_{L^\infty(J,L^\infty(I))} \le \varepsilon$ i.e. $u_N^n \in S(\varepsilon)$. (The definition of $S(\varepsilon)$ may be found in [4]). The proof is completed.

**Theorem 2.** *Assume that* $u \in C^3(J, H_p^2(I))$ *and* $u_N^n \in S_N$ *solve equations* (1) *and* (2) *respectively. Under the conditions of Theorem* 1, *then*

$$\sup_{0\le n\le K}\|u_N^n - u(t_n)\|_\infty \le c(N^{1-s} + k^2). \tag{11}$$

*Proof.* Similarly to the proof of Theorem 1, subtracting (3.1) from (6) with the choice $\chi = \Delta(\eta_{n+1} + \eta_{n-1})$ yields

$$\frac{1}{2k}(|\eta_{n+1}|_1^2 - |\eta_{n-1}|_1^2) = MQ - \frac{M\gamma}{2}\|\nabla\Delta(\eta_{n+1} + \eta_{n-1})\|^2 + \frac{k^2}{6}(\nabla u_{ttt}(t_n + k\theta), \nabla(\eta_{n+1} + \eta_{n-1})) \tag{12}$$

where $Q = (\nabla(\phi(u(t_n)) - I_N\phi(u_N^n)),\ \nabla\Delta(\eta_{n+1} + \eta_{n-1}))$.

We make a temporary assumption that $\phi \in C_b^2(R)$, by Lemma 2 and similar (8)

$$Q \le \|\nabla(\phi(u(t_n)) - I_N\phi(u_N^n))\|\|\nabla\Delta(\eta_{n+1} + \eta_{n-1})\| \le \varepsilon\|\nabla\Delta(\eta_{n+1} + \eta_{n-1})\|^2 + \frac{3c^2}{4\varepsilon}(4N^{2(1-s)} + 2|\eta_n|_1^2).$$

Taking $\varepsilon = \dfrac{\gamma}{2}$ in above, and $\|\Delta u_{ttt}\| \le c$. (12) following that

$$|\eta_{n+1}|_1^2 - |\eta_{n-1}|_1^2 \le kc_3(|\eta_{n+1}|_1^2 + |\eta_{n-1}|_1^2 + |\eta_n|_1^2 + N^{2(1-s)} + k^4) \tag{13}$$

where $c_3 = \max\left\{\dfrac{6Mc^2}{\gamma}, \dfrac{c}{18}, c\right\}$.

Therefore $|\eta_n|_1 \le c(N^{1-s} + k^2)$.

$$\sup_{0\le n\le K}|u_N^n - u(t_n)|_1 \le c(N^{1-s} + k^2) \tag{14}$$

Finally, similarly to the argument in Theorem 1, we can remove the hypothesis that $\phi \in C_b^2(R)$. By using Theorem 1, (14) and Sobolev's inequality, (11) is obtained. We completes the proof of the theorem.

Similarly to the proof of the Theorem 1, we have the following stable result.

**Theorem 3.** *Assume that $u_N^n$ and $\tilde{u}_N^n$ are solutions of the equation* (2) *and* (2) *with the perturbation $\tilde{u_0}$ respectively, under the conditions of Theorem* 1, *then*

$$\sup_{0\le n\le K} \|u_N^n - \tilde{u}_N^n\| \le c\|u_0 - \tilde{u_0}\|.$$

## 3. Analysis to Linear Stability of Critical Points

In this section, in order to simulate the properties of the solution of the equation (1), we shall discuss the stabilities of the critical points of the equation (1) and then in the next section, we shall use the pseudo-spectral scheme proposed in this paper to check our the theoretical results.

It is clear that $u = u_0$ is a critical point of the equation (1.1) for any constant $u_0$. By the method in [8], the linearized equation of (1.1) around the critical point $u_0$ can be given by

$$(\partial_t - 3Mu_0^2\partial_x^2 + M\partial_x^2 + M\gamma\partial_x^4)\partial u = 0 \tag{15}$$

In the further theoretical analysis, we search for the unstable wave-number $q$ and the linear growth rate. Introducing the very small perturbation to the critical point $u_0$ of (1.1), is modulated by

$$u(x,t) = u_0 + \delta u e^{i(qx+\Omega t)}, \tag{16}$$

where $\delta u$ is much smaller than $u_0$ and $q$ is a real wave-number.

Substitution of (16) into (15), we have the linear disperse relation

$$\Omega = -i(Mq^2 - 3Mu_0^2q^2 - M\gamma q^4).$$

$\Omega$ are pure imagery numbers and write $\Omega = i\rho$, the $\rho$ is called as the linear growth rate of the modulational instability.

$$\rho = 3Mu_0^2q^2 - Mq^2 + M\gamma q^4.$$

For any fixed $u_0$, $M$ and $\gamma$, we assume that the $\rho$ is a function of the wave-number.

Let $q_{\max} = \sqrt{\dfrac{1-3u_0^2}{\gamma}}$, then, i.e. $\rho(q_{\max}) = 0$. (The geometrical significance can be seen in Figure 1, where we take a special case with $M = 1$, $\gamma = 1$ and $u_0 = 0.1$). Hence when $0 < q < q_{\max}$, $\rho(q) < 0$, it corresponds to the instability mode. When $q > q_{\max}, \rho(q) > 0$. Thus $u_0$ is an asymptotic stable fixed point of (1.1).

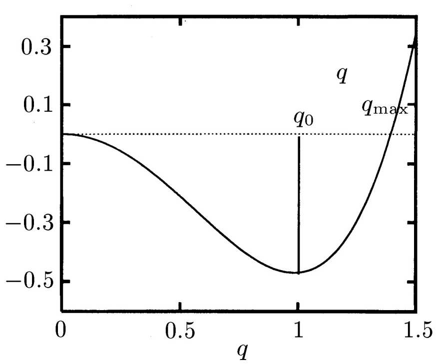



Fig. 1. The relation between the wave number $q$ and linear growth rate $\rho$, here $M = 1$, $\gamma = 0.5$ and $u_0 = 0.1$

**Remark 1.** It is clear that the instability of the critical points $u_0$ is not dependent with the mobility $M$.

Setting

$$q_0 = \sqrt{\frac{1-3u_0^2}{2\gamma}}.$$

Then $\rho$ takes minimum value at $q = q_0$ i.e.

$$\min_{0\le q\le q_{\max}} \rho = -\frac{M}{4\gamma}(3u_0^2 - 1)^2. \tag{17}$$

## 4. Algorithm and Numerical Analysis

In order to check the theoretical results of the proposed scheme (3), in the following we consider a generalized Cahn-Hilliard equation:

$$\begin{cases} u_t = M(\Delta\phi(u) - \gamma\Delta^2 u) + f(x,t) & (x,t) \in R \times J \quad (18.1) \\ u(x,0) = \sin(x) & x \in R \quad (18.2) \\ u(x+2\pi, t) = u(x,t) & (x,t) \in R \times J \quad (18.3) \end{cases}$$

It is not difficult to prove that Theorems 1, 2 are still valid for the equation (18).

The pseudo-spectral scheme corresponding the equation (18), by taking $\chi = \chi_j$ in the corresponding the equation (3), can be rewritten as.

$$\begin{cases} \overset{\sim n+1}{u}_{N_j} = a_1 \overset{\sim n-1}{u}_{N_j} - c_1 \overset{\sim}{\phi}(u_N^n)_j + d_1 \overset{\sim n}{f}_{N_j} & (19.1) \\ \overset{\sim 1}{u}_{N_j} = a_2 \overset{\sim 0}{u}_{N_j} - c_2 \overset{\sim}{\phi}(u_N^0)_j + k \overset{\sim 0}{f}_{N_j} & (19.2) \\ \overset{\sim 0}{u}_{N_j} = (\tilde{u}_0)_j & (19.3) \end{cases}$$

where $a_1 = (1 - M\gamma kj^4)/(1 + M\gamma kj^4)$, $a_2 = 1 - M\gamma kj^4$, $c_1 = 2kMj^2/(1 + M\gamma kj^4)$, $c_2 = kMj^2$ and $d_1 = 2k/(1 + M\gamma kj^4)$.

Therefore (19) is an explicit pseudo-spectral scheme which can be solved by the iterative method step by step. It is clear if the nonlinear terms are proposed, then we can get $\overset{\sim n+1}{u}_{N_j}$ from both $\overset{\sim n-1}{u}_{N_j}$ and $\overset{\sim n}{u}_{N_j}$. In the following, we shall give the algorithm of the nonlinear term $\tilde{\phi}_j$.

The calculation of $\tilde{\phi}_j$ is carried out by the transform method according to the diagram

$$\begin{array}{ccc} \{\overset{\sim n}{u}_{N_j} \mid -\frac{N}{2} \le j \le \frac{N}{2} - 1\} & \overset{(FFT)^{-1}}{\longrightarrow} & \{u_N^n(x_j) \mid 0 < j \le N\} \\ & & \downarrow \\ \{\overset{\sim}{\phi}(u_N^n)_j \mid -\frac{N}{2} \le j \le \frac{N}{2} - 1\} & \overset{FFT}{\longleftarrow} & \{\phi(u_N^n(x_j)) \mid 0 < j \le N\} \end{array}$$

in which two discrete Fourier transforms (one direct and one inverse) are required and $4N \ln N$ complex multiplications are needed. Since $f$ and $\phi$ are variable functions, we shall omit the operating numbers to calculate the functions $f$ and $\phi$ from $u_N^n(x_j)$ to both $\phi(u_N^n(x_j))$ and $f(u_N^n(x_j))$. The calculation of $\tilde{f}_j$ needs a direct fast Fourier transform, plus $2N \ln N$ complex multiplications. Hence

$$(K-1)(6N \ln N + 5N)$$

complex multiplications are needed in total computation by (19).

**Experiment 1.** Set $M = 0.001$, $\gamma = 1, \phi(u) = u(u^2 - 1)$, and $f(x,t) = (M\gamma - M - 1)e^{-t}\sin x + 3Me^{-3t}\sin x \cdot (3\sin^2 x - 2)$. It is not difficult to check that $U(x,t) = e^{-t}\sin x$ solves equation (18).

In this test we take that $T = 100$, $N = 32$, $k = 0.001$ and compute the equation (18) by UNIX C on SGI workstation by using the pseudo–spectral scheme (19):

The error $\|u^n - u_N^n\|$ is drown in the Figure 2. It shows that the error of the pseudo-spectral solution (19) is about order 6, which is same as the theoretical results in Theorems 1–3. Thus the theoretical results are belivable. In the following we shall simulate some properties of the solution by using of the pseudo-spectral scheme (19).

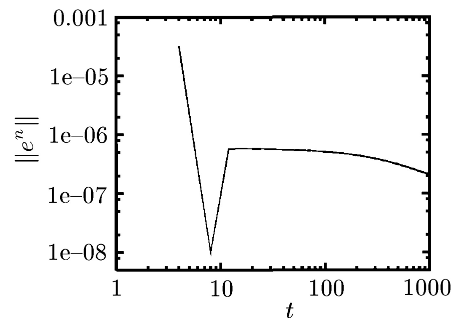


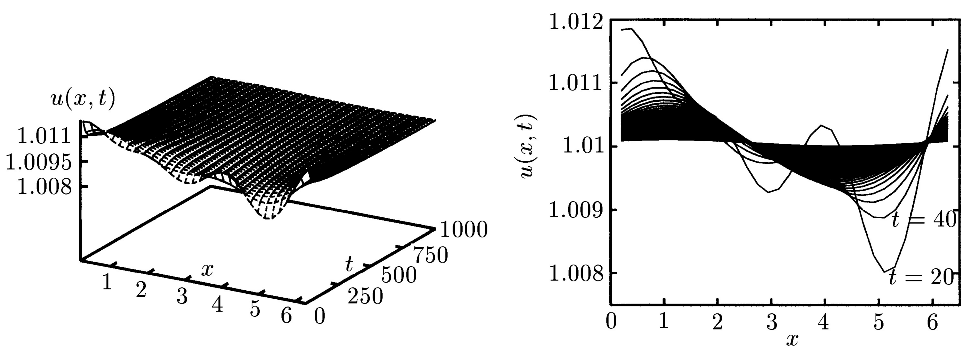

Fig. 2. Error of the pseudo-spectral scheme (19). Here we take logscale.

**Remark 2.** The pseudo–spectral method (19) is much better than with the finite difference method[3]. Numerical simulation by the difference scheme[3] provides only first order of $k$.

**Experiment 2.** We take $M = 0.001$, $k = 0.01, q = 1, \gamma = 1$, $T = 1000$ and $N = 32$ with the initial data

$$u_j(0) = 1 + 0.02\delta_j \tag{20}$$

where $\delta_j$ is random noise. The evolution of the solution is shown in Figure 3. The solution at that point in Figure 3 appears to converge to a steady state.




(a) Figure of $u(x,t)$

(b) The project of $u(x,t)$

Fig. 3. Numerical simulation with initial data (20) and $k = 0.01$, $N = 32$, $q = 1$, $\gamma = 0.5$ and $u_0 = 1$.

But at $T \approx 100$ two phases collapse and four phases transform into two phases. This pattern of very slow evolution and the relatively quick collapse of domains. Figure 3 shows that the equilibrium state finally is attained at around $T = 1000$. In Figure 3(a), the evolution of the solution is shown and in the Figure 3(b)the projection of the solution $u(x,t)$ is drown in the plane $(x, u(x,t))$.

**Experiment 3.** We take $M = 0.001$, $k = 0.01$, $\gamma = 0.5$, $q = q_0 = \sqrt{\frac{1-3u_0^2}{2\gamma}}$, and $N = 32$ with the initial data

$$u_0(x) = 0.3333 + 0.02\cos(q_0 x).$$

It is clear that in this special case, the $u_0 = 0.3333$ is unstable from the theoretical analysis in section 3. The numerical simulations in this case show absolutely different behaviour from those outlined in the experiment 2 (see Figure 4a). The solution rapidly evolves into the pattern of positive and negative domains which are shown in Figure 4a at $t = 10^5$ and then it's amplitude slow increase from $0.3333 \pm 0.03$ to $-7.8$–$1.18$. After $t > 1.5 \times 10^6$, the solution attains its a special solution and does not change. Thus the fixed solution is an element of it's attractor.

By using of the Lyapunov function method[9] to check that the solution enters into its the attractor after $t > 1.5 \times 10^6$. It is not difficult to check that

$$I(u) = M \int_0^{2\pi} \Big[\frac{\gamma}{2}(\Delta u)^2 - \Delta\psi(u)\Big]\, dx$$

is a Lyapunov function. Figure 4b shows that the Lyapunov function arrive to a fixed value after $t > 1.5 \times 10^6$.

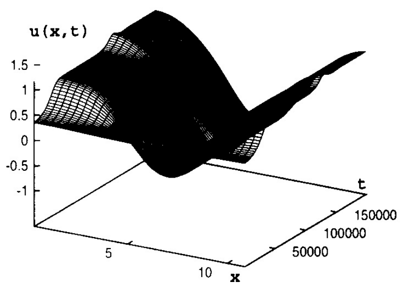



Fig. 4a. Evolution of the solution with the maximum unstable wave number $q_0$

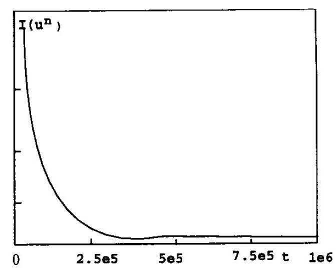



Fig. 4b. Lyapunov function $I(u^n)$

**Remark 3.** From the view of the numerical simulation, the scheme (19) can be used to the long time behaviour simulation. Because it is almost unconditionally stable, the time step lengthen can be taken larger. Therefore we can save a lot of computer time.

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