Journal of Computational Mathematics, Vol.18, No.2, 2000, 113–122.

# ON CONVERGENCE OF NOUREIN ITERATIONS FOR SIMULTANEOUS FINDING ALL ZEROS OF A POLYNOMIAL*1)

Shi-ming Zheng Zheng-da Huang
(*Department of Mathematics, Xixi Campus, Zhejiang University, Hangzhou 310028, China*)

**Abstract**

In this paper, we give the first estimation conditions for Nourein iterations for simultaneous finding all zeros of a polynomial under which the iteration processes are guaranteed to converge.

*Key words*: Polynomial zeros, Parallel iteration, Nourein iterations, Point estimation, Convergence.

## 1. Introduction

Suppose that

$$f(t)=\sum_{i=0}^{n}a_it^{n-i}=\prod_{j=1}^{n}(t-\xi_i),\quad a_0=1 \tag{1}$$

is a monic polynomial of degree $n$ with complex coefficients. Some authors have studied the parallel iterations without derivatives for simultaneous finding all zeros $\xi_1,\xi_2,\cdots,\xi_n$ of $f(t)$ (see [1]-[10],[13], [14], [16]). The famous one is Durand-Kerner iteration with the form

$$x_i^{k+1}=x_i^k-u_i^k \quad i=1,2,\cdots,n,\quad k=0,1,\cdots, \tag{2}$$

where $x_i^k$ is the k-th approximation of $\xi_i(1\le i\le n)$ and

$$u_i^k=\frac{f(x_i^k)}{\prod\limits_{j\neq i}(x_i^k-x_j^k)},\quad i=1,\cdots,n,\quad k=0,1,\cdots, \tag{3}$$

which does not require any information about the derivatives (see[3], [4], [6], [10], [14]). However, formula (2) appeared for the first time in Weierstrass' work [13],p.258] connected with a constructive proof of the fundamental theorem of algebra. The convergence of (2) is quadratic if $\xi_i\neq\xi_j$ for $i\neq j$, which was first proved by K. Dochev [3] and later by other authors.

* Received July 2, 1996.
1)The Project Supported by National Natural Science Foundation of China and by Natural Science Foundation of Zhejiang Province.

The other two iterations without derivatives are of the form

$$x_i^{k+1} = x_i^k - \frac{u_i^k}{1 + \sum\limits_{j \neq i} \frac{u_j^k}{x_i^k - x_j^k}} \qquad i = 1, 2, \cdots, n, \quad k = 0, 1, \cdots, \tag{4}$$

and

$$x_i^{k+1} = x_i^k - \frac{u_i^k}{1 + \sum\limits_{j \neq i} \frac{u_j^k}{x_i^k - u_i^k - x_j^k}} \qquad i = 1, 2, \cdots, n, \quad k = 0, 1, \cdots. \tag{5}$$

The iteration formula (4) was derived by Börsch-Supan [1], and later by Nourein [7], and (5) by Nourein [8]. As in Carstensen & Petkovic [2], (4) and (5) are both called Nourein iterations, and the order of convergence of them is three and four, respectively, if $\xi_i \neq \xi_j$ for $i \neq j$.

The concept of "point estimation", which gives the convergence criteria for iterations from data at initial points, was first proposed by S. Smale[11], and has attracted many authors (see, for example, [10], [12], [15] and references therein). In [16], the first author of this paper gave the conditions of the initials for the Durand-Kerner iteration under which the iteration (2) is guaranteed to converge to the zeros of $f(t)$, which is actually a point estimation convergence criterion.

In this paper we consider the point estimation for Nourein iterations (4) and (5). Our Theorems are different from that in [2], where a local convergence theorem is given, in which the conditions of convergence are concerning with the zeros of the polynomial. However, the zeros are unknown in advance. Therefore, the conditions are unable to verify. But all conditions in the following theorems depend only on the initials and can be verified.

## 2. Main Results and A Numerical Example

For purposes of brevity, all formulas, sums and products (such as in (2), (3), (4) and (5) above) involving indices $i, j$ and $\nu$ will assume the range $1, 2, \cdots, n$ and the iterative index $k = 0, 1, \cdots$, unless explicitly stated otherwise. Throughout this paper we will assume that $n \geq 3$.

Let

$$\delta_k = \max_{1 \leq i \leq n} |u_i^k|, B_k = \max_{j \neq i} |x_i^k - x_j^k|^{-1}, s_k = B_k \delta_k, \epsilon_n = \frac{1}{2(n+1)},$$

$$\phi_1(s) = \frac{1}{1-(n-1)s}, \qquad \phi_2(s) = \frac{1-s}{1-ns},$$

$$g_1(s) = \frac{(n-1)s^2}{[1-(n+1)s]^2} [1 + \frac{s}{1-(n+1)s}]^{n-2},$$

$$g_2(s) = \frac{(n-1)^2 s^3}{[1-(n+2)s+2s^2]^2} [1 + \frac{s(1-s)}{1-(n+2)s+2s^2}]^{n-2}.$$

$$h_\lambda(s) = (1 - 2s\phi_\lambda(s))g_\lambda(s) \quad (\lambda = 1, 2).$$

Our main results are contained in the following tow theorems.

**Theorem 1.** *Suppose that the initials* $x_i^{(0)}, i = 1, \cdots, n$, *satisfy* $s_0 < \epsilon_n$, *then the iteration (4) is well defined,*

$$\lim_{k\to\infty} x_i^k = \xi_i$$

*and*

$$|\xi_i - x_i^k| \le \frac{\phi_1(s_k)}{1 - h_1(s_k)}\delta_k$$
$$\le \frac{\phi_1(s_0 g_1(s_0)^{\frac{3^k-1}{2}})[1 - 2s_0\phi_1(s_0)]^k g_1(s_0)^{\frac{3^k}{2}-k}}{1 - [1 - 2s_0\phi_1(s_0)]g_1(s_0)^{3^k-1}}\delta_0.$$

**Theorem 2.** *Suppose that the initials* $x_i^{(0)}, i = 1, \cdots, n$, *satisfy* $s_0 < \epsilon_n$, *then the iteration (5) is well defined,*

$$\lim_{k\to\infty} x_i^k = \xi_i$$

*and*

$$|\xi_i - x_i^k| \le \frac{\phi_2(s_k)}{1 - h_2(s_k)}\delta_k$$
$$\le \frac{\phi_2(s_0 g_2(s_0)^{\frac{4^k-1}{3}})[1 - 2s_0\phi_2(s_0)]^k g_2(s_0)^{\frac{4^k}{3}-k}}{1 - [1 - 2s_0\phi_2(s_0)]g_2(s_0)^{4^k-1}}\delta_0,$$

**Remark 1.** The bounds in Theorems 1 and 2, which depend on the initial data, behave as $g_1(s_0)^{3^k}$ and $g_2(s_0)^{4^k}$, that is $|\xi_i - x_i^k| \sim g_1(s_0)^{3^k}$ for (4) and $|\xi_i - x_i^k| \sim g_2(s_0)^{4^k}$ for (5). These facts indicate that the order of convergence of the iteration methods (4) and (5) is *three* and *four*, respectively.

**Example.** Take $x_1^{(0)} = 2.035 + 0.03i$, $x_2^{(0)} = 1.035 + 0.03i$, $x_3^{(0)} = -0.975 - 0.03i$, $x_4^{(0)} = 0.03 + 1.035i$, $x_5^{(0)} = -0.03 - 0.975i$, $x_6^{(0)} = -1.035 + 2.035i$, $x_7^{(0)} = -1.035 - 1.975i$ as initial approximations to the zeros of polynomial $f(x) = x^7 + x^5 - 10x^4 - x^3 - x + 10$. We obtain $B_0 = 1, \delta_0 = 0.04944, s_0 = 0.04944, \epsilon_n = 0.0625$. Therefore, according to Theorems 1 and 2, both the iteration processes (4) and (5) are convergent starting from $x_i^{(0)}$ $(1 \le i \le 7)$.

## 3. Some Lemmas

To prove Theorems 1 and 2 we first give some Lemmas.

**Lemma 1.** *The functions* $g_\lambda(t), \phi_\lambda(t), h_\lambda(t)$, *defined in the previous section, are increasing continuous functions on interval* $[0, \epsilon_n)$.

**Lemma 2.** *Let* $x \mapsto w(x)$ *be a real monotonically increasing functions on the interval* $[0, a]$ *with some* $a > 0$, *and let* $z(x) = x^r w(x)(r = 1, 2, \cdots)$. *Then*

$$z(cx) \le c^r z(x) \quad \forall c \in [0, 1], \quad x \in [0, a]. \tag{6}$$

The proofs of Lemmas 1 and 2 are elementary and, for this reason, they are omitted.

**Lemma 3.** *If $s \in [0, \epsilon_n)$, then $g_\lambda(s) < 1 (\lambda = 1, 2)$.*

*Proof.* Since the functions $g_1(s)$ and $g_2(s)$ are monotonically on the interval $[0, \epsilon_n)$(see Lemma 1), we have

$$g_\lambda(s) \leq \max_{s \in [0, \epsilon_n)} g_\lambda(s) < g_\lambda(\epsilon_n).$$

For $g_1(s)$ we find

$$g_1(\epsilon_n) = \frac{n-1}{(n+1)^2}(1 + \frac{1}{n+1})^{n-2}.$$

Since

$$\frac{n-1}{(n+1)^2} \leq \frac{1}{8}$$

and

$$(1 + \frac{1}{n+1})^{n-2} < (1 + \frac{1}{n+1})^{n+1} < e,$$

we obtain $g_1(s) < \frac{e}{8} \approx 0.34 < 1$.

Considering $g_2(s)$ we write $g_2(\epsilon_n) = a(n)b(n)$, where

$$a(n) = \frac{(n-1)^2(n+1)}{2(n^2+n+1)^2}, \ b(n) = [1 + \frac{2n+1}{2n^2+8n+10}]^{n-2}.$$

The sequence$\{b(n)\}$ is monotonically increasing so that $b(n) < \lim_{n \to \infty} b(n) = e$. Since

$$\frac{n^2-1}{n^2+n+1} \leq 1 \ and \ \frac{n+1}{n^2+n+1} \leq \frac{4}{13},$$

we estimate $a(n) < \frac{2}{13}$ so that $g_2(s) < \frac{2e}{13} \approx 0.418 < 1$.

**Lemma 4.** *Let the sequence of approximation $\{x_i^k\}_{k=0}^\infty$ be defined by an iteration formula of the form*

$$x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{q_i(x_i^k)}, \quad i = 1, \cdots, n, \quad k = 0, 1, \cdots,$$

*where $q_i(x)$ is a real or complex function which does not vanish for any $x_i^k$. If $\lim\limits_{k \to \infty} x_i^k = x_i^*(i = 1, \cdots, n)$, then $x_i^*$ are the zeros of the polynomial $f(t)$.*

*Proof.* Letting $k \to \infty$ in the above iteration formula for $i \in \{1, \cdots, n\}$, we obtain

$$x_i^* = x_i^* - \frac{f(x_i^*)}{q_i(x_i^*)},$$

wherefrom $f(x_i^*) = 0$. Therefore, $x_i^*$ is a zero of the polynomial $f$.

**Remark 2.** In particular, if

$$q_i(x) = \left( \prod_{j=1}^n (x - x_j^k) \right)'_{x = x_i^k}, \quad \textit{that is}, \quad q_i(x_i^k) = \prod_{j \neq i} (x_i^k - x_j^k),$$

then we get the Durand-Kerner method (2) with $u_i^k = f(x_i^k)/q_i(x_i^k)$. Formulae (4) and (5) have also the form of the iteration formula given in Lemma 4. Let us note that $\lim\limits_{k\to\infty} x_i^k = x_i^*$ in (2), (4) and (5) implies $\lim\limits_{k\to\infty} \delta_k = 0$.

**Lemma 5.** *Let* $\{x_i^k\}_{k=0}^{\infty}$ *be the sequence of approximations generated by* (4) *or* (5). *If* $s_k \in [0, \epsilon_n)$, *then*

$$|x_i^{k+1} - x_i^k| \le \phi_\lambda(s_k)\delta_k, \quad (\lambda = 1, 2) \tag{7}$$

*and*

$$|x_i^{k+1} - x_i^k + u_i^k| \le \psi(s_k)\delta_k, \tag{8}$$

*where* $\psi(s) = \dfrac{(n-1)s}{1-ns}$.

*Proof.* Let us prove first the inequality (7) for the method (4) ($\lambda = 1$). From (4) we obtain

$$|x_i^{k+1} - x_i^k| \le \left| \frac{u_i^k}{1 + \sum\limits_{j\ne i} \dfrac{u_j^k}{x_i^k - x_j^k}} \right| \le \frac{\delta_k}{1-(n-1)\delta_k B_k} = \frac{\delta_k}{1-(n-1)s_k} = \phi_1(s_k)\delta_k.$$

In the case of the improved Nourein's method (5) we arrange (5) in the form

$$x_i^{k+1} = x_i^k - \frac{u_i^k}{1 + \sum\limits_{j\ne i} \dfrac{u_j^k/(x_i^k - x_j^k)}{1 - \dfrac{u_i^k}{x_i^k - x_j^k}}}$$

and estimate

$$|x_i^{k+1} - x_i^k| \le \left| \frac{u_i^k}{1 + \sum\limits_{j\ne i} \dfrac{u_j^k/(x_i^k - x_j^k)}{1 - u_i^k/(x_i^k - x_j^k)}} \right| \le \frac{\delta_k}{1 - \dfrac{(n-1)s_k}{1-s_k}} = \frac{1-s_k}{1-ns_k}\delta_k = \phi_2(s_k)\delta_k.$$

which is the inequality (7) for $\lambda = 2$.

To prove (8) we first arrange the expression for $x_i^{k+1} - x_i^k + u_i^k$,

$$x_i^{k+1} - x_i^k + u_i^k = u_i^k - \frac{u_i^k}{1 + \sum\limits_{j\ne i} \dfrac{u_i^k}{x_i^k - u_i^k - x_j^k}} = \frac{\sum\limits_{j\ne i} \dfrac{u_i^k/(x_i^k - x_j^k)}{1 - u_i^k/(x_i^k - x_j^k)}}{1 + \sum\limits_{j\ne i} \dfrac{u_i^k/(x_i^k - x_j^k)}{1 - u_i^k/(x_i^k - x_j^k)}} u_i,$$

Hence

$$|x_i^{k+1} - x_i^k + u_i^k| \le \frac{\dfrac{(n-1)s_k}{1-s_k}}{1 - \dfrac{(n-1)s_k}{1-s_k}}\delta_k = \frac{(n-1)s_k}{1-ns_k}\delta_k = \psi(s_k)\delta_k.$$

This completes the proof of (8).

**Lemma 6.** *Let $y_i^k = x_i^k$ in the case of the method* (4) *and $y_i^k = x_i^k - u_i^k$ for the method* (5). *Then*

$$u_i^{k+1} = (x_i^{k+1} - x_i^k)(y_i^k - x_i^{k+1}) \sum_{j \neq i} \frac{u_j^k}{(x_i^{k+1} - x_j^k)(y_i^k - x_j^k)} \prod_{j \neq i} (1 + \frac{x_j^{k+1} - x_j^k}{x_i^{k+1} - x_j^{k+1}}). \quad (9)$$

*Proof.* Using the introduced quantity $y_i^k$, both methods (4) and (5) can be represented by unique formula,

$$x_i^{k+1} = x_i^k - \frac{u_i^k}{1 + \sum_{j \neq i} \frac{u_j^k}{y_i^k - x_j^k}}.$$

Hence

$$\frac{u_i^k}{x_i^{k+1} - x_i^k} + \sum_{j \neq i} \frac{u_j^k}{y_i^k - x_j^k} + 1 = 0. \quad (10)$$

The polynomial $f(t)$ is identical to its Lagrangean interpolation polynomial for the points $x_1^k, \cdots, x_n^k$, that is

$$f(t) = \sum_{j=1}^n u_j^k \prod_{\nu \neq j} (t - x_\nu^k) + \prod_{j=1}^n (t - x_j^k) = \Big[\sum_{j=1}^n \frac{u_j^k}{t - x_j^k} + 1\Big] \prod_{j=1}^n (t - x_j^k). \quad (11)$$

Combining (10) and (11) we obtain for $t = x_i^{k+1}$

$$\begin{aligned} f(x_i^{k+1}) &= (x_i^{k+1} - x_i^k) \sum_{j \neq i} [\frac{u_j^k}{x_i^{k+1} - x_j^k} - \frac{u_j^k}{y_i^k - x_j^k}] \prod_{j \neq i} (x_i^{k+1} - x_j^k) \\ &= (x_i^{k+1} - x_i^k)(y_i^k - x_i^{k+1}) \sum_{j \neq i} \frac{u_j^k}{(x_i^{k+1} - x_j^k)(y_i^k - x_j^k)} \prod_{j \neq i} (x_i^{k+1} - x_j^k). \end{aligned} \quad (12)$$

Dividing both sides of (12) by $\prod_{j \neq i} (x_i^{k+1} - x_j^{k+1})$ and taking into account that

$$\frac{x_i^{k+1} - x_j^k}{x_i^{k+1} - x_j^{k+1}} = 1 + \frac{x_j^{k+1} - x_j^k}{x_i^{k+1} - x_j^{k+1}} \qquad (j \neq i),$$

we obtain (9).

**Lemma 7.** *If the sequence of approximations $\{x_i^k\}_{k=0}^\infty$ is generated by* (4) *or* (5) *and if $s_0 < \epsilon_n$, then*

$$s_k < \epsilon_n, \quad (13)$$

*and*

$$\delta_{k+1} \leq h_\lambda(s_k)\delta_k, \quad s_{k+1} \leq g_\lambda(s_k)s_k \quad (\lambda = 1, 2) \quad (14)$$

*hold for all $k = 0, 1, \cdots$.*

*Proof.* We will prove this lemma only for the method (5) since the proof for (4) is similar and goes in a similar way. We will use the complete induction. Let us assume that (13) holds for $k = m$, we obtain the estimates

$$
\begin{aligned}
&|x_i^{m+1} - x_i^m| \le \phi_2(s_m)\delta_m, \ |x_i^{m+1} - x_i^m + u_i^m| \le \psi(s_m)\delta_m,\\
&|x_i^{m+1} - x_j^m| \ge |x_i^m - x_j^m| - |x_i^{m+1} - x_i^m| \ge \frac{1}{B_m} - \phi_2(s_m)\delta_m = \frac{1}{B_m}(1 - s_m\phi_2(s_m)),\\
&|x_i^m - u_i^m - x_j^m| \ge |x_i^m - x_j^m| - |u_j^m| \ge \frac{1}{B_m}(1 - s_m),\\
&|x_i^{m+1} - x_j^{m+1}| \ge |x_i^m - x_j^m| - |x_i^{m+1} - x_i^m| - |x_j^{m+1} - x_j^m| \ge \frac{1}{B_m}(1 - 2s_m\phi_2(s_m)).
\end{aligned}
$$

In view of those bounds, from (9) for $k = m$ we find

$$
\begin{aligned}
|u_i^{m+1}| &\le |x_i^{m+1} - x_i^m||x_i^m - u_i^m - x_i^{m+1}|\\
&\quad\times \sum_{j\ne i} \frac{|u_j^m|}{|x_i^{m+1} - x_j^m||x_i^m - u_i^m - x_j^m|} \prod_{j\ne i}\Big(1 + \frac{|x_j^{m+1} - x_j^m|}{|x_i^{m+1} - x_j^{m+1}|}\Big)\\
&\le \frac{\phi_2(s_m)\psi(s_m)s_m^2(n-1)\delta_m}{(1 - s_m\phi_2(s_m))(1 - s_m)} \prod_{j\ne i}\Big(1 + \frac{s_m\phi_2(s_m)}{1 - 2s_m\phi_2(s_m)}\Big)\\
&\le \frac{\phi_2(s_m)\psi(s_m)s_m^2(n-1)\delta_m}{(1 - s_m\phi_2(s_m))(1 - s_m)} \Big(1 + \frac{s_m\phi_2(s_m)}{1 - 2s_m\phi_2(s_m)}\Big)^{n-1}\\
&= (1 - 2s_m\phi_2(s_m))\frac{\phi_2(s_m)\psi(s_m)s_m^2(n-1)\delta_m}{(1 - 2s_m\phi_2(s_m))^2(1 - s_m)} \Big(1 + \frac{s_m\phi_2(s_m)}{1 - 2s_m\phi_2(s_m)}\Big)^{n-2}\\
&= (1 - 2s_m\phi_2(s_m))\frac{(n-1)^2 s_m^3}{[1 - (n+2)s_m + 2s_m^2]^2} \Big(1 + \frac{s_m(1 - s_m)}{1 - (n+2)s_m + 2s_m^2}\Big)^{n-2}\delta_m\\
&= (1 - 2s_m\phi_2(s_m))g_2(s_m)\delta_m = h_2(s_m)\delta_m,
\end{aligned}
$$

which gives

$$\delta_{m+1} \le h_2(s_m)\delta_m. \tag{15}$$

Under the above assumption $s_m < \epsilon_n$ and by Lemma 3, we get $g_2(s_m) < 1$. Furthermore, it is easy to prove the inequality $2s_m\phi_2(s_m) < 1$, which implies $h_2(s_m) = (1 - 2s_m\phi_2(s_m))g_2(s_m) < 1$.

Having in mind the inequality (7) for $k = m$, we find

$$
\begin{aligned}
|x_i^{m+1} - x_j^{m+1}| &\ge |x_i^m - x_j^m| - |x_i^{m+1} - x_i^m| - |x_j^{m+1} - x_j^m|\\
&\ge \frac{1}{B_m}(1 - 2\phi_2(s_m)\delta_m B_m) = \frac{1}{B_m}(1 - 2s_m\phi_2(s_m)),
\end{aligned}
$$

so that

$$B_{m+1} = \max_{j\ne i} \frac{1}{|x_i^{m+1} - x_j^{m+1}|} \le \frac{B_m}{1 - 2s_m\phi_2(s_m)}. \tag{16}$$

According to (15), (16), the inequality $g_2(s_m) < 1$ and the assumption $s_m < \epsilon_n$, we finally get

$$s_{m+1} = B_m \delta_{m+1} \le \frac{h_2(s_m) B_m \delta_m}{1 - 2 s_m \phi_2(s_m)} = g_2(s_m) s_m < s_m < \epsilon_n.$$

Obviously, the above consideration can be performed for $k = 0$ in the quite same way so that the inequality (13) of Lemma 7 follows by induction. Besides, since $s_{m+1} < \epsilon_n$ and $g_2(s_{m+1}) < g_2(s_m) < 1$, we derive by the same procedure the inequality $s_{m+2} \le g_2(s_{m+1}) s_{m+1} < s_m < \epsilon_n$, which proves (14).

## 4. Proofs of Theorems

The proofs of Theorems 1 and 2 are similar and they are based on the previous lemmas given for both methods (4) and (5). We will only prove Theorem 2 because the proof of Theorem 1 is similar and uses the same procedure.

**Proof of Theorem 2.** The functions $h_2(s)$ and $g_2(s)$ satisfy the conditions of Lemma 2 (with $r = 3$), so that we have by (6)

$$g_2(0) = 0, h_2(0) = 0, g_2(cs) \le c^3 g_2(s),\ h_2(cs) \le c^3 h_2(s), \qquad \forall c \in [0,1]\ s \in [0, \epsilon_n). \tag{17}$$

Since $g_2(s)$ and $h_2(s)$ are increasing functions on $[0,\epsilon_n)$ (Lemma 1) and the sequence $\{s_k\}$ is monotonically decreasing and bounded by $\epsilon_n$ (Lemma 7), for all $k = 0, 1, \cdots$ we have the following inequalities

$$\begin{aligned} &g_2(s_{k+1}) < g_2(s_k) < 1, \quad h_2(s_{k+1}) < h_2(s_k) < 1 \\ &s_{k+1} \le g_2(s_k) s_k < s_k, \quad \delta_{k+1} \le h_2(s_k) \delta_k < \delta_k. \end{aligned} \tag{18}$$

The last inequality follows according to the proof of Lemma 7 and the inequality $s_k < \epsilon_n$ for all $k = 0, 1, \cdots$.

Let us note that if $s_k < \epsilon_n$ then the equality

$$B_{m+1} = \max_{j \ne i} \frac{1}{|x_i^{m+1} - x_j^m|} \le \frac{B_m}{1 - 2 s_m \phi_2(s_m)},$$

derived in the proof of Lemma 7, is valid for all $k = 0, 1, \cdots$, which yields that $x_i^{k+1}$ are well defined for $1 \le i \le n$ and $k = 0, 1 \cdots$.

Define $h_k = h_2(s_k),\ g_k = g_2(s_k)$, then the inequalities

$$h_{k+1} \le g_k^3 h_k, \qquad g_{k+1} \le g_k^4$$

follow from (17) and (18), which yields

$$h_{k+p} \le h_k g_k^{4^p - 1}, \quad g_{k+p} \le g_k^{4^p}, \quad \delta_{k+p} \le \delta_k h_k^p g_k^{\frac{4^p-1}{3} - p}, \quad s_{k+p} \le s_k g_k^{\frac{4^p-1}{3}} \tag{19}$$

for $p = 0, 1, \ldots, \infty$ by induction method. By using (19) it is easy to verify that $\{\delta_k\}$, $\{s_k\}$, $\{h_k\}$, $\{g_k\}$ are all nonincreasing sequences with $g_k < 1$ and $h_k < 1$. In addition, $h_k^p < 1$ and $\lim\limits_{p \to \infty} h_k^p = 0$, so that for every $k, m \ge 0$ we have

$$\sum_{p=k}^{k+m} \delta_p = \sum_{p=0}^{m} \delta_{p+k} \leq \delta_k \sum_{p=0}^{\infty} h_k^p = \frac{\delta_k}{1-h_k}.$$

Therefore $\sum_{k=0}^{\infty} \delta_k$ converges and

$$\lim_{k\to\infty} \delta_k = 0, \qquad \sum_{p=k}^{\infty} \delta_p \leq \frac{\delta_k}{1-h_k}. \tag{20}$$

Then, taking into account that $\phi_2(s)$ is monotonically increasing on $[0,\epsilon_n)$ (Lemma 1), it follows

$$\begin{aligned} |x_i^{k+m} - x_i^k| &= |\sum_{p=0}^{m-1} (x_i^{k+p+1} - x_i^{k+p})| \leq \sum_{p=0}^{\infty} \phi_2(s_{k+1})\delta_{k+p} \\ &\leq \phi_2(s_k) \sum_{p=0}^{\infty} \delta_{k+p} \leq \frac{\phi_2(s_k)\delta_k}{1-h_k}, \end{aligned} \tag{21}$$

which shows that $\{x_i^k\}_{k=0}^{\infty}$ is a Cauchy sequence by means of (20) and the definition of $\phi_2(s)$. Furthermore, taking $k=0$ in (21) we obtain $|x_i^m - x_i^0| < \phi_2(s_0)\delta_0/(1-h_0)$ which means that all $x_i^m (m = 0, 1, \cdots)$, produced by (5), lie in the disk $D_i = \{x : |x - x_i^0| < \phi_2(s_0)\delta_0/(1-h_0)\}$. Since the metric subspace $D_i$ is complete (as a closed set in $\mathbf{C}$), the Cauchy sequence $\{x_i^k\}_{k=0}^{\infty}$ converges to a unique limit $x_i^*$. According to Lemma 4 this limit is a zero of the polynomial $f(t)$, that is

$$\lim_{k\to\infty} x_i^k = \xi_i.$$

Letting $m \to \infty$ in (21), we obtain the first formula in Theorem 2. Exchanging the position of $k$ and $p$ in (19) and letting $p = 0$, we obtain the second formula in Theorem 2 because of $\frac{h_0}{g_0} = 1 - 2s_0\phi_2(s_0)$, which completes the proof of Theorem 2.

## References


[1] W. Börsch-Supan, Residuenabschätzung für Polynom-Nullstellen Mittels Lagrange-Interpolation, *Numer. Math.,* **14** (1970), 287-297.

[2] C. Carstensen, M. S. Petković, On iteration methods without derivatives for the simultaneous determination of polynomial zeros, *J. Comput Appl. Math.,* **45** (1993), 251-267.

[3] K. Dochev, Modified Newton method for the simultaneous approximate calculation of all roots of a given algebraic equation (in Bulgarian), *Mat. Spis. Bulgar. Akad. Nauk.,* **5** (1962), 136-139.

[4] E. Durand, Solutions Numerigues des Équations Algébrigues Tome I: Équations du Type $F(x) = 0$, Racines dún Ployôme Masson, Paris, 1960.

[5] L.W. Ehrlich, A modified Newton method for polynomials, *Comm. ACM.,* **10** (1967), 107-108.

[6] I.O. Kerner, Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen, *Numer. Math.,* **8** (1966), 290-294.

[7] A.W.M. Nourein, An iteration formula for the simultaneous determination of the zeros of a polynomial, *J. Comput. Appl. Math.,* **1** (1975), 251-254.

[8] A.W.M. Nourein, An improvement on Nourein's method for the simultaneous determination of the zeros of a polynomial (An algorithm), *J. Comput. Appl. Math.,* **3** (1977), 109-110.

[9] M.S. Petković, Iteration Methods for the Simultaneous Inclusion of Polynomial Zeros, Springer, Berlin, 1989.

[10] M.S. Petković, C. Carstensen, M. Trajković, Weierstrass formula and zerofinding methods, *Numer. Math.,* **69** (1995), 353-372.

[11] S. Smale, International Congress of Mathematicians', Berkeley, California., U.S.A., 1986, 1-26.

[12] X. Wang, S. Zheng, D. Han, On convergence of Euler's series, Euler's iterative family and Halley's iterative family from data at one point (in Chinese), *ACTA Math. Sinica,* **33** (1990), 721-738.

[13] Weierstrass, Neuer beweis des Satzes, dass jede ganze rationale Funktion einer Verändelichen dargestellt werden kann als ein Product aus linearen Funktionen derstelben Veränderrlichin, *Ges. Werke,* **3** (1903), 251-269 (Johnson Reprint Corp. New York 1967).

[14] T. Yamamoto, S. Nanno, L. Atanassova, Validated computation of polynomial zeros by the Durand-Kerner method, in: Topics in Validated Computations (ed. J. Herzberger), North Holland, Amsterdam, 1994.

[15] F. Zhao, D. Wang, The theory of Smale's point estimation and the convergence of Durand-Kerner Program (in Chinese), *Math. Numer. Sinica,* **7** (1993), 196-206.

[16] S. Zheng, On convergence of Durand-Kerner method for finding all roots of a polynomial simultaneously, *Kexue Tongbao,* **27,** (1982), 1262-1265.