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A FINITE DIFFERENCE SCHEME FOR THE GENERALIZED NONLINEAR SCHRÖDINGER EQUATION WITH VARIABLE COEFFICIENTS*

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Abstract

A finite difference scheme for the generalized nonlinear Schrödinger equation with variable coefficients is developed. The scheme is shown to satisfy two conservation laws. Numerical results show that the scheme is accurate and efficient.

Key words: Finite difference scheme, Schrödinger equation, Discrete energy method.

1. Generalized Nonlinear Schrödinger Equation

The Schrödinger equation has been extensively used in physics research, particularly in the modeling of nonlinear dispersion waves [8]. Numerical methods for solving the Schrödinger equation have been discussed in the literature. In this article, we consider a generalized nonlinear Schrödinger equation with variable coefficients

$$i\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}(A(x)\frac{\partial u}{\partial x})+iF(t)u+B(x)\left|u\right|^{p-1}u=0,\quad i^2=-1,\quad p>1,\tag{1}$$

where $u(x,0)=\phi(x)$. The coefficients $A(x), F(t)$ and $B(x)$ are real functions with $A(x)>0$, and $\phi(x)$ a sufficiently smooth function which vanishes for sufficiently large $|x|$. The solution $u(x,t)$ is a complex-valued function defined over the whole real line R. The above equation is a generalized case of those equations described in the literature [2,3,7]. In [2,3] the authors considered that the coefficient of u (which is the third term on the left-hand side of the Eq. (1)) was a real function rather than a complex function $iF(t)$. We find in the next text that the conservation laws for these two cases are different. In [7] the authors considered that the coefficient of u was a constant complex number iv rather than a complex function $iF(t)$. When $F(t)=v>0$, there is a strong dissipative term resulting in amplitude decay of the soliton for the problem of propagation of a single soliton. However, the obtained numerical scheme produced small ripples for solving the propagation of a soliton [7]. Authors in [7] pointed out

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that as a result of the ripple effect in the numerical solution other methods should be explored.

To derive two conservation laws for Eq. (1), we first let $u = we^{-\int F(t)dt}$ to eliminate the term $iF(t)u$. For convenience, we assume here that $\int F(t)dt = 0$ when $F(t) = 0$. As such, one obtains

$$i\frac{\partial w}{\partial t} - \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x}) + B(x)e^{-(p-1)\int F(t)dt}\,|w|^{p-1}\,w = 0. \tag{2}$$

Multiplying Eq. (2) by $\bar{w}$ (which is the conjugate of w), integrating over the whole real line and taking the imaginary part, one obtains

$$\mathrm{Im}\int_R \left(i\frac{\partial w}{\partial t}\bar{w} - \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x})\bar{w} + B(x)e^{-(p-1)\int F(t)dt}\,|w|^{p+1}\right)dx = 0.$$

Since

$$\mathrm{Re}(\bar{w}\frac{\partial w}{\partial t}) = \frac{1}{2}(\bar{w}\frac{\partial w}{\partial t} + w\overline{\frac{\partial w}{\partial t}}) = \frac{1}{2}\frac{\partial |w|^2}{\partial t}$$

and

$$\begin{aligned}\int_R \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x})\bar{w}dx &= \int_R \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x}\bar{w})dx - \int_R A(x)\frac{\partial w}{\partial x}\frac{\partial \bar{w}}{\partial x}dx \\ &= -\int_R A(x)\frac{\partial w}{\partial x}\frac{\partial \bar{w}}{\partial x}dx \\ &= -\int_R A(x)\left|\frac{\partial w}{\partial x}\right|^2 dx\end{aligned}$$

then $\frac{d}{dt}\int_R |w|^2\,dx = 0$ from the imaginary part. Here, w is zero in the limit at $\pm\infty$ since the initial condition $\phi(x)$ is a sufficiently smooth function and vanishes for sufficiently large $|x|$. Replacing w by $ue^{\int F(t)dt}$, we obtain $\frac{d}{dt}\left(e^{2\int F(t)dt}\int_R |u|^2\,dx\right) = 0$. Hence, the first conservation law can be written as follows:

$$\int_R |u(x,t)|^2\,dx = \int_R |\phi(x)|^2\,dx \cdot e^{-2\int F(t)dt}. \tag{3}$$

It can be seen from Eq. (3) that the first conservation law is the same as that obtained in the literature [7,8,10,11,12] if $F(t) = 0$ or constant v.

We now multiply Eq. (1) by $\frac{\partial \bar{w}}{\partial t}$, integrate over R and take the real part to obtain

$$\mathrm{Re}\int_R \left(i\left|\frac{\partial w}{\partial t}\right|^2 - \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x})\frac{\partial \bar{w}}{\partial t} + B(x)e^{-(p-1)\int F(t)dt}\,|w|^{p-1}\,w\frac{\partial \bar{w}}{\partial t}\right)dx = 0. \tag{4}$$

It can be shown that

$$\begin{aligned}\mathrm{Re}\int_R B(x)\,|w|^{p-1}\,w\frac{\partial \bar{w}}{\partial t}dx &= \frac{1}{2}\int_R B(x)\,|w|^{p-1}\left(w\frac{\partial \bar{w}}{\partial t}+\overline{w\frac{\partial \bar{w}}{\partial t}}\right)dx\\ &= \frac{1}{p+1}\frac{d}{dt}\int_R B(x)\,|w|^{p+1}\,dx,\end{aligned} \tag{5}$$

and

$$\begin{aligned}\mathrm{Re}\int_R \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x})\frac{\partial \bar{w}}{\partial t}dx &= \mathrm{Re}\int_R \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x}\frac{\partial \bar{w}}{\partial t})dx\\ &\quad -\mathrm{Re}\int_R A(x)\frac{\partial w}{\partial x}\frac{\partial}{\partial t}(\frac{\partial \bar{w}}{\partial x})dx.\end{aligned} \tag{6}$$

Here, the first term on the right-hand side of Eq. (6) is zero since it is evaluated in the limit at $\pm\infty$, while the second term gives

$$\begin{aligned}-\mathrm{Re}\int_R A(x)\frac{\partial w}{\partial x}\frac{\partial}{\partial t}(\frac{\partial \bar{w}}{\partial x})dx &= -\frac{1}{2}\int_R A(x)\left(\frac{\partial w}{\partial x}\frac{\partial}{\partial t}(\frac{\partial \bar{w}}{\partial x})+\overline{\frac{\partial w}{\partial x}\frac{\partial}{\partial t}(\frac{\partial \bar{w}}{\partial x})}\right)dx\\ &= -\frac{1}{2}\int_R A(x)\frac{\partial}{\partial t}(\left|\frac{\partial w}{\partial x}\right|^2)dx.\end{aligned}$$

Hence, we obtain

$$\frac{d}{dt}\int_R A(x)\left|\frac{\partial w}{\partial x}\right|^2 dx + e^{-(p-1)\int F(t)dt}\frac{1}{p+1}\frac{d}{dt}\int_R B(x)\,|w|^{p+1}\,dx = 0.$$

We rewrite it as follows:

$$\begin{aligned}&\frac{d}{dt}\int_R A(x)\left|\frac{\partial w}{\partial x}\right|^2 dx + \frac{d}{dt}\left[\frac{1}{p+1}e^{-(p-1)\int F(t)dt}\int_R B(x)\,|w|^{p+1}\,dx\right]\\ &+\frac{p-1}{p+1}F(t)e^{-(p-1)\int F(t)dt}\int_R B(x)\,|w|^{p+1}\,dx\\ = &\ 0.\end{aligned}$$

Integrating the above equation, one obtains

$$\begin{aligned}&\int_R A(x)\left|\frac{\partial w}{\partial x}\right|^2 dx + \frac{1}{p+1}e^{-(p-1)\int F(t)dt}\int_R B(x)\,|w|^{p+1}\,dx\\ &+\frac{p-1}{p+1}\int_0^t\left(F(s)e^{-(p-1)\int F(s)ds}\int_R B(x)\,|w|^{p+1}\,dx\right)ds\\ = &\int_R A(x)\left|\frac{\partial w(x,0)}{\partial x}\right|^2 dx + \frac{1}{p+1}\int_R B(x)\,|w(x,0)|^{p+1}\,dx\\ \equiv &\ Const.\end{aligned}$$

Replacing w by $ue^{\int F(t)dt}$, we obtain the second conservation law for Eq. (1) as follows:

$$\begin{aligned} & e^{2\int F(t)dt}\int_R A(x)\left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx + \frac{1}{p+1}e^{2\int F(t)dt}\int_R B(x)\left|u(x,t)\right|^{p+1}dx \\ & +\frac{p-1}{p+1}\int_0^t\left(F(s)e^{2\int F(s)ds}\int_R B(x)\left|u(x,s)\right|^{p+1}dx\right)ds \\ = & \int_R A(x)\left|\frac{\partial u(x,0)}{\partial x}\right|^2 dx + \frac{1}{p+1}\int_R B(x)\left|u(x,0)\right|^{p+1}dx \\ \equiv & \ Const. \end{aligned} \tag{7}$$

One can simplify the second conservation law, Eq. (7), for some special cases. Obviously, if $F(t) = 0$, then Eq. (7) becomes

$$\begin{aligned} & \int_R A(x)\left|\frac{\partial u}{\partial x}\right|^2 dx + \frac{1}{p+1}\int_R B(x)\left|u\right|^{p+1}dx \\ = & \int_R A(x)\left|\frac{\partial \phi}{\partial x}\right|^2 dx + \frac{1}{p+1}\int_R B(x)\left|\phi\right|^{p+1}dx. \end{aligned} \tag{8}$$

This is the same as the second conservation law obtained from [2]. Further, if $F(t) \neq 0$ and $B(x) \equiv 0$, then Eq. (7) can be reduced to

$$\int_R A(x)\left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx = e^{-2\int F(t)dt}\int_R A(x)\left|\frac{\partial \phi}{\partial x}\right|^2 dx. \tag{9}$$

Particularly, when $A(x)$ is a constant and $F(t) = v$, Eq. (9) gives

$$\int_R \left|\frac{\partial u}{\partial x}\right|^2 dx = e^{-2vt}\int_R \left|\frac{\partial \phi}{\partial x}\right|^2 dx. \tag{10}$$

From Eq. (10), it can be seen that the conservation law $\int_R \left|\frac{\partial u}{\partial x}\right|^2 dx = \int_R \left|\frac{\partial \phi}{\partial x}\right|^2 dx$ obtained in [7] is incorrect when $v \neq 0$.

In what follows, we develop a finite difference scheme for Eq. (1). This scheme will be shown to satisfy the discrete analogues of the conservation laws (3) and (7). Specifically, the scheme satisfies the discrete analogue of the conservation law (8) when $F(t) = 0$ and the discrete analogue of the conservation law (9) when $B(x) = 0$.

2. Numerical Scheme

To develop a numerical scheme for solving Eq. (1), we first introduce some notations. Let w_j^n be the approximation of $w(x_j, t_n)$, where $x_j = j\Delta x, t_n = n\Delta t$, Δx is the grid size and Δt is the time increment. Here we consider the whole space, that is $j \in Z$, where Z is the set of all positive and negative integers. We also denote the first order forward/backward differences $(w_j^n)_x = \frac{w_{j+1}^n - w_j^n}{\Delta x}$ and $(w_j^n)_{\bar{x}} = \frac{w_j^n - w_{j-1}^n}{\Delta x}$, respectively. As such, the numerical scheme can be written as follows:

$$\begin{aligned} & i\frac{w_j^{n+1}-w_j^n}{\Delta t}-\frac{1}{2}\left[A_{j+\frac{1}{2}}\left((w_j^{n+1})_x+(w_j^n)_x\right)\right]_{\bar{x}} \\ & +\frac{B_j}{p+1}e^{-(p-1)\int F(t)dt\rfloor_{t=t_n+\frac{1}{2}}}\cdot\frac{\left|w_j^{n+1}\right|^{p+1}-\left|w_j^n\right|^{p+1}}{\left|w_j^{n+1}\right|^2-\left|w_j^n\right|^2}(w_j^{n+1}+w_j^n) \\ = & \ 0, \end{aligned} \tag{11}$$

$$w_j^n=\phi(j\Delta x), \tag{12}$$

and

$$u_j^{n+1}=w_j^{n+1}e^{-\int F(t)dt\rfloor_{t=t_n+1}}, \tag{13}$$

where $A_{j+\frac{1}{2}}=A((j+\frac{1}{2})\Delta x)$ and $B_j=B(j\Delta x)$. To discuss the discrete conservation laws for the scheme (11-13), we will use the following lemma.

Lemma. *For any mesh functions* $\left\{w_j^n\right\}_{j=-\infty}^{\infty}$ *and* $\left\{v_j^n\right\}_{j=-\infty}^{\infty}$ *with zero in the limit at* $\pm\infty$,

$$\sum_{j\in Z}\{A_{j+\frac{1}{2}}[(w_j)_x]\}_{\bar{x}}\cdot v_j=-\sum_{j\in Z}A_{j+\frac{1}{2}}(w_j)_x\cdot(v_j)_x.$$

The discrete analogues of the conservation laws (3) and (7) are obtained in the following theorem.

Theorem. *For all* $n\geq 0$, *the scheme (11-13) satisfies*

$$\Delta x\sum_{j\in Z}\left|u_j^n\right|^2=e^{-\int F(t)dt\rfloor_{t=t_n}}\cdot\Delta x\sum_{j\in Z}\left|\phi(j\Delta x)\right|^2, \tag{14}$$

and

$$\begin{aligned} & \frac{\Delta x}{2}e^{2\int F(t)dt\rfloor_{t=t_n}}\cdot\sum_{j\in Z}A_{j+\frac{1}{2}}\left|\frac{u_{j+1}^n-u_j^n}{\Delta x}\right|^2+\frac{\Delta x}{p+1}e^{2\int F(t)dt\rfloor_{t=t_n}}\cdot\sum_{j\in Z}B_j\left|u_j^n\right|^{p+1} \\ & +\Delta x\frac{p-1}{p+1}\sum_{k=0}^{n-1}\sum_{j\in Z}B_j[\left|u_j^{k+1}\right|^{p+1}e^{2\int F(t)dt\rfloor_{t=t_{k+1}}} \\ & \cdot(e^{(p-1)\int F(t)dt\rfloor_{t=t_{k+1}}-(p-1)\int F(t)dt\rfloor_{t=t_k+\frac{1}{2}}}-1) \\ & +\left|u_j^k\right|^{p+1}e^{2\int F(t)dt\rfloor_{t=t_k}}(1-e^{(p-1)\int F(t)dt\rfloor_{t=t_k}-(p-1)\int F(t)dt\rfloor_{t=t_k+\frac{1}{2}}})] \\ = & \ \frac{\Delta x}{2}\sum_{j\in Z}A_{j+\frac{1}{2}}\left|\frac{u_{j+1}^0-u_j^0}{\Delta x}\right|^2+\frac{\Delta x}{p+1}\sum_{j\in Z}B_j\left|u_j^0\right|^{p+1} \\ \equiv & \ Const. \end{aligned} \tag{15}$$

Proof. In order to obtain (14) one multiplies (11) by $\bar{w}_j^{n+1}+\bar{w}_j^n$, then sums over j

and takes the imaginary part. Explicitly, the first term gives

$$\begin{aligned}\sum_{j\in Z}\frac{i}{\Delta t}(w_j^{n+1}-w_j^n)(\bar{w}_j^{n+1}+\bar{w}_j^n) &= \sum_{j\in Z}\frac{i}{\Delta t}(\left|w_j^{n+1}\right|^2-\left|w_j^n\right|^2+w_j^{n+1}\bar{w}_j^n-w_j^n\bar{w}_j^{n+1}) \\ &= \sum_{j\in Z}\frac{i}{\Delta t}(\left|w_j^{n+1}\right|^2-\left|w_j^n\right|^2-2i\mathrm{Im}(w_j^n\bar{w}_j^{n+1})),\end{aligned}$$

the second term (multiplied by 2) becomes, by the above lemma,

$$\begin{aligned}&\sum_{j\in Z}\left[A_{j+\frac{1}{2}}\left((w_j^{n+1})_x+(w_j^n)_x\right)\right]_{\bar{x}}\cdot[\bar{w}_j^{n+1}+\bar{w}_j^n] \\ =\ & -\sum_{j\in Z}A_{j+\frac{1}{2}}[(w_j^{n+1})_x+(w_j^n)_x]\cdot[\bar{w}_j^{n+1}+\bar{w}_j^n]_x \\ =\ & -\sum_{j\in Z}A_{j+\frac{1}{2}}\left|[(w_j^{n+1})_x+(w_j^n)_x]\right|^2,\end{aligned}$$

and the third term becomes a real number. Thus, keeping the imaginary part, one obtains $\sum_{j\in Z}\left|w_j^{n+1}\right|^2=\sum_{j\in Z}\left|w_j^n\right|^2=\cdots=\sum_{j\in Z}\left|w_j^0\right|^2$. Hence, we obtain $\Delta x\sum_{j\in Z}\left|w_j^n\right|^2=\Delta x\sum_{j\in Z}\left|w_j^0\right|^2$. If w_j^n is replaced by $w_j^n=u_j^n e^{\int F(t)dt|_{t=t_n}}$, then Eq. (14) is obtained. To obtain Eq. (15), one multiplies Eq. (11) by $\bar{w}_j^{n+1}-\bar{w}_j^n$, then sums over j and take the real part. Explicitly, the first term gives

$$\sum_j\frac{i}{\Delta t}(w_j^{n+1}-w_j^n)(\bar{w}_j^{n+1}-\bar{w}_j^n)=\frac{i}{\Delta t}\sum_j\left|w_j^{n+1}-w_j^n\right|^2,$$

the second term becomes, by the lemma,

$$\begin{aligned}&\frac{1}{2}\sum_{j\in Z}\left[A_{j+\frac{1}{2}}\left((w_j^{n+1})_x+(w_j^n)_x\right)\right]_{\bar{x}}\cdot[\bar{w}_j^{n+1}-\bar{w}_j^n] \\ =\ & -\frac{1}{2}\sum_{j\in Z}A_{j+\frac{1}{2}}[(w_j^{n+1})_x+(w_j^n)_x]\cdot[(\bar{w}_j^{n+1})_x-(\bar{w}_j^n)_x] \\ =\ & -\frac{1}{2}\sum_{j\in Z}A_{j+\frac{1}{2}}\{\left|(w_j^{n+1})_x\right|^2-\left|(w_j^n)_x\right|^2-(w_j^{n+1})_x(\bar{w}_j^n)_x+(w_j^n)_x(\bar{w}_j^{n+1})_x\} \\ =\ & -\frac{1}{2}\sum_{j\in Z}A_{j+\frac{1}{2}}\{\left|(w_j^{n+1})_x\right|^2-\left|(w_j^n)_x\right|^2-2i\mathrm{Im}[(w_j^{n+1})_x(\bar{w}_j^n)_x]\},\end{aligned}$$

and the third term becomes

$$\begin{aligned}
& \sum_{j\in Z}\frac{B_j}{p+1}e^{-(p-1)\int F(t)dt\rfloor_{t=t_n+\frac{1}{2}}}\cdot[\left|w_j^{n+1}\right|^{p+1}-\left|w_j^n\right|^{p+1}] \\
= & \sum_{j\in Z}\frac{B_j}{p+1}[\left|w_j^{n+1}\right|^{p+1}\cdot e^{-(p-1)\int F(t)dt\rfloor_{t=t_n+1}}-\left|w_j^n\right|^{p+1}\cdot e^{-(p-1)\int F(t)dt\rfloor_{t=t_n}}] \\
& +\sum_{j\in Z}\frac{B_j}{p+1}[\left|w_j^{n+1}\right|^{p+1}\cdot(e^{-(p-1)\int F(t)dt\rfloor_{t=t_n+\frac{1}{2}}}-e^{-(p-1)\int F(t)dt\rfloor_{t=t_n+1}}) \\
& +\left|w_j^n\right|^{p+1}\cdot(e^{-(p-1)\int F(t)dt\rfloor_{t=t_n}}-e^{-(p-1)\int F(t)dt\rfloor_{t=t_n+\frac{1}{2}}})].
\end{aligned}$$

Thus, keeping only the real part, one obtains

$$\begin{aligned}
& \frac{1}{2}\sum_{j\in Z}A_{j+\frac{1}{2}}\left|\frac{w_{j+1}^n-w_j^n}{\Delta x}\right|^2+\frac{1}{p+1}\sum_{j\in Z}B_j\left|w_j^n\right|^{p+1}\cdot e^{-(p-1)\int F(t)dt\rfloor_{t=t_n+1}} \\
& +\frac{1}{p+1}\sum_{k=0}^{n}\sum_{j\in Z}B_j[\left|w_j^{k+1}\right|^{p+1}\cdot(e^{(p-1)\int F(t)dt\rfloor_{t=t_k+\frac{1}{2}}}-e^{(p-1)\int F(t)dt\rfloor_{t=t_{k+1}}}) \\
& +\left|w_j^k\right|^{p+1}\cdot(e^{(p-1)\int F(t)dt\rfloor_{t=t_k}}-e^{(p-1)\int F(t)dt\rfloor_{t=t_k+\frac{1}{2}}})] \\
= & \frac{1}{2}\sum_{j\in Z}A_{j+\frac{1}{2}}\left|\frac{w_{j+1}^0-w_j^0}{\Delta x}\right|^2+\frac{1}{p+1}\sum_{j\in Z}B_j\left|w_j^0\right|^{p+1}.
\end{aligned}$$

If w_j^n is replaced by $w_j^n=u_j^n e^{\int F(t)dt\rfloor_{t=t_n}}$, then Eq. (15) is obtained.

From the above theorem, it can be seen that i) the numerical scheme (11-13) with $F(t)=0$ satisfies the discrete analogues of the conservation laws (3) and (8); ii) equations (14) and (15) with $B(x)\equiv 0$ are the discrete analogues of the conservation law (3) and (9), respectively; and iii) when $F(t)=v\equiv Const$, the second conservation law (15) is the discrete analogue of the conservation law (10). This shows that the discrete analogue of the second conservation law obtained in [7] is incorrect.

3. Numerical Examples

In this section, we will apply the scheme (11-13) to solve two Schrödinger equations. We first consider the following linear Schrödinger equation with variable coefficients

$$i\frac{\partial u}{\partial t}+0.01tiu-\frac{\partial}{\partial x}((x^2+1)\frac{\partial u}{\partial x})=0,\quad 0\le x\le 5. \tag{16}$$

The initial and boundary conditions are

$$u(x,0)=x,\quad u(0,t)=0,\quad u(5,t)=5e^{-0.005t^2-2it}. \tag{17}$$

The exact solution for this problem is $u(x,t)=xe^{-0.005t^2-2it}$.

We chose $\Delta t = 0.01$, $\Delta x = 0.1$ for the scheme (11-13). Compared with the exact solution, the numerical solution $\left|u_j^n\right|$ at $t = 20$ is seen to be accurate as shown in Table 1.

Table 1. Comparison of the numerical soltuion at $t = 20$ with the exact solution

x	$\lvert u_j^n \rvert$	$\lvert u(x,t) \rvert$	$\lvert u_j^n - u(x,t) \rvert$
0.00	0.0	0.0	0.0
0.50	0.073845	0.067668	0.008680
1.00	0.143663	0.135335	0.011270
1.50	0.211746	0.203003	0.011416
2.00	0.278616	0.276671	0.010523
2.50	0.345230	0.338338	0.009060
3.00	0.411777	0.406006	0.007422
3.50	0.478219	0.473674	0.005689
4.00	0.544464	0.541341	0.003808
4.50	0.610593	0.609009	0.001891
5.00	0.676676	0.676676	0.0

We then consider the propagation of a soliton

$$i\frac{\partial u}{\partial t} + iF(t)u - \frac{\partial^2 u}{\partial x^2} - 2u\left|u\right|^2 = 0 \tag{18}$$

with $u(x,0) = 2\eta e^{2xi}\operatorname{sech}\{2\eta(x-x_c)\}$. In order to obtain a solution, we must suppose as in [7] that our solution has a compact support and that it is zero outside some interval $[x_0, x_{N+1}]$. As such, we use the artificial boundary condition $u(x_0) = u(x_{N+1}) = 0$. We now write our system of nonlinear equations in "matrix" form as in [7]. Let A be the tridiagonal matrix associated with the discrete second derivative in Eq. (11). We define

$$\begin{aligned} F_n(w_j) &= \frac{B_j}{p+1}e^{-(p-1)\int F(t)dt\rfloor_{t=t_n+\frac{1}{2}}} \cdot \frac{\left|w_j^{n+1}\right|^{p+1} - \left|w_j^n\right|^{p+1}}{\left|w_j^{n+1}\right|^2 - \left|w_j^n\right|^2}(w_j^{n+1}+w_j^n), \ if \ w_j \neq w_j^n, \\ &= \left|w_j\right|^{p-1} B_j e^{-(p-1)\int F(t)dt\rfloor_{t=t_n+\frac{1}{2}}} \cdot w_j, \quad if \quad w_j = w_j^n. \end{aligned}$$

We now rewrite system (11) in the more compact form

$$i(w^{n+1} - w^n) + \Delta t A(w^{n+1} + w^n) + \Delta t F_n(w^{n+1}) = 0.$$

This is a nonlinear system and it must be solved by some iterative technique. We follow the method in [7]. Let $w_0 = w_j^n$ be given. We compute a sequence $\{w_p\}_{p=0,1,\cdots}$ by the inductive relation $w_{p+1} = [iI + \frac{\Delta t A}{2}]^{-1}[(iI - \frac{\Delta t A}{2})w^n - \Delta t F_n(w_p)]$. The iteration will be continued until $|w_{p+1} - w_p| \leq 10^{-6}$ is satisfied.

Let $F(t) = v$. We chose $\Delta t = 0.02, \Delta x = 0.1$, $\eta = 0.75$, $x_c = -5$, and an interval $[-30, 30]$, which is an example in [7]. With $v = 0.1$ (Fig. 1), there is a strong dissipative term resulting in amplitude decay of the soliton. From Fig. 1 (for $v = 0.1$), it can be seen that the curves are smooth with no ripples. In [7], numerical results show small ripples for solving the propagation of a soliton. Authors in [7] pointed out that as a result of the ripple effect in the numerical solution other methods should be explored. Therefore, the scheme (11-13) under the special case of constant coefficients is an improvement on a similar scheme in the literature [7].

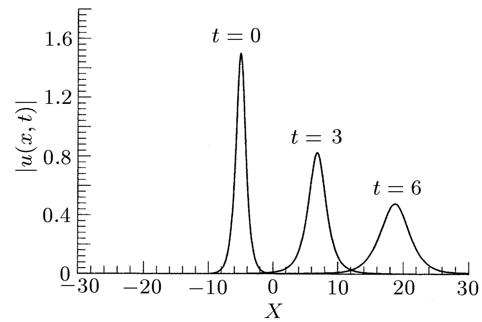

Fig.1 Propagation of a single soliton ($\nu = 0.1, \eta = 0.75$ and $x_c - 5$).

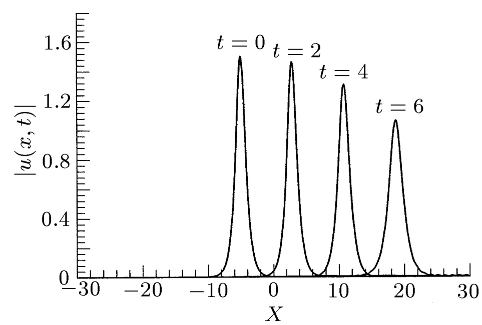

Fig.2 Propagation of a single soliton ($F(t) = 0.01t$, $\eta = 0.75$ and $x_c = -5$).

We then let $F(t) = 0.01t$ and chose $\Delta t = 0.02, \Delta x = 0.1$, $\eta = 0.75$, $x_c = -5$, and an interval $[-30, 30]$. The numerical results are shown in Fig. 2. Again, the curves are smooth with no ripples.

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