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WAVELET METHOD FOR BOUNDARY INTEGRAL EQUATIONS*

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Abstract

In this paper, we show how to use wavelet to discretize the boundary integral equations which are both singular and ill-conditioned. By using an explicit diagonal preconditioning, the condition number of the corresponding matrix is bounded by a constant, while the sparse structure speed up the iterative solving process. Using an iterative method, one thus obtains a fast numerical algorithm to solve the boundary integral equations.

Key words: Wavelet bases, Boundary integral equation, Preconditioning.

1. Introduction

The application of wavelets to signal and image processing has been successful. But there are few results about numerical solution for the partial differential equations. We think that it is enough only to use the coefficients $\{h_n\}$ (as a filter) in signal and image processing, but it is not sufficient for numerical computation[7][8][10], where we consider wavelet as a special function series instead of a filter. Then it brings some problems such as the complication of function value computation and the difficulty to handle the boundary conditions[8][13], etc. For the first problem, we have solved in [16]. For the second problem, some researchers consider the periodic problems to escape the complicated boundary [3], others consider the boundary integral problems. In [3], the author discussed the potential integral equation of the 2D Laplace operator, as we know, the kernel is no-singular and well-conditioned.

In this paper we examine the feasibility of applying wavelet based numerical methods to solve elliptic equations. We use compactly supported wavelets and develop a wavelet boundary element method to handle the boundary conditions. The boundary element method has been firmly established as an important alternative technique to the prevailing numerical methods of analysis in continuum mechanics[14][17], and many others which can be written as a function of a potential and whose governing equation is the classical Laplace or Poisson equation. It reduces the problem's dimension and can be used when the domain is infinite. In contrast to the finite difference methods

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and finite element methods, the classical discretization of the boundary integral usually has slow decay away from the diagonal and leads to a large non-sparse linear system. As we know, directly applying a dense matrix to a vector requires roughly N^2 operations. Therefore, the efficiency induced by lowering the dimensions is distroyed by the cost brought by the dense matrices. This problem can be solved by the wavelet method proposed by Beylkin, Coifman and Rokhlin in [1]. According to the framework of Zygmund-Calderon operators studied extensively in [1], if we project the integral operator in a wavelet basis, since the basis function satisfies the vanishing moment condition, the coefficients away from the diagonal will be small. Neglecting these small entries yields a finger-like sparse linear system.

In this paper we mainly discuss two kinds of integral induction methods: single-layer induction and natural boundary induction. Both of them are ill-conditioned and have singularity in their kernels. In order to avoid these problems, we construct numerical algorithm to avoid the singularity, and find an easily inheritable matrix D such that $D^{-1}MD^{-1}$ will have a better condition number κ. We prove that using the wavelet basis, a diagonal matrix D yields $\kappa = O(1)$.

2. Boundary Integral Equations

Let Ω be a domain in the plane $\mathbf{R}^2$ and Γ its boundary, we discuss the Laplace equation:

$$\triangle u = 0, \qquad x \in \Omega, \tag{2.1}$$

with boundary conditions of Dirichlet type

$$u(x) = f(x) \qquad \text{on } \Gamma, \tag{2.2}$$

or the Neumann type

$$\frac{\partial u(x)}{\partial n} = g(x) \qquad \text{on } \Gamma, \tag{2.3}$$

where n is the unit outward normal to surface Γ.

The Neumann problem (2.1) and (2.3) has a solution only if the consistency condition

$$\int_\Gamma g(x) dS_x = 0 \tag{2.4}$$

holds and this solution is unique only to within an arbitrary additive constant.

It is well known that the above elliptic boundary-value problem can be reduced into several kinds of integral equations. To obtain an integral equation for the solution of the Dirichlet problem, the classical approach is to assume that the unknown function u may be expressed solely as a single-layer potential with unknown density σ,

$$u(x) = \int_\Gamma \sigma(y) \ln \frac{1}{|x-y|} dS_y + C, \qquad x \in \Omega, \tag{2.5}$$

where C is a constant which will be determined later. Since the kernel in this equation is continuous as x passes through the surface, the limit of equation (2.5) as x is taken

to Γ gives

$$f(x) = \int_\Gamma \sigma(y) \ln \frac{1}{|x-y|} dS_y + C, \qquad \text{on } \Gamma. \tag{2.6}$$

and the density σ satisfy the consistency condition

$$\int_\Gamma \sigma(x) dS_x = 0. \tag{2.7}$$

Equation (2.6) is a Fredholm equation of the first kind, as the unknown appears only inside the integral. For many Dirichlet problems, formulations using such equations have been proven to be more illuminating physically and more convenient mathematically than using equations of the second kind.

Taking the derivative of equation (2.5) in the direction of the outward normal to Γ, as x is taken to Γ, we get the boundary relation for the Neumann problem

$$g(x) = -\pi\sigma(x) + \int_\Gamma \sigma(y) \frac{\partial}{\partial n_x} \ln \frac{1}{|x-y|} dS_y. \tag{2.8}$$

Equation (2.8) is a Fredholm equation of the second kind, as the unknown appears both inside and outside the integral. After solving the system of corresponding algebraic equations, values of u at any interior or boundary points can be calculated by using equation (2.5).

The following Theorem and its proof can be found in Zhu's book [17].

Theorem 2.1. *$b(\sigma, \mu)$ is an elliptic symmetric bilinear form on the quotient space $\mathcal{V}(\Gamma) = H_*^{-\frac{1}{2}}(\Gamma)$, there are constants $C_2 \geq C_1 > 0$, such that*

$$b(\sigma, \sigma) \geq C_1 \|\sigma\|^2_{\mathcal{V}(\Gamma)}, \qquad \forall \sigma \in \mathcal{V}(\Gamma). \tag{2.9}$$

$$b(\sigma, \mu) \leq C_2 \|\sigma\|_{\mathcal{V}(\Gamma)} \|\mu\|_{\mathcal{V}(\Gamma)}, \qquad \forall \sigma, \mu \in \mathcal{V}(\Gamma). \tag{2.10}$$

where

$$b(\sigma, \mu) = -\frac{1}{2\pi} \int_\Gamma \int_\Gamma \sigma(x)\mu(y) \ln|x-y| dS_x dS_y$$

$$H_*^{-\frac{1}{2}}(\Gamma) = \left\{ \sigma \in H^{-\frac{1}{2}}(\Gamma), \int_\Gamma \sigma dS_x = 0 \right\}.$$

A conceptual disadvantage of single-layer or double-layer potentials is the introduction of formal source densities which usually bear no physical relation to the problem. This can be overcome by using the direct reduction of the boundary element method, where values of the function and its normal derivative over Γ play the role of the source densities in generating u through Ω. Among all the methods of direct reduction, the natural boundary reduction introduced by Feng[5] seems a good one for it faithfully preserves the essential characteristics of the original problem, such as the self-adjointness, the coerciveness and the variational principle, and can be fully compatible with the Finite Element Method.

Applying the natural boundary reduction suggested by Feng and Yu [6], Neumann problem (2.1) and (2.3) can be converted to the following natural integral equation on the boundary Γ:

$$\mathcal{K}\sigma = g, \qquad \text{on } \Gamma \tag{2.11}$$

and the equivalent variational formulation is: find $\sigma \in H^{\frac{1}{2}}(\Gamma)$, such that

$$D(\sigma, \mu) = F(\mu), \qquad \forall \mu \in H^{\frac{1}{2}}(\Gamma), \tag{2.12}$$

where

$$D(\sigma, \mu) = \int_\Gamma (\mathcal{K}\sigma)(x) \cdot \mu(x) dx,$$

$$F(\mu) = \int_\Gamma g(x)\mu(x) dx,$$

then the solution of the Neumann problem can be given by the Poisson integral formula

$$u(x) = -\int_\Gamma \frac{\partial}{\partial n_y} G(x, y)\sigma(y) dy, \tag{2.13}$$

where $G(x, y)$ is the Green function.

The following Theorem and its proof can be found in Yu's book [14].

Theorem 2.2. *$D(\sigma, \mu)$ is an elliptic symmetric bilinear form on the quotient space $\mathcal{U}(\Gamma) = H_*^{\frac{1}{2}}(\Gamma)$, there are constants $C_2 \geq C_1 > 0$, such that*

$$D(\sigma, \sigma) \geq C_1 \|\sigma\|^2_{\mathcal{U}(\Gamma)}, \qquad \forall \sigma \in \mathcal{U}(\Gamma), \tag{2.14}$$

$$D(\sigma, \mu) \leq C_2 \|\sigma\|_{\mathcal{U}(\Gamma)} \|\mu\|_{\mathcal{U}(\Gamma)}, \qquad \forall \sigma, \mu \in \mathcal{U}(\Gamma), \tag{2.15}$$

where

$$H_*^{\frac{1}{2}}(\Gamma) = \left\{ \sigma \in H^{\frac{1}{2}}(\Gamma), \int_\Gamma \sigma dS_x = 0 \right\}.$$

For the natural boundary method, we will mainly consider two cases:

1. When Ω is the half plane, the natural integral operator is

$$\mathcal{K}\sigma = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma(x')}{(x - x')^2} dx',$$

and the Poission integral formula is

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - x')^2 + y^2} \sigma(x') dx', \qquad y > 0. \tag{2.16}$$

2. When Ω is the domain outside a unit circle, the natural integral operator is

$$\mathcal{K}\sigma = -\frac{1}{4\pi} \int_0^{2\pi} \frac{\sigma(\theta')}{\sin^2\left(\dfrac{\theta - \theta'}{2}\right)} d\theta',$$

and the Poission integral formula is

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2-1)}{1+r^2-2r\cos(\theta-\theta')}\sigma(\theta')d\theta', \qquad r>1. \tag{2.17}$$

3. Compactly Supported Wavelets

We begin with a function $\varphi(x)$ belonging to $L^2(R)$[2][11] such that

$$\varphi(x-k), \qquad k \in Z$$

is an orthonormal sequence in $L^2(R)$. Let V_0 denote the closed linear subspace of $L^2(R)$ generated by this sequence. More generally, define V_j in terms of V_0 by simply changing scales, that is to say,

$$f(x) \in V_0 \Longleftrightarrow f(2^j x) \in V_j,$$

for function $f \in L^2(R)$.

Other hypotheses on V_j are as follows: 1. the V_j, $j \in Z$, form a nested sequence; 2. their intersection $\bigcap_{-\infty}^{\infty} V_j$ reduces to $\{0\}$; and 3. the union $\bigcup_{-\infty}^{\infty} V_j$ is dense in $L^2(R)$.

To analyze further details, let's denote the orthogonal complement of V_j in V_{j+1} with W_j, i.e. $V_{j+1} = V_j \oplus W_j$. Then there exists at least one function ψ belonging to W_0 such that $\psi(x-k)$, $k \in Z$, is an orthonormal basis of W_0. The function $\psi(x)$ is called the mother wavelet. Denote

$$\begin{aligned} \varphi_{j,k}(x) &:= 2^{j/2}\varphi(2^j x - k), \qquad j,k \in Z, \\ \psi_{j,k}(x) &:= 2^{j/2}\psi(2^j x - k), \qquad j,k \in Z, \end{aligned}$$

and let h_n, g_n be the wavelet filters, i.e.

$$\begin{aligned} \varphi_{j-1,k}(x) &= \sum_{k\in Z} h_k \varphi_{j,k}(x) \\ \psi_{j-1,k}(x) &= \sum_{k\in Z} g_k \varphi_{j,k}(x). \end{aligned} \tag{3.1}$$

In the space spanned by wavelets, we have two orthogonal basis, namely $\mathcal{B}$ and $\mathcal{B}'$, where

$$\begin{aligned} \mathcal{B}' &= (\varphi_{J,k})_k, \\ \mathcal{B} &= \bigcup_{j<J} (\psi_{j,k})_k. \end{aligned}$$

3.1 A periodic wavelet

To apply wavelet theory on an integral equation along a closed boundary, we should construct an orthonormal basis on the closed curve $\Gamma \subset R^2$. The following "brutal periodization" of an orthonormal wavelet was introduced in Dorobantu's paper [4].

Assume the supports of φ and ψ are no longer than the length of Γ, otherwise we substitute them with a scaled version $2^{j_0/2}\varphi(2^{j_0}x)$ and $2^{j_0/2}\psi(2^{j_0}x)$. For convenience, let's consider the interval $I_0 = [0, 2\pi]$. We only retain all the basis functions who are fully supported in I_0. If the support of a basis function, say $\psi_{j,k}$, is out of I_0 at the left end, the missing part is matched by $\psi_{j,k+2^j}$ at the right end. Add the two functions and restrict the sum to I_0. It is easy to see that the new function is also orthonormal on all the other basis functions and satisfies the two-scale relation (3.1) which is the key to the fast wavelet transformation (FWT). Therefore, we get periodized versions of $\varphi_{j,k}$ and $\psi_{j,k}$. Let's denote them with $\widetilde{\varphi}_{j,k}$ and $\widetilde{\psi}_{j,k}$.

Theorem 3.3. *$\widetilde{\varphi}_{j,k}$ and $\widetilde{\psi}_{j,k}$ defined above form an orthonormal basis in $L^2(I_0)$. For each $j \geq 0$ they generate a sequence of subspaces $\widetilde{V}_j$ and $\widetilde{W}_j$ such that*

$$\widetilde{V}_{j+1} = \widetilde{V}_j + \widetilde{W}_j, \qquad \widetilde{V}_j \perp \widetilde{W}_j, \qquad \overline{\bigcup_{-\infty}^{\infty} \widetilde{V}_j} = L^2(I_0).$$

Let f be the periodic extension to the whole real axis of $\widetilde{f} \in L^2(I_0)$, then

$$\begin{aligned} \langle f, \varphi_{j,k} \rangle &= \langle \widetilde{f}, \widetilde{\varphi}_{j,k} \rangle \\ \langle f, \psi_{j,k} \rangle &= \langle \widetilde{f}, \widetilde{\psi}_{j,k} \rangle. \end{aligned} \tag{3.2}$$

The proof can be found in [4].

(3.2) shows that the wavelet decomposition of $\tilde{f} \in L^2(I_0)$ can be regarded as the standard decomposition of f by those wavelets whose supports intersect I_0. According to it, we can also define the number of vanishing moments of $\widetilde{\psi}_{j,k}$, which is actually equal to that of $\psi_{j,k}$. From now on we drop the notation "$\sim$" and directly call the periodized basis a wavelet basis.

3.2 Characterization of Sobolev spaces

The wavelet $\psi_{j,k}$ constitute an unconditional basis for $H^s(R), s \geq 0$, i.e. there exists a characterization for function $f \in H^s(R)$ using only the absolute values of the wavelet coefficients of f. In other words, given f, we can decide whether $f \in H^s(R)$ by looking only at the $|\langle f, \psi_{j,k} \rangle|$. The characterization of Sobolev spaces by means of wavelet coefficient is

$$f \in H^s(R) \Longleftrightarrow \sum_{j,k} |\langle f, \psi_{j,k} \rangle|^2 (1 + 4^{js}) < +\infty, \qquad s \geq 0 \tag{3.3}$$

for $s < r$, where r is the vanishing moment of the wavelet. The proof of (3.3) can be found in Meyer's book [12].

We have a same result for the Sobolev space $H^s(I_0)$ with the periodic wavelet instead of the orthonormal wavelet basis.

Theorem 3.4. *The characterization of Sobolev spaces $H^s(I_0)$ by means of wavelet coefficient is*

$$\widetilde{f} \in H^s(I_0) \Longleftrightarrow \sum_{j,k} |\langle \widetilde{f}, \widetilde{\psi}_{j,k} \rangle|^2 (1 + 4^{js}) < +\infty, \quad s \geq 0 \tag{3.4}$$

for $s < r$.

Proof. Let J be the set of sum. We define

$$J_1 = \{\{j,k\}; \quad supp\psi_{jk} \subset I_0\},$$

$$J_2 = \{\{j,k\}; \quad 0 \in supp\psi_{jk}\}$$

and

$$J_3 = \{\{j, k+2^j\}; \quad 0 \in supp\psi_{jk}\},$$

where $I_0 = [0,l]$, we have $J = J_1 \bigcup J_2$. Then

$$\begin{aligned} & \sum_{J \bigcup J_s} |\langle \bar{f}, \psi_{j,k}\rangle|^2 (1+4^{js}) \\ = & \sum_{J \bigcup J_s} |\langle \widetilde{f}, \widetilde{\psi}_{j,k}\rangle|^2 (1+4^{js}) \\ \leq & \ 2\sum_{j} |\langle \widetilde{f}, \widetilde{\psi}_{j,k}\rangle|^2 (1+4^{js}) \\ < & \ +\infty \end{aligned}$$

it is easy to know

$$\bar{f} = \widetilde{f}, \qquad \text{on} \quad \mathrm{I}_0,$$

by (3.3), we have $\bar{f} \in H^s(R)$, and so $\widetilde{f} \in H^s(I_0)$.

On the other hand, let f be the periodic extension to the whole real axis of $\widetilde{f}$, and $\phi \in C_0^\infty(R)$ such that

$$\phi(x) = 1, \qquad -l \leq x \leq 2l,$$

then $\phi f \in H^s(R)$. Using (3.3), we have

$$\begin{aligned} & \sum_{J} |\langle \widetilde{f}, \widetilde{\psi}_{j,k}\rangle|^2 (1+4^{js}) \\ \leq & \sum_{j,k} |\langle \phi f, \psi_{j,k}\rangle|^2 (1+4^{js}) \\ < & \ \infty. \end{aligned}$$

This completes the proof.

4. Sample in the Wavelet Basis

In the following four sections, we show how to use wavelet to discretize boundary integral equations for the Laplace equation. We only consider two kinds of boundary reduction: single-layer and natural reduction which are discussed in section 2.

4.3 Sample for single-layer boundary integral

If we suppose that Γ is smooth, the kernel of Fredholm equation of second kind (2.8)

$$\mathcal{S}(x,y) = \frac{\partial}{\partial n_x} \ln \frac{1}{|x-y|}$$

will lose singularity. The sample in wavelet basis $\mathcal{B}'$ is similar to the Nyström's method[3].

Here, we mainly study the kernel of Fredholm equation of first kind (2.6). Suppose Γ can be expressed as $x(\alpha), \alpha \in [0, 2\pi]$, rewrite (2.6) as

$$f(x(\alpha)) = \int_0^{2\pi} \mathcal{S}(\alpha, \alpha')\sigma(x(\alpha'))d\alpha' + C \tag{4.1}$$

where

$$\mathcal{S}(\alpha, \alpha') = \ln \frac{1}{|x(\alpha) - x(\alpha')|}. \tag{4.2}$$

Project equation (4.1) onto V_J

$$\mathcal{S}_J P_J \sigma = P_J \mathcal{S} P_J P_J \sigma = P_J f,$$

or

$$\sum_q s_{p,q}\sigma_q = v_p, \tag{4.3}$$

where

$$\begin{aligned} &\sigma_p = \int \sigma(x(\alpha))\varphi_{J,p}(\alpha)d\alpha, \qquad v_p = \int f(x(\alpha))\varphi_{J,p}(\alpha)d\alpha, \\ &s_{p,q} = \int\int \ln \frac{1}{|x(\alpha) - x(\alpha')|}\varphi_{J,p}(\alpha)\varphi_{J,q}(\alpha')d\alpha d\alpha'. \end{aligned} \tag{4.4}$$

At first, let's consider the simple case when Γ is an ellipse,

$$\begin{cases} x_1(\alpha) = a\cos\alpha, \\ x_2(\alpha) = b\sin\alpha. \end{cases}$$

We have

$$\begin{aligned} \mathcal{S}(\alpha, \alpha') &= \ln \frac{1}{|x(\alpha) - x(\alpha')|} \\ &= -\ln |(a\cos\alpha - a\cos\alpha')^2 + (b\sin\alpha - b\sin\alpha')^2|^{\frac{1}{2}} \\ &= \mathcal{S}_1(\alpha, \alpha') + \mathcal{S}_2(\alpha, \alpha'), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \mathcal{S}_1(\alpha, \alpha') &= -\ln \left| 2\sin\frac{\alpha - \alpha'}{2} \right| \\ \mathcal{S}_2(\alpha, \alpha') &= -\ln \left(a^2 \sin^2 \frac{\alpha + \alpha'}{2} + b^2 \cos^2 \frac{\alpha + \alpha'}{2} \right)^{\frac{1}{2}}. \end{aligned}$$

Notice that $\mathcal{S}_1(\alpha,\alpha')$ is sigular at $\alpha=\alpha'$, whereas $\mathcal{S}_2(\alpha,\alpha')$ is smooth. Such decomposition can be extended to general cases.

For a smooth boundary Γ, let

$$\mathcal{S}(\alpha,\alpha')=\mathcal{S}_1(\alpha,\alpha')+\mathcal{S}_2(\alpha,\alpha'), \tag{4.6}$$

where

$$\mathcal{S}_1(\alpha,\alpha')=-\ln\left|2\sin\frac{\alpha-\alpha'}{2}\right|$$
$$\mathcal{S}_2(\alpha,\alpha')=\mathcal{S}(\alpha,\alpha')-\mathcal{S}_1(\alpha,\alpha').$$

As we know, $\mathcal{S}(\alpha,\alpha')$ and $\mathcal{S}_1(\alpha,\alpha')$ are sigular only when $\alpha=\alpha'$, so $\mathcal{S}_2(\alpha,\alpha')$ is smooth except $\alpha=\alpha'$, when $\alpha'\to\alpha$, we have

$$\begin{aligned}\lim_{\alpha'\to\alpha}\mathcal{S}_2(\alpha,\alpha') &= -\lim_{\alpha'\to\alpha}\ln\frac{|x(\alpha)-x(\alpha')|}{|2\sin\dfrac{\alpha-\alpha'}{2}|}\\ &= -\ln\left(\left(\frac{dx_1(\alpha)}{d\alpha}\right)^2+\left(\frac{dx_2(\alpha)}{d\alpha}\right)^2\right)^{\frac{1}{2}},\end{aligned}$$

thus we have proved that $\mathcal{S}_2(\alpha,\alpha')$ is smooth.

Therefore, we have

$$\begin{aligned}s_{p,q} &= \int\int\mathcal{S}_1(\alpha,\alpha')\varphi_{J,p}(\alpha)\varphi_{J,q}(\alpha')d\alpha d\alpha'+\int\int\mathcal{S}_2(\alpha,\alpha')\varphi_{J,p}(\alpha)\varphi_{J,q}(\alpha')d\alpha d\alpha'\\ &= s_{p,q}^{(1)}+s_{p,q}^{(2)}\end{aligned}$$

From Yu's book [14], the kernel $\mathcal{S}_1(\alpha,\alpha')$ can be expressed as

$$-\ln|2\sin\frac{\alpha}{2}|=\frac{1}{2}\sum_{-\infty,n\neq0}^{\infty}\frac{1}{|n|}e^{in\alpha}.$$

If we suppose that the wavelet basis $\varphi_{J,k}(\alpha)$ can also be expanded in Fourier series:

$$\varphi_{J,k}(\alpha)=\sum_{-\infty,n\neq0}^{\infty}b_n^{(k)}e^{in\alpha},$$

then we have

$$s_{p,q}^{(1)}=2\pi^2\sum_{-\infty,n\neq0}^{\infty}\frac{1}{|n|}b_n^{(p)}b_{-n}^{(q)}.$$

By the smoothness of the kernel $\mathcal{S}_2(\alpha,\alpha')$

$$\begin{aligned}s_{p,q}^{(2)} &= \int\int\mathcal{S}_2(\alpha,\alpha')\varphi_{J,p}(\alpha)\varphi_{J,q}(\alpha')d\alpha d\alpha'\\ &\approx 2^{-J}\mathcal{S}_2(C_p,C_q)\approx\frac{1}{N}\mathcal{S}_2(x_p,x_q)\end{aligned}$$

when $N=2^J$ is sufficiently large. It is not surprised since the basis $\mathcal{B}'$ is very "near" the uniform grid basis on fine enough scales.

4.4 Sample for natural boundary integral

Wu studied the sample in the wavelet basis $\mathcal{B}'$ on the half plane in [15]. Project equation (2.11) onto V_J

$$\mathcal{K}_J P_J \sigma = P_J \mathcal{K} P_J P_J \sigma = P_J g$$

or

$$\sum_q a_{p,q}\sigma_q = g_p, \tag{4.7}$$

where

$$\begin{aligned} &\sigma_p = \int \sigma(x)\varphi_{J,p}(x)dx, \qquad g_p = \int g(x)\varphi_{J,p}(x)dx, \\ &a_{p,q} = \int\int -\frac{1}{\pi(x-y)^2}\varphi_{J,p}(x)\varphi_{J,q}(y)dxdy. \end{aligned} \tag{4.8}$$

(4.8) is a Hadamard integral, we will get the matrix by Fourier method.

For the half plane case

$$\begin{aligned} a_{p,q} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{1}{\pi(x-y)^2}\varphi_{J,p}(x)\varphi_{J,q}(y)dxdy \\ &= 2^J \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{1}{\pi(x-y)^2}\varphi(2^J x-p)\varphi(2^J y-q)dxdy \\ &= 2^J \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{1}{\pi(x-y+p-q)^2}\varphi(x)\varphi(y)dxdy \\ &= 2^J \int_{-\infty}^{\infty} \varphi(x)(-\frac{1}{\pi[x+(p-q)]^2} * \varphi(x))dx \\ &= \frac{2^J}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\zeta)\overline{|\zeta|\hat{\varphi}(\zeta)e^{-i(p-q)\zeta}}d\zeta \\ &= \frac{2^J}{2\pi}\int_{-\infty}^{\infty} |\zeta||\hat{\varphi}(\zeta)|^2 e^{i(p-q)\zeta}d\zeta, \end{aligned}$$

where we have used the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-i\xi x}dx.$$

Since $|\hat{\varphi}(\zeta)|$ is an even function, let

$$\mathcal{R}(j) = \frac{1}{2\pi}\int_0^{\infty} |\zeta||\hat{\varphi}(\zeta)|^2\cos(j\zeta)d\zeta$$

then

$$a_{p,q} = 2^{J+1}\mathcal{R}(p-q).$$

For the domain outside the unit circle, the natural integral operator is

$$\mathcal{K}\sigma = -\frac{1}{4\pi}\int_0^{2\pi} \frac{\sigma(\theta')}{\sin^2\dfrac{\theta-\theta'}{2}}d\theta',$$

we will get the matrix through Fourier series

$$a_{p,q} = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{\sin^2\dfrac{\theta-\theta'}{2}}\varphi_{J,p}(\theta)\varphi_{J,q}(\theta')d\theta d\theta'$$

Since the natural integral kernel can be expressed as [14]

$$-\frac{1}{4\pi\sin^2\dfrac{\theta}{2}} = \sum_{-\infty}^{\infty}\frac{1}{2\pi}|n|e^{in\theta}.$$

so we have

$$a_{p,q} = 2\pi\sum_{-\infty}^{\infty}|n|b_n^{(p)}b_{-n}^{(q)}.$$

5. Condition Numbers of the Matrices

Let's first consider the single-layer integral equation (2.6). Denote by $\mathcal{W}$ the discrete wavelet transform from the space $\mathcal{B}'$ to $\mathcal{B}$. Let $\widetilde{\mathcal{S}}_J$ be the matrix of S_J in the basis $\mathcal{B}$. Then the wavelet discretization of the equation (2.6) is

$$\widetilde{S}_J w = \tilde{f}, \qquad \widetilde{S}_J = \mathcal{W}\mathcal{S}_J\mathcal{W}^{-1}, \tag{5.1}$$

where $\tilde{f}$ is the wavelet transform of $P_J f$,

$$w = (w_{j,k})^T.$$

As an orthonormal transformation, $\mathcal{W}$ doesn't change the condition number, so

$$\kappa(\widetilde{\mathcal{S}}_J) = \kappa(\mathcal{S}_J) = O(N).$$

We are going to show that the simplest conceivable preconditioning, namely a diagonal matrix $D_{\mathcal{S}}$ yields

$$\kappa(D_{\mathcal{S}}^{-1}\widetilde{S}_J D_{\mathcal{S}}^{-1}) = O(1).$$

Let $\phi = \sum w_{j,k}\psi_{jk}$, by (5.1)

$$w^T\widetilde{\mathcal{S}}_J w = \langle\phi, \mathcal{S}\phi\rangle = b(\sigma,\sigma).$$

By theorem 2.1, we have

$$C_1\|\phi\|^2_{\mathcal{V}(\Gamma)} \le w^T\widetilde{\mathcal{S}}_J w \le C_2\|\phi\|^2_{\mathcal{V}(\Gamma)}.$$

Using (3.3), we get

$$C_1\sum_{j,k}|2^{-j/2}w_{j,k}|^2 \le w^T\widetilde{\mathcal{S}}_J w \le C_2\sum_{j,k}|2^{-j/2}w_{j,k}|^2.$$

Hence, if $D_{\mathcal{S}}$ is the diagonal matrix defined by

$$(D_{\mathcal{S}})_{(j,k),(j',k')} = 2^{-j/2}\delta_{(j,j')}\delta_{(k,k')}, \tag{5.2}$$

for any vector w,

$$C_1\|w\|^2 \le w^T D_S^{-1} S_J D_S^{-1} w \le C_2\|w\|^2.$$

Therefore, we have proved the following Theorem.

Theorem 5.5. *Using standard wavelet basis to approximate the solution of integral equation (2.6), the condition number of $D_{\mathcal{S}}^{-1}\widetilde{\mathcal{S}}_J D_{\mathcal{S}}^{-1}$ is bounded, independent of the step size.*

For the natural boundary equation (2.11), Yu [14] has proved its condition number is $O(N)$. Wu [15] proved the same result for the sample matrix with the wavelet basis $\mathcal{B}'$. Let $\widetilde{\mathcal{K}}_J$ be the matrix of $\mathcal{K}_J$ in the basis $\mathcal{B}$. Using theorem 2.2 and noticing that σ in (2.11) belongs to the space $H^{\frac{1}{2}}(\Gamma)$, with some obvious modification to the proof of theorem 5.1, we can find a diagonal matrix $D_{\mathcal{K}}$ with the elements

$$(D_{\mathcal{K}})_{(j,k),(j',k')} = 2^{-j/2}\delta_{(j,j')}\delta_{(k,k')}, \tag{5.3}$$

such that

$$\kappa(D_{\mathcal{K}}^{-1}\widetilde{\mathcal{K}}_J D_{\mathcal{K}}^{-1}) = O(1),$$

where

$$\widetilde{\mathcal{K}}_J = \mathcal{W}\mathcal{K}_J\mathcal{W}^{-1}.$$

So, we have the following theorem:

Theorem 5.6. *Using standard wavelet basis to approximate the solution of natural boundary integral equation (2.11), there is a diagonal matrix $D_{\mathcal{K}}$ defined by (5.3), such that the condition number of $D_{\mathcal{K}}^{-1}\widetilde{\mathcal{K}}_J D_{\mathcal{K}}^{-1}$ is bounded, independent of the step size.*

Remark. For the finite element method using wavelet basis to solve elliptic equations, Jaffard proved the similar results, see [8].

6. Sparsity of the Matrices in the the Standard Wavelet Basis

In this section, we show that the integral kernels under wavelet transform are sparse. The result is by no means new, it is known for Calderon-Zygmund kernels.

For a Calderon-Zygmund kernel $\mathcal{T}$, the elements of the matrix representing $\mathcal{T}$ in wavelet basis $\mathcal{B}$ are of the form

$$\widetilde{\mathcal{T}}_{I,I'} = \langle \mathcal{T}\psi_{j,k}, \psi_{j',k'}\rangle, \tag{6.1}$$

where I and I' denote the supports of the wavelet function $\psi_{j,k}$ and $\psi_{j',k'}$ respectively. In [1], Beylkin etc. proved the following estimate for $\widetilde{\mathcal{T}}$, the standard form of $\mathcal{T}$ in wavelet transform

$$|\widetilde{\mathcal{T}}_{I,I'}| \le C_M \left(\frac{|I|}{|I'|}\right)^{\frac{1}{2}} \left(\frac{|I|}{d(I,I')}\right)^{M+1}, \tag{6.2}$$

where C_M is a constant depending on the property of $\mathcal{T}$ and the choice of the wavelets, M is the vanish moments of the wavelets, $d(I,J)$ denotes the distance between I and I', and it is assumed that $|I| \leq |I'|$ and $d(I,I') \neq 0$.

A similar estimate also holds for the periodic wavelets, but $d(I,I')$ must be considered as the distance between two intervals on a unit circle.

Given a threshold $\varepsilon > 0$, for large N, only $O(N \log N)$ elements of the matrix (6.1) which concentrate near the diagonal of each block (for fixed j and j') will be greater than ε. If we discard all the elements that are smaller than a predetermined threshold, we compress it to $O(N \log N)$ elements. Directly applying the estimates to the boundary integral kernels discussed in the above sections, we see that the matrices $\widetilde{\mathcal{S}}_J$ and $\widetilde{\mathcal{K}}_J$ are sparse.

Remark. For the periodic wavelets, we view the matrices as periodic and thus a block diagonal band also contains the lower-left and upper-right corners of the block.

7. Numerical Methods and Experiments

In this section we discuss the numerical methods for solving boundary integral equations by wavelets and present some numerical results.

7.5 Description of the numerical methods

Rearrange the order of wavelets, we have a set of orthonormal wavelet basis functions $\psi_i(x), (i = 1, \cdots, N)$. Here, we only describe the algorithm for the integral equation (2.6). The numerical method for natural integral equation is similar.

By wavelet transform, the singular part of the integral operator (2.6) is

$$\widetilde{\mathcal{S}}_J^1 = \mathcal{W}\mathcal{S}_1\mathcal{W}^{-1}$$

whose elements are

$$\begin{aligned} \tilde{s}_{i,j}^1 &= \int\int -\log|2\sin\frac{\alpha-\alpha'}{2}|\psi_i(\alpha)\psi_j(\alpha')d\alpha d\alpha' \\ &= \int \psi_i(\alpha)\left(\frac{1}{2}\sum_{-\infty,n\neq 0}^{\infty}\frac{1}{|n|}e^{in\alpha} * \psi_j\right)(\alpha)d\alpha \\ &= \int \psi_i(\alpha)\mathcal{F}^{-1}\left(\frac{\pi}{|n|}\mathcal{F}\psi_j(n)\right)(\alpha)d\alpha \\ &= \mathcal{W}_i\mathcal{F}^{-1}\left(\frac{\pi}{|n|}(\mathcal{F}\psi_j)(n)\right) \end{aligned} \tag{7.1}$$

where, $\mathcal{W}$ and $\mathcal{F}$ denote the discrete wavelet and Fourier transform respectively, and $\mathcal{W}_i f$ is the ith component of $\mathcal{W}f$. As we know, the numbers of computations for FWT and FFT are $O(N)$ and $O(N \log_2 N)$ respectively, so using (7.1) to compute the matrix $\widetilde{\mathcal{S}}_J^1$ only need $O(N^2 \log_2 N)$ computations, much less than the ordinary method which is $O(N^4)$.

By the discussion in section 6, if a threshold ε is given, only $O(N \log N)$ elements in the matrix $\widetilde{\mathcal{S}}_J^1$ whose absolute values are greater than ε are to be computed out. Under this circumstance, we would like to compute these elements by the following method

$$\tilde{s}_{i,j}^{(1)} = 2\pi^2 \sum_{-\infty, n\neq 0}^{\infty} \frac{1}{|n|} b_n^{(i)} b_{-n}^{(j)}, \tag{7.2}$$

where $b_n^{(i)}$ are coffecients of the Fourier series for $\psi_i(t)$

$$\psi_i(t) = \sum_{-\infty, n\neq 0}^{\infty} b_n^{(i)} e^{in\tau}.$$

Compared with (7.1), it also needs $O(N^2 \log_2 N)$ computations to produce the kernel $\widetilde{\mathcal{S}}_J^1$, but some inverse Fourier transforms are saved.

Set

$$\widetilde{\mathcal{S}}_J^2 = \mathcal{W}\mathcal{S}_2\mathcal{W}^{-1},$$
$$\widetilde{\mathcal{S}}_J = \widetilde{\mathcal{S}}_J^1 + \widetilde{\mathcal{S}}_J^2,$$

and

$$A = D_{\mathcal{S}}^{-1} \widetilde{\mathcal{S}}_J D_{\mathcal{S}}^{-1},$$

we have the linear algebraic equation

$$Ax = y, \tag{7.3}$$

where

$$y = D_{\mathcal{S}}^{-1} \mathcal{W} f$$

$D_{\mathcal{S}}$ is the preconditioning which is introduces in section 5. Then we can derive the unknown density function by

$$\sigma = \mathcal{W}^{-1} D_{\mathcal{S}} x.$$

We use the preconditioned conjugate gradient method to solve equation (7.3) and have the following iteration algorithm:

$$\begin{cases} r^{(0)} = y - Ax^{(0)}, \\ p^{(1)} = r^{(0)}, \\ \alpha_k = \dfrac{(r^{(k-1)}, p^{(k)})}{(p^{(k)}, Ap^{(k)})}, \\ x^{(k)} = x^{(k-1)} + \alpha_k p^{(k)}, \\ r^{(k)} = y - Ax^{(k)}, \\ \beta_{k+1} = -\dfrac{(r^{(k)}, Ap^{(k)})}{(p^{(k)}, Ap^{(k)})}, \\ p^{(k+1)} = r^{(k)} + \beta_{k+1} p^{(k)}. \end{cases}$$

Remark. Because of the sparsity of the matrix S_J, the number of computations for the matrix-vector multiplication Ax_n can also be reduced from $O(N^2)$ to $O(N \log_2 N)$. So truncation outside the diagonal bands can make the algorithm very efficiency.

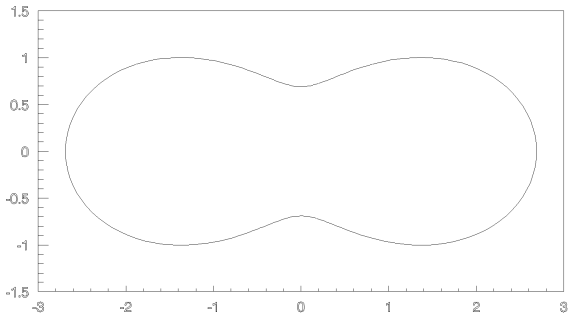

Figure 1. The curve Γ used in example 1.

7.6 Numerical experiments

Example 1: In the first example, we compute the Dirichlet problem

$$\begin{cases} \triangle u(x,y) = 0, & \text{in } \Omega, \\ u(x,y) = f, & \text{on } \Gamma, \end{cases}$$

by solving the single-layer integral equation (2.6). Γ is an analytic curve (see Figure 1), determined by

$$r = \cos 2\alpha + \sqrt{1.3^4 - \sin^2 2\alpha} \qquad 0 \le \alpha \le 2\pi,$$

Ω is the outside area of Γ, and

$$f(\alpha) = \frac{\cos\alpha}{\cos 2\alpha + \sqrt{1.3^4 - \sin^2 2\alpha}} \qquad 0 \le \alpha \le 2\pi.$$

by some simple computations, we can find

$$\mathcal{S}_2(\alpha,\alpha') = -\ln\left[4\sin^2(\alpha+\alpha')\cos^2\frac{\alpha-\alpha'}{2}\left(1+\frac{2\cdot\cos(\alpha+\alpha')}{f(\alpha)+f(\alpha')}\right)^2 + (\cos 2\alpha + f(\alpha))(\cos 2\alpha' + f(\alpha'))\right]^{1/2},$$

where

$$f(\alpha) = \sqrt{1.3^4 - \sin^2 2\alpha}.$$

The analytic solution for this problem is:

$$u(x,y) = \frac{x}{x^2+y^2} \qquad \text{in } \Omega.$$

For this example, we compute the kernel of $\widetilde{S}_J$. Setting to zero all the entries whose absolute value are smaller than 10^{-7}, we obtain the martrix shown in Figure 2. In the figure, the grey level represents the logarithm of each element's absolute value.

For different discrete increments $d\alpha$ and numbers of points N, the computing results are shown in Table 1. We select the 10th wavelet in Daubechies orthonormal wavelet series[2], always chose the initial $x_0 \equiv 0$, and the iterations are stopped when the residual's norm is less than 10^{-6}. From Table 1, we see the number of iterations are only slightly variant when N is changed. This due to the preconditioning D which makes the matrix A well-conditioned.

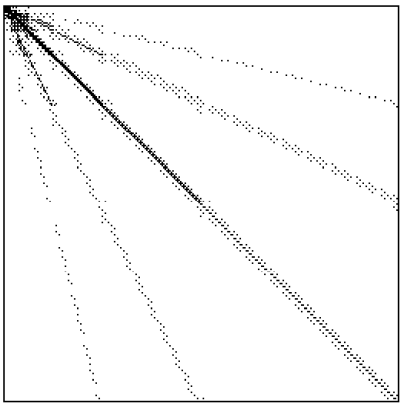

Figure 2. The matrix $\widetilde{\mathcal{S}}_J$ in example 1.

Number of points N	Increment $d\alpha$	Maxmum of the relative error	Average of the relative error	Number of iterations
32	0.196	1.48×10^{-3}	8.75×10^{-6}	9
64	0.098	1.28×10^{-5}	3.52×10^{-7}	10
128	0.049	5.21×10^{-6}	1.29×10^{-6}	10
256	0.025	4.67×10^{-7}	9.42×10^{-8}	11

Table 1: Computing errors of $u(x, y)$ at 3200 points outside Γ.

For $N = 256$, we truncate all the entries of matrices below a given threshold ε and solve the problem. The results are shown in Table 2.

Truncation threshold ε	Density of matrix	Maxmum of the relative error	Average of the relative error	Number of iterations
10^{-2}	0.55%	4.85×10^{-2}	2.08×10^{-2}	8
10^{-3}	2.46%	4.39×10^{-3}	7.78×10^{-4}	10
10^{-4}	7.85%	9.07×10^{-5}	7.45×10^{-6}	10
10^{-5}	13.4%	8.23×10^{-6}	9.24×10^{-7}	10

Table 2: Computing results for different truncation threshold.

Example 2: In the second example, we compute the Neumann problem outside the unit circle

$$\begin{cases} \triangle u = 0, & r > 1, 0 \le \alpha \le 2\pi \\ \dfrac{\partial u}{\partial n}\Big|_{r=1} = g(\alpha), & 0 \le \alpha \le 2\pi \end{cases}$$

by natural boundary method, where

$$g(\alpha) = -\cos(\alpha).$$

The analytic solution for this problem is

$$u(r, \alpha) = \frac{\cos(\alpha)}{r},$$

$$\sigma(\alpha) = u(1, \alpha) = \cos(\alpha).$$

For different discrete increments $d\alpha$ and N, the computing results are shown in table 3.

Number of points N	Increment $d\alpha$	Maxmum of the relative error	Average of the relative error	Number of iterations
32	0.196	9.73×10^{-5}	3.59×10^{-5}	4
64	0.0982	4.55×10^{-5}	2.41×10^{-6}	4
128	0.0491	3.75×10^{-5}	1.38×10^{-6}	4
256	0.0245	4.61×10^{-6}	6.88×10^{-8}	5

Table 3: Computing errors of $u(x, y)$ at 3200 points outside Γ.

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