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# A TRUST REGION-TYPE METHOD FOR SOLVING MONOTONE VARIATIONAL INEQUALITY*1)

Xi-ming Liang
(*Institute of Industrial Process Control, National Lab of Industrial Control Technology, Zhejiang University, Hangzhou 310027, China*)

Cheng-xian Xu
(*School of Science, Xi'an Jiaotong University, Xi'an 710049, China*)

Ji-xin Qian
(*Institute of Industrial Process Control, National Lab of Industrial Control Technology, Zhejiang University, Hangzhou 310027, China*)

**Abstract**

The Newton method for variational inequality problem is locally and quadratically convergent. By using a differentiable merit function, Taji, Fukushima and Ibaraki[1] have given a globally convergent modified Newton method for the strongly monotone variational inequality problem and proved their method to be quadratically convergent under some additional assumptions. In this paper we propose to present a trust region-type modification of Newton method for the strictly monotone variational inequality problem using the same merit function as that in [1]. It is then shown that our method is well defined and globally convergent and that, under the same assumptions as those in [1], our algorithm reduces to the basic Newton method and hence the rate of convergence is quadratic. Computational experimence indicates the efficiency of the proposed method.

*Key words*: Variational inequality problem, Trust region method, Global convergence, Quadratic convergence.

## 1. Introduction

Let $S$ be a nonempty closed convex subset of $R^n$ and let $F : R^n \to R^n$ be a continuous mapping. The variational inequality problem

$$\text{Find } x^* \in S \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in S \qquad \text{(VIP)}$$

is widely used to study various equilibrium models arising in economic, operations research, transportation and regional sciences[2,3], where $\langle \cdot, \cdot \rangle$ denotes the inner product in $R^n$. Many iterative methods for (VIP) have been developed, for example, projection methods[7,8], the nonlinear Jacobi method[5], the successive overrelaxation method[9]

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and generalized gradient method[10,11]. These methods usually converge to a solution of (VIP) under certain conditions on the mapping $F$ and rates of convergence are generally linear[2,5,12].

It is well known that Newton method for nonlinear equations and unconstrained minimization problems converges locally and quadratically. For (VIP), Newton method generates a sequence of iterates $\{x^k\}$, where $x^{k+1}$ is a solution of the linearized variational inequality problem

$$\text{Find } x \in S \text{ such that } \langle F(x^k) + \nabla F(x^k)^T(x - x^k), y - x\rangle \geq 0 \text{ for all } y \in S. \qquad (0)$$

It has been shown that[5], under assumptions that $x^*$ is a regular solution of (VIP) and $\nabla F(x)$ is Lipschitz continuous around $x^*$, the sequence converges quadratically to $x^*$ if the starting point $x^0$ is sufficiently close to $x^*$.

Recently, Marcotte and Dussault (1989), Taji, Fukushima and Ibaraki (1993) presented a globally convergent Newton method for (VIP) by incorporating line search strategies. Marcotte and Dussault's method uses the gap function $g(x) = \max\{\langle F(x), x - y\rangle | y \in S\}$ as a merit function. The function $g$ is generally nondifferentiable and achieves its minimum at a solution of (VIP) on $S$. The set $S$ is assumed to be compact in order that the function $g$ is well-defined. It is shown that when $F$ is monotone, the method is globally convergent when line searches are exact and that under the joint assumptions of strong monotonicity and strict complementarity, the rate of convergence is quadratic. Taji, Fukushima and Ibaraki's method employes a differentiable merit function proposed by Fukushima[6], whose minimizer on $S$ coincides with the solution of (VIP). The method allows inexact line searches and does not rely on the compact assumption of the set $S$. When $F$ is strongly monotone, the method is globally convergent and, under additional assumptions that the set $S$ is polyhedral convex, $\nabla F(x)$ is locally Lipschitz continuous and strictly complementarity condition holds at the unique solution $x^*$ of (VIP), the rate of convergence is quadratic.

In this paper, we propose a trust region modification of Newton method for (VIP). Fukushima's differentiable mertit function is used. When $F$ is strictly monotone rather than strongly monotone, the proposed algorithm is well-defined and converges globally to the unique solution of (VIP). Under the same assumptions as made in [1], it is shown that for $k$ sufficiently large, no trust region subproblem involves, therefore, the algorithm reduces to the basic Newton method and hence the rate of convergence is also quadratic.

The paper is organized as follows. In section 2, we review some preliminary results of the monotone mapping $F$ and the merit function that are useful in the subsequent sections. In section 3, we present our trust region-type modification of Newton method for solving monotone variational inequality and prove that it is well defined. In section 4, we establish a global convergence theorem without the assumptions that $F$ is strongly monotone and $S$ is compact. The rate of convergence of our algorithm is given in section 5. In section 6, we present some computational results.

## 2. Preliminaries

In this section, we summarize some basic concepts of monotone mapping $F$ and Fukushima's differentiable merit function and their properties used in subsequent sections.

A mapping $F : R^n \to R^n$ is said to be monotone on $S$ if

$$\langle F(x) - F(x'), x - x' \rangle \geq 0 \text{ for all } x, x' \in S$$

and strictly monotone on $S$ if strict inequality holds whenever $x \neq x'$. If $F$ is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in S$, i.e., $\langle d, \nabla F(x)d \rangle > 0$ for all $x \in S$ and $d \in R^n$ $(d \neq 0)$, then $F$ is strictly monotone on $S$. Note that $\nabla F(x)$ may not be symmetric. A mapping $F$ is said to be strongly (or uniformly) monotone with modulus $\mu > 0$ on $S$ if

$$\langle F(x) - F(x'), x - x' \rangle \geq \mu \|x - x'\|^2 \text{ for all } x, x' \in S \tag{1}$$

When $F$ is continuously differentiable, a necessary and sufficient condition for (1) is

$$\langle d, \nabla F(x)d \rangle \geq \mu \|d\|^2 \text{ for all } x \in S \text{ and } d \in R^n.$$

It is clear that strongly monotone implies strictly monotone.

Let $G$ be any given $n \times n$ symmetric positive definite matrix. The $G$-norm projection of a point $x \in R^n$ onto a set $S$, denoted by $Proj_{S,G}(x)$, is defined as the unique solution to the following constrained optimization problem

$$\text{minimize } \|y - x\|_G \text{ subject to } y \in S,$$

where $\|x\|_G = \langle x, Gx \rangle^{1/2}$ denotes the $G$-norm of a vector $x$ in $R^n$. The projection operator $Proj_{S,G}(\cdot)$ is nonexpansive [14, Proposition 3.2], i.e.,

$$\|Proj_{S,G}(x) - Proj_{S,G}(x')\|_G \leq \|x - x'\|_G \text{ for all } x, x' \in R^n \tag{2}$$

Suppose that an $n \times n$ symmetric positive definite matrix $G$ is given and $x \in R^n$ is any given point. Since the problem

$$\text{minimize } \langle F(x), y - x \rangle + \frac{1}{2} \langle y - x, G(y - x) \rangle \text{ subject to } y \in S \tag{3}$$

is essentially equivalent to the problem

$$\text{minimize } \|y - (x - G^{-1}F(x))\|_G^2 \text{ subject to } y \in S,$$

$H(x) = Proj_{S,G}(x - G^{-1}F(x))$ is the unique optimal solution of problem (3). It follows from (2) that $H : R^n \to S$ is continuous whenever $F$ is continuous. The mapping $H$ yields a fixed point characterization of the solution of (VIP).

**Proposition 2.1**[1,6]. *Let $G$ an $n \times n$ symmetric positive definite matrix and let $H(x)$ be the unique optimal solution of problem* (3) *for each given $x \in R^n$. Then $x$ solves* (VIP) *if and only if $x$ is a fixed point of the mapping $H$, i.e., $x = H(x)$.*

For any given $x \in S$, the linearized variational inequality problem of (VIP) at $x$ is

$$\text{Find } z \in S \text{ such that } \langle F(x) + \nabla F(x)^T(z-x), y-z\rangle \geq 0 \text{ for all } y \in S.$$

When $F$ is continuously differentiable and $\nabla F(x)$ is positive definite, a unique solution, denoted by $z(x)$, exists. The mapping $z : S \to S$ has the following property:

**Proposition 2.2**[1]**.** *If $F$ is continuously differentiable and strongly monotone on $S$, then the mapping $z : S \to S$ is continuous on $S$. Furthermore, $x$ is the solution of* (VIP) *if and only if $x$ satisfies $x = z(x)$.*

In fact from the proof of this proposition (see [1]) it can be concluded that when $F$ is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite on $S$, the second part of the proposition is also true.

For given mapping $F : R^n \to R^n$ and given $n \times n$ positive definite symmetric matrix $G$, define the function $f : R^n \to R$ as

$$f(x) = -\langle F(x), H(x) - x\rangle - \frac{1}{2}\langle H(x) - x, G(H(x) - x)\rangle, \tag{4}$$

where $H(x)$ is the unique solution of problem (3). It has been shown that, for any nonempty closed convex set $S$, the function $f$ has the following property:

**Proposition 2.3**[6]**.** *If the mapping $F : R^n \to R^n$ is continuous, then the function $f : R^n \to R$ is also continuous. Furthermore, if $F$ is continuously differentiable, then $f$ is also continuously differentiable and its gradient is given by*

$$\nabla f(x) = F(x) - [\nabla F(x) - G](H(x) - x). \tag{5}$$

Using the function $f$, an equivalent optimization problem can be formulated for any variational inequality problem.

**Proposition 2.4**[6]**.** *$f(x) \geq 0$ for all $x \in S$ and $f(x^*) = 0$ if and only if $x^*$ solves* (VIP)*. Hence $x^*$ solves* (VIP) *if and only if $x^*$ solves the following optimization problem and $f(x^*) = 0$:*

$$\textit{minimize } f(x) \textit{ subject to } x \in S. \tag{6}$$

Though the function $f$ generally is not convex, it has the desirable property that, if $\nabla F(x)$ is positive definite for all $x \in S$, any stationary point of problem (6) is also a global optimal solution of problem (6).

**Proposition 2.5**[6]**.** *Assume that the mapping $F : R^n \to R^n$ is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in S$. If $x$ is a stationary point of problem* (6)*, i.e., $\langle \nabla f(x), y - x\rangle \geq 0$ for all $y \in S$, then $x$ is a global optimal solution of* (6) *and hence solves* (VIP)*.*

This proposition indicates that the function $f$ can be used as a merit function for a descent method to solve a kind of strictly monotone variational inequality problems.

## 3. Trust Region Method

In this section we present a trust region-type modification of Newton method for solving monotone variational inequality problem. Throughout this section we assume that the set $S$ is nonempty, closed and convex and that the mapping $F : R^n \to R^n$ is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in S$. We denote $g(x) = \nabla f(x)$.

For given $x^k \in S$, consider the following linearized variational inequality problem:

$$\text{Find } z \in S \text{ such that } \langle F(x^k) + \nabla F(x^k)^T(z - x^k), x - z\rangle \geq 0 \text{ for all } x \in S. \quad (\text{LVIP } (x^k))$$

The assumptions on $F$ ensure that the linearized problem (LVIP($x^k$)) always has a unique solution $z(x^k)$ in $S$. If the set $S$ is polyhedral convex, the problem (LVIP($x^k$)) can be rewritten as a linear complementarity problem and can be solved in a finite number of steps by Lemke's complementary pivoting method[13].

Trust region algorithms in [15, 16] for solving

$$\text{minimize } f(x) \text{ subject to } x \in S,$$

first solve an "easy" subproblem

$$\text{minimize } Q_k(y) \equiv \frac{1}{2}M\|y\|^2 + g_k^T y \text{ subject to } x^k + y \in S, \ \|y\| \leq \Delta,$$

to get a solution $y^k$, then compute a $\bar{y}^k$, satisfying

$$\psi_k(\bar{y}^k) \leq \gamma Q_k(y^k), \ x^k + \bar{y}^k \in S \text{ and } \|\bar{y}^k\| \leq \Delta, \ \gamma \in (0,1),$$

as a trial step, where $g^k = g(x^k)$, $M > 0$, $\psi_k(y) = \frac{1}{2}y^T B_k y + g_k^T y$ and $B_k$ is a symmetric matrix such that $\|B_k\| \leq M$. Note that $y^k$ is an admissible choice of $\bar{y}^k$.

Now we state the trust region-type modification of Newton method for solving monotone variational inequality problem as follows:

**Algorithm VITR.**

Choose $x^0 \in S$, $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in (0,1)$, $M > 0$ and let $k = 0$.

Step 1. If $f(x^k) = 0$, stop.

Step 2. Find the solution $z(x^k) \in S$ of problem LVIP($x^k$), let $d^k = z(x^k) - x^k$.

Step 3. If $f(x^k + d^k) = f(z(x^k)) \leq \alpha f(x^k)$, then $x^{k+1} = z(x^k)$, $k = k+1$ and go to Step 1.

Step 4. Let $\Delta = \|d^k\|$.

Step 5. Compute a global solution $y^k$ of the following problem:

$$\text{minimize} \quad Q_k(y) = \frac{M}{2}\|y\|^2 + g_k^T y \text{ subject to } x^k + y \in S, \ \|y\| \leq \Delta \tag{7}$$

Step 6. If

$$f(x^k + y^k) \leq f(x^k) + \beta g_k^T y_k \tag{8}$$

then $x^{k+1} = x^k + y^k$, $\Delta_k = \Delta$, $k = k+1$ and go to Step 1,
else let $\Delta = \gamma\Delta$ and go to Step 5.

**Remarks.** (i). If $f(x^k) \neq 0$ then $x^k$ does not solve (VIP) by proposition 2.4 and hence $d^k \neq 0$ by proposition 2.2.

(ii). Note that $Q_k(y^k) = 0$ if and only if $x^k$ is a stationary point of (6). In fact, $Q_k(y^k) = 0$ if and only if 0 is a minimizer of (7). By the optimality conditions [14, Proposition 3.1], 0 minimizes (7) if and only if $\langle \nabla Q_k(0), y - 0 \rangle \geq 0$ for every feasible point $y$ of (7), i.e.,

$$\langle g_k, y \rangle \geq 0 \text{ for all } y \text{ such that } x^k + y \in S, \ \|y\| \leq \Delta,$$

which is equivalent to

$$\langle \nabla f(x^k), x - x^k \rangle \geq 0 \text{ for all } x \in S.$$

When $f(x^k) \neq 0$, $x^k$ is not a stationary point of (6) by propositions 2.4 and 2.5, and hence $Q_k(y^k) \neq 0$, so we have $Q_k(y^k) < 0$ and $g_k^T y^k < 0$ because $y = 0$ is feasible to (7).

**Lemma 3.1.** *Algorithm* VITR *is well defined. That is, if $f(x^k) \neq 0$, then $x^{k+1}$ is obtained either in Step* 3 *or by repeating Steps* 5 *and* 6 *a finite number of times.*

*Proof.* It is clear that we only need to prove that $x^{k+1}$ can be computed by repeating Steps 5 and 6 a finite number of times if $x^{k+1}$ is not obtained at Step 3.

Since $f(x^k) \neq 0$, $x^k$ is not a solution and hence not a stationary point of (6) by propositions 2.4 and 2.5. There exists a feasible descent direction $d \in R^n \backslash \{0\}$ such that, for some $\overline{\lambda} > 0$, $g_k^T d < 0$ and $x^k + \lambda d \in S$ for all $\lambda \in [0, \overline{\lambda}]$. For small enough $\Delta > 0$ such that $\dfrac{\Delta}{\|d\|} < \overline{\lambda}$, it follows from (7) that

$$Q_k(y^k) \leq Q_k\Big(\frac{\Delta d}{\|d\|}\Big) = \frac{M}{2}\Delta^2 + \frac{\Delta}{\|d\|} g_k^T d.$$

Thus,

$$\overline{\lim_{\Delta \to 0}} \frac{g_k^T y^k}{\Delta} \leq \overline{\lim_{\Delta \to 0}} \frac{Q_k(y^k)}{\Delta} \leq \frac{g_k^T d}{\|d\|} < 0.$$

By the definition of supremum, there exists $\overline{\Delta} > 0$ such that

$$\frac{g_k^T y^k}{\Delta} \leq \frac{g_k^T d}{2\|d\|} \triangleq c^1 < 0, \text{ for all } \Delta \in (0, \overline{\Delta}].$$

Define $\rho(\Delta) = \dfrac{f(x^k + y^k) - f(x^k)}{g_k^T y^k}$ for $\Delta \in (0, \overline{\Delta}]$, then

$$\begin{aligned} |\rho(\Delta) - 1| &= \Big| \frac{f(x^k + y^k) - f(x^k) - g_k^T y^k}{g_k^T y^k} \Big| \leq \Big| \frac{f(x^k + y^k) - f(x^k) - g_k^T y^k}{c^1 \Delta} \Big| \\ &\leq \Big| \frac{f(x^k + y^k) - f(x^k) - g_k^T y^k}{c^1 \|y^k\|} \Big| \end{aligned}$$

Therefore, by the continuous differentiability of $f$, we have

$$\lim_{\Delta \to 0} \rho(\Delta) = 1, \tag{9}$$

which implies that after a finite number of reduction of $\Delta$, condition (8) must be satisfied and $x^{k+1}$ is well defined.

## 4. Global Convergence

Since $x^k$ solves (VIP) if algorithm VITR stops at Step 1, it is assumed, without loss of generality, that algorithm VITR generates an infinite sequence $\{x^k\}$. The global convergence of algorithm VITR is proved in the following theorem, where the notation $\lim_{k\in K}$ denotes the limit when $k \to \infty$ for all $k$ belonging to an infinite subset of indices $K \subset N$.

**Theorem 4.1 (global convergence).** *Suppose that the mapping $F$ is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in S$ and that the set $S$ is nonempty, polyhedral, closed and convex. Then, for any starting point $x^0 \in S$, the sequence $\{x^k\}$ converges to the unique solution of* (VIP) *whenever $\{x^k\}$ is bounded.*

*Proof.* If $f(x^k + d^k) \le \alpha f(x^k)$ holds infinitely often, then $\lim\limits_{k\to\infty} f(x^k) = 0$. Since $f$ is continuous by proposition 2.3, $f(\bar{x}) = 0$ for any accumulation point $\bar{x}$ of $\{x^k\}$ and hence $\bar{x}$ is a solution of (VIP). Since (VIP) has at most a solution (see [2, proposition 3.2]), it follows that $\bar{x}$ is the unique solution of (VIP) and that the entire sequence $\{x^k\}$ has a unique accumulation point $\bar{x}$ and necessarily converges to $\bar{x}$.

Now we consider the case when $f(x^k + d^k) \le \alpha f(x^k)$ hold for only finitely many $k$. In this case, the sequence $\{x^k\}$ is generated by Step 6 and satisfies (8) for $k$ sufficiently large. Let $\{x^k\}_{k\in K}$ be any convergent subsequence of $\{x^k\}$ and let $\bar{x} \in S$ be its limit point. If $\inf\limits_{k\in K} \|d^k\| = 0$, then there exists an infinite subset $K_0$ of $K$ such that $\lim\limits_{k\in K_0} \|d^k\| = 0$. Since $d^k = z(x^k) - x^k$ and $\lim\limits_{k\in K_0} x^k = \bar{x}$, we can obtain, if necessary, take the subset of $K_0$, that $\lim\limits_{k\in K_0} z(x^k) = \bar{x}$. From the continuity of $f$ by proposition 2.3 and the fact that $f(z(x^k)) > \alpha f(x^k)$ by the algorithm, we have $f(\bar{x}) \ge \alpha f(\bar{x})$. Therefore, as $\alpha \in (0, 1)$ and $f(\bar{x}) \ge 0$, $f(\bar{x}) = 0$ and $\bar{x}$ solves (VIP) by proposition 2.4. If $\inf\limits_{k\in K} \|a^k\| = c > 0$, then there are two posibilities:

$$\text{(i).} \qquad \inf_{k\in K} \Delta_k = 0, \tag{10}$$

$$\text{(ii).} \qquad \inf_{k\in K} \Delta_k > 0. \tag{11}$$

If (10) holds, then there exists an infinite subset $K_1$ of $K$ such that

$$\lim_{k\in K_1} \Delta_k = 0 \tag{12}$$

Let $k_1 \in K_1$ be such that $\Delta_k < c$ for all $k \in K_2 = \{k \in K_1 : k \ge k_1\}$. Since we obtain each $\Delta_k$ by reducing the radius $\|d^k\| > c$, for all $k \in K_2$ before $y^k$ is accepted, there must exist $\overline{\Delta}_k = \dfrac{\Delta_k}{\gamma}$ and the solution $\bar{y}^k$ of

$$\text{minimize } Q_k(y) = \frac{M}{2}\|y\|^2 + g_k^T y \text{ subject to } x^k + y \in S, \ \|y\| \le \bar{\Delta}_k$$

does not satisfy (8), i.e.,

$$f(x^k + \bar{y}^k) > f(x^k) + \beta g_k^T \bar{y}^k \tag{13}$$

By (12) and $\|\bar{y}^k\| \le \overline{\Delta}_k = \dfrac{\Delta_k}{\gamma}$ $(k \in K_2)$, we have

$$\lim_{k \in K_2} \|\bar{y}^k\| = 0. \tag{14}$$

Assume that $\bar{x}$ is not a solution of (VIP). Then there exists a feasible descent direction $d \in R^n \backslash \{0\}$ such that $g(\bar{x})^T d < 0$ and $\bar{x} + \lambda d \in S$ for all $\lambda \in [0,1]$. The polyhedral convexity of $S$ and convergence of $\{x^k\}_{k \in K}$ to $\bar{x}$ imply that there exists $k_2 \in K_2$ such that

$$x^k + \frac{\lambda d}{2} \in S \text{ for all } k \in K_3 = \{k \in K_2 : k \ge k_2\} \text{ and } \lambda \in [0,1]. \tag{15}$$

Let $u^k = \dfrac{\|\bar{y}^k\| d}{\|d\|}$, then by (14) and (15), there exists $k_3 \in K_3$ such that $x^k + u^k \in S$ for all $k \in K_4 = \{k \in K_3 : k \ge k_3\}$.

Since $\|u^k\| = \|\bar{y}^k\| \le \overline{\Delta}_k$, from Step 5 we have, for all $k \in K_4$,

$$g_k^T \bar{y}^k \le Q_k(\bar{y}^k) \le Q_k(u^k) = \frac{M}{2}\|\bar{y}^k\|^2 + \frac{\|\bar{y}^k\|}{\|d\|} g_k^T d$$

$$\frac{g_k^T \bar{y}^k}{\|\bar{y}^k\|} \le \frac{M}{2}\|\bar{y}^k\| + \frac{g_k^T d}{\|d\|}.$$

Then by the continuity of $g(x)$ and (14), we obtain

$$\lim_{k \in K_4} \frac{g_k^T \bar{y}^k}{\|\bar{y}^k\|} \le \frac{g(\bar{x})^T d}{\|d\|} < 0.$$

Hence, there exists $c_2 < 0$, $k_4 \in K_4$ such that

$$\frac{g_k^T \bar{y}^k}{\|\bar{y}^k\|} \le c_2 < 0 \text{ for all } k \in K_5 = \{k \in K_4 : k \ge k_4\}.$$

For $k \in K_5$, let $\rho_k = \dfrac{f(x^k + \bar{y}^k) - f(x^k)}{g_k^T \bar{y}^k}$, then

$$|\rho_k - 1| = \Big|\frac{f(x^k + \bar{y}^k) - f(x^k) - g_k^T \bar{y}^k}{g_k^T \bar{y}^k}\Big| \le \Big|\frac{f(x^k + \bar{y}^k) - f(x^k) - g_k^T \bar{y}^k}{c_2 \|\bar{y}^k\|}\Big|.$$

The continuity of $g(x)$ and (14) implies $\lim_{k \in K_5} \rho_k = 1$, which contradicts with (13). Therefore $\bar{x}$ must be a solution of (VIP).

Now, assume that (11) holds. Since $\lim_{k \in K} x^k = \bar{x}$ and $f(x^k)$ is monotonically decreasing, we have that $\lim_{k \in K}[f(x^{k+1}) - f(x^k)] = 0$. Moreover, we have, from Step 6, $f(x^{k+1}) \le f(x^k) + \beta g_k^T y^k \le f(x^k) + \beta Q_k(y^k)$. Hence

$$\lim_{k \in K} Q_k(y^k) = 0 \tag{16}$$

Define $\overline{\Delta} = \inf_{k \in K} \Delta_k > 0$, let $\bar{y}$ be the solution of problem

$$\text{minimize} \bar{Q}(y) = \frac{M}{2}\|y\|^2 + g(\bar{x})^T y \text{ subject to } \bar{x} + y \in S, \; \|y\| \leq \frac{\overline{\Delta}}{2} \tag{17}$$

and $k_5 \in K$ be such that $\|x^k - \bar{x}\| \leq \dfrac{\overline{\Delta}}{2}$ for all $k \in K_6 = \{k \in K : k > k_5\}$. If we define $\bar{x}^k = \bar{x} - x^k + \bar{y}$, then we have

$$\|\bar{x}^k\| = \|\bar{x} - x^k + \bar{y}\| \leq \|\bar{x} - x^k\| + \|\bar{y}\| \leq \overline{\Delta} \leq \Delta_k \text{ for all } k \in K_6.$$

Since $x^k + \bar{x}^k = \bar{x} + \bar{y} \in S$, from (7) we have

$$Q_k(y^k) \leq Q_k(\bar{x}^k) \text{ for all } k \in K_6. \tag{18}$$

Combining (16) and (18), we have

$$\begin{aligned} \bar{Q}(\bar{y}) = \frac{M}{2}\|\bar{y}\|^2 + g(\bar{x})^T \bar{y} &= \lim_{k \in K_6} \Big(\frac{M}{2}\|\bar{x}^k\|^2 + g(x^k)^T \bar{x}^k\Big) = \lim_{k \in K_6} Q_k(\bar{x}^k) \\ &\geq \lim_{k \in K_6} Q_k(y^k) = 0 \end{aligned}$$

which shows $\bar{Q}(\bar{y}) = 0$ since 0 is feasible to (17). This proves that $\bar{x}$ is a solution to (VIP).

From the uniqueness of the solution to (VIP), we can conclude that the entire sequence $\{x^k\}$ has a unique accumulation point $\bar{x}$ and converges to $\bar{x}$. □

## 5. Rate of Convergence

In this section, we will show that, under assumptions that the set $S$ is polyhedral convex and that $F$ is strongly monotone on $S$, the algorithm VITR is locally quadratically convergent.

**Proposition 5.1**[1]. *Let $x^*$ be a solution to* (VIP). *If $F$ is strongly monotone with modulus $\mu$ on $S$, then $f$ of* (4) *satisfies the inequality*

$$f(x) \geq \Big(\mu - \frac{1}{2}\|G\|\Big)\|x - x^*\|^2 \text{ for all } x \in S. \tag{19}$$

*In paticular, if the matrix $G$ is chosen sufficiently small to satisfy $\|G\| < 2\mu$, then*

$$\lim_{x \in S, \|x\| \to \infty} f(x) = +\infty.$$

It is obvious that the decreasing of $\{f(x^k)\}$ and proposition 5.1 imply that when $F$ is strongly monotone on $S$ and when the matrix $G$ is sufficiently small the sequence $\{x^k\}$ generated by algorithm VITR is bounded. To obtain the second order convergence result, we need the following strict complementarity condition [1], which is a generalization of the strict complementarity condition for inequality constraints and corresponding Lagrange multiplies that appear in the Karush-Kuhn-Tucker conditions in nonlinear programming.

**Definition 5.1**[1]. *Suppose that $S$ is polyhedral and that* (VIP) *has a unique solution $x^*$. Let $T^*$ denote the minimal face of $S$ containing $x^*$. Then strict complementarity holds at $x^*$ if $x \in S$ and $\langle F(x^*), x - x^* \rangle = 0$ imply $x \in T^*$.*

Now we give the following rate of convergence result.

**Theorem 5.1 (quadratic convergence).** *Suppose that the set $S$ is polyhedral convex, the mapping $F$ is strongly monotone with modulus $\mu$ on $S$ and $\nabla F(\cdot)$ is Lipschitz continuous on a neighborhood $N$ of the unique solution $x^*$ of* (VIP)*. If the matrix $G$ is sufficiently small such that $\|G\| < 2\mu$ and the strict complementarity condition holds at $x^*$, then the sequence $\{x^k\}$ generated by algorithm* VITR *converges quadratically to $x^*$.*

*Proof.* Under the assumptions on $F$, it is not difficult to show that $\nabla F(\cdot)$ is also Lipschitz continuous on the neighborhood $N$ of $x^*$, i.e., there exists a constant $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \text{ for all } x, y \in N.$$

From the proof of [1, Theorem 5.1], we obtain that there exists $\zeta > 0$ such that

$$f(z(x^k)) \leq \frac{L\zeta^2}{2\mu - \|G\|}\|x^k - x^*\|^2 f(x^k)$$

for all $k$ sufficiently large. So, $f(x^k + d^k) = f(z(x^k)) \leq \alpha f(x^k)$ holds for all $k$ large enough to satisfy

$$\frac{L\zeta^2}{2\mu - \|G\|}\|x^k - x^*\|^2 \leq \alpha.$$

Therefore, after a finite number of steps, the Steps 4, 5 and 6 will no longer be involved and the algorithm VITR is exactly the basic Newton iteration $x^{k+1} = z(x^k)$. Hence, the result of the theorem follows from the fact that the convergence rate of the basic Newton iteration $x^{k+1} = z(x^k)$ is quadratic[2,5].

## 6. Computational Results

In this section, we report numerical results for the proposed algorithm. The algorithm is programmed in Turbo C 2.0 with double precision and the numerical experiments are implemented on personal computer. The parameters used in the algorithm are set as $\alpha = 0.5$, $\beta = 0.01$, $\gamma = 0.4$ and $M = 1$. The symmetric positive definite matrix $G$ was chosen to be the identity matrix. The convergence criterion $f(x^k) = 0$ in Step 1 is replaced by $f(x^k) \leq \varepsilon$ with $\varepsilon = 10^{-6}$ in practice.

For comparison, we also code the Newton method (cf.(0)) and the algorithm GCNM of Taji, Fukushima and Ibaraki[1]. In all test examples, the feasible region $S$ are polyhedral convex sets specified by linear inequalities. In solving the linearized subproblem LVIP $(x^k)$ at each iteration of the three algorithms, we first transform them into linear complementarity problems, and then Lemke's complementary pivoting method is applied.

The experiments are made on several test problems which are modifications of the test problem used by Taji, Fukushima and Ibaraki[1]. In each problem, the constraint set $S$ takes the form $S = \{x \in R^5 | Ax \le b,\ x \ge 0\}$ and the mapping $F$ is given by $F(x) = Mx + \rho D(x) + q$, where $M$ is a $5 \times 5$ asymmetric positive definite matrix and $D_i(x)$ is a nonlinear mapping with components $D_i(x) = d_i x_i^4$, where $d_i$ are positive constants. The parameter $\rho$ is used to vary the degree of asymmetry and nonlinearity. The data of the example are given in Table 1. Numerical results are shown in Table 2.

**Table 1. Data for the example**

$$F(x) = \begin{bmatrix} 3 & -4 & -16 & -15 & -4 \\ 4 & 1 & -5 & -10 & -11 \\ 16 & 5 & 2 & -11 & -7 \\ 15 & 10 & 11 & 3 & -10 \\ 4 & 11 & 7 & 10 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \rho \begin{bmatrix} 0.004x_1^4 \\ 0.007x_2^4 \\ 0.005x_3^4 \\ 0.009x_4^4 \\ 0.008x_5^4 \end{bmatrix} + \begin{bmatrix} -15 \\ 10 \\ -50 \\ -30 \\ -25 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & -0.5 & 0 & -2 \\ -2 & -2 & 0 & -0.5 & -2 \\ 2 & 2 & -4 & 2 & -3 \\ -5 & 3 & -2 & 0 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} -10 \\ -10 \\ 13 \\ 18 \end{bmatrix}$$

Solution for $\rho = 0.01 : \{11.44, 0.00, 0.00, 0.00, 5.00\}^T$;

Solution for $\rho = 0.1 : \{11.01, 0.97, 0.00, 0.00, 5.00\}^T$;

Solution for $\rho = 1 : \{9.08, 4.84, 0.00, 0.00, 5.00\}^T$;

Solution for $\rho = 10 : \{5.51, 4.07, 0.15, 0.00, 4.96\}^T$;

Solution for $\rho = 100 : \{3.82, 2.65, 3.42, 0.00, 4.14\}^T$.

**Table 2. Results for the test example (Number of iterations)**

| | $\rho = 0.01$ | | | $\rho = 0.1$ | | | $\rho = 1$ | | | $\rho = 10$ | | | $\rho = 100$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| initial point $x^0$ | V | G | N | V | G | N | V | G | N | V | G | N | V | G | N |
| $\{0, 0, 0, 0, 0\}$ | 2 | 3 | 2 | 3 | 3 | 3 | 4 | 5 | 4 | 6 | 7 | 6 | 9 | 10 | 9 |
| $\{100, 0, 0, 0, 0\}$ | 7 | 7 | 7 | 9 | 9 | 9 | 11 | 11 | 11 | 13 | 13 | 13 | 14 | 15 | 14 |
| $\{0, 0, 100, 0, 0\}$ | 9 | 9 | 9 | 10 | 12 | 11 | 11 | 13 | 13 | 13 | 15 | 15 | 15 | 17 | 18 |
| $\{0, 0, 0, 0, 100\}$ | 7 | 8 | 7 | 10 | 10 | 10 | 11 | 12 | 11 | 13 | 14 | 13 | 15 | 15 | 15 |
| $\{100, 0, 0, 0, 100\}$ | 8 | 8 | 8 | 10 | 10 | 10 | 11 | 12 | 11 | 14 | 14 | 14 | 16 | 16 | 16 |
| $\{0, 100, 0, 100, 0\}$ | 9 | 10 | 13 | 5 | 12 | 16 | 6 | 13 | 18 | 14 | 16 | 20 | 15 | 17 | 21 |
| $\{100, 0, 100, 0, 100\}$ | 3 | 9 | 10 | 4 | 11 | 11 | 5 | 12 | 14 | 15 | 15 | 16 | 16 | 16 | 18 |
| $\{100, 100, 100, 100, 100\}$ | 8 | 8 | 8 | 10 | 10 | 10 | 12 | 12 | 12 | 14 | 15 | 14 | 16 | 16 | 16 |

V: Algorithm VITR; G: Algorithm in [1]; N: Newton method.

The results show that the three versions of Newton method all converge to the solution for each test problem. However it is observed that the trust region strategy has a good effect. We may conclude that, as far as our limited computational experience is concerned, the proposed algorithm VITR is comparable to the algorithm of Taji, Fukushima and Ibaraki [1] and is stable and robust for solving monotone variational inequalities.

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