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ON A CELL ENTROPY INEQUALITY OF THE RELAXING SCHEMES FOR SCALAR CONSERVATION LAWS*1)

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Abstract

In this paper we study a cell entropy inequality for a class of the local relaxation approximation –The Relaxing Schemes for scalar conservation laws presented by Jin and Xin in [1], which implies convergence for the one-dimensional scalar case.

Key words: Hyperbolic conservation laws, the relaxing schemes, cell entropy inequality.

1. Introduction

In [1], Jin and Xin constructed a class of the local relaxation approximation-The Relaxing Schemes for systems of nonlinear conservation laws or single nonlinear conservation laws

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{d}\frac{\partial f_i(u)}{\partial x_i}=0, \tag{1.1}$$

with initial data $u(0,x)=u_0(x),\ x=(x_1,...,x_d)$, by using the idea of the local relaxation approximation[1-4].

The scheme is obtained in the following way: first a linear hyperbolic system with a stiff source term is constructed to approximate the original system (1.1) with a small dissipative correction; Then, the new linear hyperbolic system can be solved easily by underresolved stable numerical discretizations without using Riemann solvers spatially.

It is well known that the above Cauchy problem (1.1) may not always have a smooth global solution even if the initial data u_0 is smooth[1-3]. Thus, we consider its weak solution so that the problem (1.1) might have a global solution allowing discontinuities(e.g. shock wave etc.). Moreover, the entropy condition must be imposed in order to single out a physically relevant solution(also called the entropy solution)[7-9].

For the numerical approximation method of the equation (1.1), the numerical entropy condition(e.g. the proper cell entropy inequality) must be imposed on it in order that the numerical solution can converges to the entropy solution of the above problem. However, the entropy conditions seems difficulty to prove for high-order finite difference schemes[10-11].

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Here, we will study the entropy condition for the semidiscrete relaxing schemes for scalar conservation laws with general flux. The paper is organized as follows. In section 2, we simply recall the relaxing system with a stiff source term, constructed by Jin et al. to approximate the equation (1.1). In section 3, we establish the relation between the entropy pair for the relaxing system and the entropy pair for the system (1.1). In section 4, we discuss the entropy conditions for the semidiscrete first order upwind relaxing scheme and second order MUSCL-type relaxing scheme.

2. Preliminaries

In this section, we will review the relaxing system with a stiff source, constructed by Jin and Xin. to approximate the equation (1.1). In the following, we will only consider single scalar conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{2.1}$$

with initial data

$$u(0,x) = u_0(x). \tag{2.2}$$

As in [1], a linear system with a stiff source term (hereafter called the **relaxing system**) can be constructed as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + a\dfrac{\partial u}{\partial x} = -\dfrac{1}{\epsilon}(v - f(u)), \end{cases} \tag{2.3}$$

where the small positive parameter ϵ is the relaxation rate, and a is a positive constant satisfying

$$| f'(u) | \leq \sqrt{a}, \text{ for all } u \in R. \tag{2.4}$$

Remark. Here, we can consider the more general $a(x,t)$ instead of the above constant a. The results in this paper are not limited by the above constant a.

In the small relaxation limit $\epsilon \to 0^+$, the relaxing system(2.3) can be approximated to leading order by the following *relaxed* equations

$$v = f(u), \tag{2.5}$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{2.6}$$

The state satisfying (2.5) is called the **local equilibrium**. By the Chapman-Enskog expansion[12], we can derive the following first order approximation to (2.3)

$$v = f(u) - \epsilon\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}, \tag{2.7}$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon\frac{\partial}{\partial x}\Big(\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}\Big), \tag{2.8}$$

It is clear that the above second equation (2.8) is dissipative under condition (2.4) (which is referred to as the *subcharacteristic condition* by T.-P. Liu in [2]). Here, we can choose the special initial condition for the relaxing system (2.3) as follows:

$$\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x) \equiv f(u_0(x)). \end{aligned} \tag{2.9}$$

The aim is to avoid the initial layer introduced by the relaxing system (2.3). In doing so the state is already in equilibrium initially.

3. Entropy

In the following, we will always assume $(\eta,\ q)$ to represent the entropy pairs for models (2.3)[3, 4], where the convex function $\eta(u,v) \in C^2(R^2)$, then they must satisfy the consistency condition:

$$(\eta_u, \eta_v)\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} = (q_u, q_v) \tag{3.1}$$

where η_u represent the partial differentiation with respect to u. Furthermore, we have

$$\begin{aligned} \eta_{uu} - a\eta_{vv} &= 0, \\ q_{uu} - aq_{vv} &= 0 \end{aligned} \tag{3.2}$$

Thus the general representation of the entropy pairs for (2.3) is

$$\begin{aligned} \eta &= G(v+\sqrt{a}u) + H(v-\sqrt{a}u), \\ q &= \sqrt{a}(G(v+\sqrt{a}u) - H(v-\sqrt{a}u)) \end{aligned} \tag{3.3}$$

for any functions G and H in $C^2(R)$.

It is easy to verify that η is convex function if and only if $H''G'' \geq 0$. At the equilibrium state v=f(u), we have

$$\begin{aligned} \eta\mid_{v=f(u)} &= G(f(u)+\sqrt{a}u) + H(f(u)-\sqrt{a}u) \equiv \bar{\eta}(u), \\ q\mid_{v=f(u)} &= \sqrt{a}(G(f(u)+\sqrt{a}u) - H(f(u)+\sqrt{a}u)) \equiv \bar{q}(u) \end{aligned} \tag{3.4}$$

and expect to have

$$\eta_v \mid_{v=f(u)} = 0 \tag{3.5}$$

which means that η_v vanishes at the equilibrium state v=f(u).

Then the pair $(\bar{\eta}(u), \bar{q}(u))$ forms an entropy pair for scalar conservation laws (2.1), that is

$$\bar{\eta}'(u)f'(u) = \bar{q}'(u), \text{ if } H'' \geq 0 \text{ and } G'' \geq 0.$$

At the equilibrium state $v = f(u)$, we expect to have the following entropy condition for equation (2.1)

$$\frac{\partial\bar{\eta}}{\partial t} + \frac{\partial\bar{q}}{\partial x} \leq 0. \tag{3.6}$$

and the numerical entropy condition (4.15) for the numerical method for solving scalar conservation laws (2.1).

4. The Relaxing Schemes and the Entropy Condition

Now, we introduce the spatial grid points $x_j,\ j \in Z$ with the uniform mesh width $\Delta x = x_{j+1} - x_j$, i.e. Δx is a constant, and denote by $w_j(t)$ the approximate point value of w(x, t) at $x = x_j$. As in [1], the relaxing scheme is obtained by discreting the system (2.3), for which it is convenient to treat the spatial and time discretization separately (This is known as the method of line). In the following, we only consider the semidiscrete schemes.

A spatial discretization to (2.3) in conservation form can be written as

$$\begin{aligned} &\frac{\partial}{\partial t}u_j + \frac{1}{\Delta x}(v_{j+1/2} - v_{j-1/2}) = 0, \\ &\frac{\partial}{\partial t}v_j + \frac{a}{\Delta x}(u_{j+1/2} - u_{j-1/2}) = -\frac{1}{\epsilon}(v_j - f(u_j)) \end{aligned} \tag{4.1}$$

where the numerical flux $u_{j+1/2}$ and $v_{j+1/2}$ will be defined in two ways specified below.

For the sake of simplicity in the presentation, define $w^+ = v + \sqrt{a}u$ and $w^- = v - \sqrt{a}u$, which imply $v = \frac{1}{2}(w^+ + w^-)$ and $u = \frac{1}{2\sqrt{a}}(w^+ - w^-)$.

Algorithm I:(First order upwind scheme) *The numerical flux in (4.1) is defined as:*

$$w^+_{j+1/2} = w^+_j, \; w^-_{j+1/2} = w^-_{j+1} \tag{4.2}$$

Algorithm II:(Second order MUSCL scheme) *The numerical flux in (4.1) is defined as:*

$$\begin{aligned} w^+_{j+1/2} &= w^+_j + \frac{1}{2}\phi(r^+_j)(w^+_{j+1} - w^+_j) = w^+_j + \frac{1}{2}\phi(\frac{1}{r^+_j})(w^+_j - w^+_{j-1}), \\ w^-_{j+1/2} &= w^-_{j+1} - \frac{1}{2}\phi(r^-_{j+1})(w^-_{j+2} - w^-_{j+1}) = w^-_j - \frac{1}{2}\phi(\frac{1}{r^-_{j+1}})(w^-_{j+1} - w^-_j) \end{aligned} \tag{4.3}$$

where

$$r^\pm_j = \frac{w^\pm_j - w^\pm_{j-1}}{w^\pm_{j+1} - w^\pm_j}, \tag{4.4}$$

and $\phi(r)$ is limiters[6].

Remark. One simpler choice of limiters is the so-called minmod limiters

$$\phi(r) = max(0, min(1, r)). \tag{4.5}$$

Sharper limiters was introduced by van Leer[5, 6] as

$$\phi(r) = (\mid r \mid + r)/(1 + \mid r \mid), \tag{4.6}$$

and *Superbee* limiter

$$\phi(r) = max(0, min(1, 2r), min(2, r)). \tag{4.7}$$

For the relaxing schemes (4.1), we want to have the following numerical entropy inequality to guarantee convergence of numerical solution to the entropy solution.

$$\frac{\partial \eta}{\partial t} + \frac{1}{\Delta x}(q_{j+1/2} - q_{j-1/2}) + \frac{\eta_v}{\epsilon}(v_j - f(u_j)) \leq 0. \tag{4.8}$$

Multipying both side of equation (4.1) by $(\eta_u, \eta_v)_j$, we have

$$\frac{\partial \eta}{\partial t} + \frac{1}{\Delta x}\eta_u \mid_j (v_{j+1/2} - v_{j-1/2}) + \frac{a}{\Delta x}\eta_v \mid_j (u_{j+1/2} - u_{j-1/2}) = -\frac{\eta_v}{\epsilon}(v_j - f(u_j)). \tag{4.9}$$

Then (denote the left hand side of inequality (4.8) by *LHS*)

$$\begin{aligned} LHS = &\frac{1}{\Delta x}(q_{j+1/2} - q_{j-1/2}) - \frac{1}{\Delta x}\eta_u \mid_j (v_{j+1/2} \\ &- v_{j-1/2}) - \frac{a}{\Delta x}\eta_v \mid_j (u_{j+1/2} - u_{j-1/2}). \end{aligned} \tag{4.10}$$

Now, substitute (3.3) in (4.10), then

$$\begin{aligned} LHS = & \frac{\sqrt{a}}{\Delta x}\{G(w^+_{j+1/2}) - H(w^-_{j+1/2}) - G(w^+_{j-1/2}) + H(w^-_{j-1/2}) \\ & -G'(w^+_j)(w^+_{j+1/2} - w^+_{j-1/2}) + H'(w^-_j)(w^-_{j+1/2} - w^-_{j-1/2})\} \\ = & \frac{\sqrt{a}}{\Delta x}\{\int_{w^-_{j-1/2}}^{w^-_{j+1/2}}[H'(w^-_j) - H'(\xi)]d\xi + \int_{w^+_{j-1/2}}^{w^+_{j+1/2}}[G'(\eta) - G'(w^+_j)]d\eta\}. \end{aligned} \tag{4.11}$$

Therefore, to guarantee the above numerical entropy inequality to be satisfied, we have

Theorem 4.1. *A sufficient condition for the inequality (4.8) to be satisfied is , for all $j \in Z$,*

$$\begin{aligned} & sign(w^+_{j+1/2} - w^+_{j-1/2})[G'(\eta) - G'(w^+_j)] \le 0, \text{for every } \eta \text{ between } w^+_{j+1/2} \text{ and } w^+_{j-1/2}, \\ & sign(w^-_{j+1/2} - w^-_{j-1/2})[H'(w^-_j) - H'(\xi)] \le 0, \text{for every } \xi \text{ between } w^-_{j+1/2} \text{ and } w^-_{j-1/2}, \end{aligned} \tag{4.12}$$

Theorem 4.2. *For the first order upwind scheme (4.1), (4.2), the entropy inequality (4.8) is valid, if H and G are all convex function.*

The above two conclusions are obvious. Their proofs are omitted.

Theorem 4.3. *For the second order MUSCL-type scheme (4.1), (4.3), the entropy inequality (4.8) is valid, if H and G are any symmetric convex function, and limiters ϕ satifies the following condition*

$$0 \ \le \ \phi(r), \ \phi(\frac{1}{r}) \ \le 2 \tag{4.13}$$

Proof. By (4.3), we have

$$\begin{aligned} w^+_{j+1/2} - w^+_{j-1/2} &= (1 + \frac{1}{2}\phi(\frac{1}{r^+_j}) - \frac{1}{2}\phi(r^+_{j-1}))(w^+_j - w^+_{j-1}), \\ w^-_{j+1/2} - w^-_{j-1/2} &= (1 - \frac{1}{2}\phi(\frac{1}{r^-_{j+1}}) + \frac{1}{2}\phi(r^-_j))(w^-_{j+1} - w^-_j). \end{aligned} \tag{4.14}$$

Under the condition (4.13), we have

$$\begin{aligned} sign(w^+_{j+1/2} - w^+_{j-1/2}) &= sign(w^+_j - w^+_{j-1}), \\ sign(w^-_{j+1/2} - w^-_{j-1/2}) &= sign(w^-_{j+1} - w^-_j) \end{aligned}$$

On the other hand, defining

$$\begin{aligned} A &= w^-_{j+1/2} - w^-_j \ = [1 - \frac{1}{2}\phi(\frac{1}{r^-_{j+1}})](w^-_{j+1} - w^-_j), \\ B &= w^-_j \ - w^-_{j-1/2} = \frac{1}{2}\phi(r^-_j)(w^-_{j+1} - w^-_j), \\ C &= w^+_{j+1/2} - w^+_j \ = \frac{1}{2}\phi(\frac{1}{r^+_j})(w^+_j - w^+_{j-1}), \\ D &= w^+_j - w^+_{j-1/2} = [1 - \frac{1}{2}\phi(r^+_j)](w^+_j - w^+_{j-1}), \end{aligned}$$

we have $\mid A \mid \ge \mid B \mid$ and $\mid D \mid \ge \mid C \mid$ under the condition (4.10).

Now, if define two new functions as follows:

$$Y_1(x) = \int_{w^+_j}^{x}[G'(\eta) - G'(w^+_j)]d\eta, \ Y_2(x) = -\int_{w^-_j}^{x}[H'(w^-_j) - H'(\xi)]d\xi,$$

then it is easy to verify the fact that function $Y_1(x)$ and $Y_2(x)$ are all convex function, and w_j^+ and w_j^- are their minium value point if H and G are two convex function.

Thus combining the previous results and the property of the symmetric convex function with (4.9), we can complete the proof of this Theorem.

Remark. 1. Condition (4.13) can guarantee scheme to be of second order accuracy and very high resolution, because it includes the critical point $\phi(1) = 1$[5, 6]. On the other hand, condition (4.13) is valid for a few TVD-type limiter, e.g. van Leer and *Superbee* limiters etc.[6], so that it can guarantee scheme (4.1)-(4.3) to be of very high resolution. 2. At the equilibrium state $v = f(u)$, the entropy inequlity (4.8) reduces to

$$\frac{\partial \bar{\eta}}{\partial t} + \frac{1}{\Delta x}(\bar{q}_{j+1/2} - \bar{q}_{j-1/2}) \leq 0. \tag{4.15}$$

where $\bar{\eta}$ and $\bar{q}$ defind in (3.4), and H and G are all convex functions. (4.15) forms a cell entropy inequality for the relaxed schemes[1], which is as follows:

$$v_j = f(u_j),$$

$$\frac{\partial}{\partial t} u_j + \frac{1}{\Delta x}(v_{j+1/2} - v_{j-1/2}) = 0.$$

Further theoretical studies, such as the entropy condition for the fully discrete relaxing schemes and the convergence of the relaxing schemes, need to be considered.

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