Journal of Computational Mathematics, Vol.19, No.6, 2001, 629–638.

GENUINE-OPTIMAL CIRCULANT PRECONDITIONERS FOR WIENER-HOPF EQUATIONS*[1)]

Fu-rong Lin
(*Department of Mathematics, Shantou University, Shantou 515063, China*)

Abstract

In this paper, we construct the genuine-optimal circulant preconditioner for finite-section Wiener-Hopf equations. The genuine-optimal circulant preconditioner is defined as the minimizer of Hilbert-Schmidt norm over certain integral operators. We prove that the difference between the genuine-optimal circulant preconditioner and the original integral operator is the sum of a small norm operator and a finite rank operator. Thus, the preconditioned conjugate gradient (PCG) method, when applied to solve the preconditioned equations, converges superlinearly. Finally, we give an efficient algorithm for the solution of Wiener-Hopf equation discretized by high order quadrature rules.

Key words: Wiener-Hopf equations, Circulant preconditioner, Preconditioned conjugate gradient method, Quadrature rules, Hilbert-Schmidt norm.

1. Introduction

Wiener-Hopf equations are integral equations defined on the half-line:

$$\sigma x(t) + \int_0^\infty a(t-s)x(s)ds = g(t), \quad t \in \mathbb{R}^+,$$

where $\sigma > 0$, $a(\cdot) \in L_1(\mathbb{R})$ and $g(\cdot) \in L_2(\mathbb{R}^+)$. Here $\mathbb{R} \equiv (-\infty, \infty)$ and $\mathbb{R}^+ \equiv [0, \infty)$. In our discussions, we assume that $a(\cdot)$ is conjugate symmetric, i.e. $a(-t) = \overline{a(t)}$. Wiener-Hopf equations arise in a variety of practical applications in mathematics and engineering, for instance, in the linear prediction problems for stationary stochastic processes [8, pp.145–146], diffusion problems and scattering problems [8, pp.186–189]. In this paper, we consider using the preconditioned conjugate gradient (PCG) method to solve finite-section Wiener-Hopf equations:

$$\sigma x(t) + \int_0^\tau a(t-s)x(s)ds = g(t), \quad 0 \le t \le \tau. \tag{1}$$

Gohberg, Hanke and Koltracht [7] have introduced two circulant integral preconditioners, i.e., wrap-around and optimal circulant preconditioners to precondition the finite-section Wiener-Hopf equations (1). Circulant integral operators are operators of the form

$$(H_\tau y)(t) = \int_0^\tau h_\tau(t-s)y(s)ds, \quad 0 \le t \le \tau,$$

where $h_\tau \in L_1[-\tau, \tau]$ are τ-periodic, i.e., $h_\tau(t-\tau) = h_\tau(t)$ for $t \in [0, \tau]$. Let

$$A_\tau x(t) = \int_0^\tau a(t-s)x(s)ds, \quad 0 \le t \le \tau, \tag{2}$$

then the preconditioned equation is given by

$$(\sigma I + H_\tau)^{-1}(\sigma I + A_\tau)x_\tau(t) = (\sigma I + H_\tau)^{-1}g(t), \quad 0 \le t \le \tau. \tag{3}$$

* Received June 18, 1998; Final revised January 15, 2001.
1) Supported in part by the natural science foundation of China No. 19901017.

It has been proved in [7] that for sufficiently large τ, the spectra of the wrap-around and optimal circulant preconditioned operators are clustered around one. Hence the PCG method converges superlinearly for sufficiently large τ, see for instance [1].

In §2, we construct the genuine-optimal circulant integral preconditioner for the finite-section Wiener-Hopf equation (1). The genuine-optimal circulant preconditioner is the minimizer of the following Hilbert-Schmidt norm

$$|||I - (\sigma I + H_\tau)^{-1/2}(\sigma I + A_\tau)(\sigma I + H_\tau)^{-1/2}|||$$

over all circulant integral operator H_τ such that $(\sigma I + H_\tau)$ is positive definite. We prove that the genuine-optimal circulant preconditioners have the property that the spectra of the preconditioned operators $(\sigma I + H_\tau)^{-1}(\sigma I + A_\tau)$ are clustered around one for sufficiently large τ.

In this paper, we also consider the discretization of the preconditioned integral equations (3) by high order quadrature rules. Let the interval $[0, \tau]$ be partitioned into N equal subintervals of step-size $\iota = \tau/N$. By Newton-Cotes quadrature rules, using $s_k = k\iota,\ k = 0, 1, \cdots, N$ as quadrature points, we get the preconditioned matrix systems

$$(\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}(\sigma \mathbf{I}_p + \mathbf{A}_p\mathbf{D}_p)\mathbf{x}_p = (\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}\mathbf{g}_p, \tag{4}$$

where $p = N + 1$ is the number of quadrature points (if the rectangular rule is used, then the quadrature points are given by $s_k = k\iota,\ k = 0, 1, \cdots, N-1$ and $p = N$). Here $\mathbf{I}_p$ is the p-by-p identity matrix, $\mathbf{A}_p$ is the Hermitian Toeplitz matrix with the first column given by $(\iota a(0), \iota a(\iota), \cdots, \iota a(N\iota))^T$ and the $N \times N$ principal submatrix of $\mathbf{C}_p$ is a circulant matrix with the first column given by $(\iota h_\tau(0), \iota h_\tau(\iota), \cdots, \iota h_\tau((N-1)\iota))^T$. We recall that a matrix $\mathbf{A}_p$ is said to be a Toeplitz matrix if $\mathbf{A}_p = [a_{i,j}]$ satisfies $a_{i,j} = a_{i-j}$ and a matrix $\mathbf{C}_p$ is a circulant matrix if it is a Toeplitz matrix and $c_{-i} = c_{p-i}$ for $i = 1, 2, \cdots, p-1$. In (4), $\mathbf{D}_p$ is a diagonal matrix that depends only on the quadrature rule used. For instance, the diagonals of $\mathbf{D}_p$ corresponding to Simpson's rule are given by $(\frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \cdots, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{3})$. We note that if $a(t)$ and $g(t)$ are smooth functions, then the accuracy of the discretized solutions $\sqrt{\sum_{i=0}^{N} \iota (x(i\iota) - [\mathbf{x}_p]_i)^2}$ of the rectangular, trapezoidal and Simpson's rule are of the order $O(\iota)$, $O(\iota^2)$ and $O(\iota^4)$ respectively, see [11] for instance.

We will give an efficient method to find an approximation of $(\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}$ in $O(p \log p)$ operations. We note that if the rectangular rule is used, (4) is basically a circulant preconditioned Toeplitz system which requires only $O(p \log p)$ operations in each iteration by means of FFTs [13] and the convergence rate of these systems has been analyzed for instances in [2, 9, 10, 14]. If high order quadrature rules are used, the discretization matrices of the circulant preconditioners $\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p$ are in general not circulant and it is difficult to find the inverse of $(\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}$ efficiently.

The outline of this paper is as follows. In §2, we construct the genuine-optimal circulant preconditioners for (1) and prove that the spectra of the preconditioned operators are clustered around one. Numerical results are given in §3 to illustrate the efficiency of circulant preconditioners. In §4, we propose an efficient algorithm for solving (4) and give numerical results to show the fast convergence and the stability of our algorithm.

2. Genuine-Optimal Circulant Integral Preconditioners

Let H_τ be a circulant preconditioner. We then solve the preconditioned equation

$$[(\sigma I + H_\tau)^{-1}(\sigma I + A_\tau)]x_\tau(t) = (\sigma I + H_\tau)^{-1}g(t).$$

A natural idea is to find the circulant operator H_τ such that the Hilbert-Schmidt norm

$$|||H_\tau - A_\tau|||^2 \equiv \int_0^\tau \int_0^\tau (a(t,s) - h_\tau(t-s))\overline{(a(t,s) - h_\tau(t-s))}dtds$$

is minimized, from which the optimal preconditioner $\mathcal{P}(A_\tau)$ is obtained [7, 4]. The kernel function of $\mathcal{P}(A_\tau)$ is given by (9). It is also natural that we expect the operator $(\sigma I+H_\tau)^{-1}(\sigma I+A_\tau)$ to be as close to the identity operator I as possible, i.e., we want to find the circulant operator H_τ which minimizing

$$|||I-(\sigma I+H_\tau)^{-1}(\sigma I+A_\tau)|||.$$

From that the super-optimal preconditioner, which is given by

$$\mathcal{P}[(\sigma I+A_\tau)(\sigma I+A_\tau^*)][\mathcal{P}(\sigma I+A_\tau^*)]^{-1}-\sigma I \tag{5}$$

see [14, 3], is constructed. Generally, $(\sigma I+H_\tau)^{-1}(\sigma I+A_\tau)$ is not self-adjoint and in practice we solve the preconditioned equation

$$(\sigma I+H_\tau)^{-1/2}(\sigma I+A_\tau)(\sigma I+H_\tau)^{-1/2}y_\tau(t)=(\sigma I+H_\tau)^{-1/2}g(t),\quad 0\le t\le\tau,$$

and $x_\tau(t)=(\sigma I+H_\tau)^{-1/2}y_\tau(t)$. Thus, it is reasonable to find the minimizer of

$$|||I-(\sigma I+H_\tau)^{-1/2}(\sigma I+A_\tau)(\sigma I+H_\tau)^{-1/2}||| \tag{6}$$

over all circulant integral operator H_τ such that $(\sigma I+H_\tau)$ is positive definite. Let V_τ be the minimizer of (6), we call $(\sigma I+V_\tau)$ the genuine-optimal preconditioner of $(\sigma I+A_\tau)$.

In the following, we will obtain the solution of (6) and prove that the genuine-optimal preconditioner is a good preconditioner. For this purpose, we need the following results.

1. Let B_1 and B_2 be integral operators defined as

$$B_ix(t)=\int_0^\tau b_i(t,s)x(s)ds,\quad 0\le t\le\tau,$$

where $b_i(t,s)$ are functions in $L_2[0,\tau]^2$. Then the kernel function of the composite operator $B=B_1B_2$ is

$$b(t,s)=\int_0^\tau b_1(t,\eta)b_2(\eta,s)d\eta. \tag{7}$$

2. Let B_τ be an integral operator with kernel function $b_\tau\in L_2[0,\tau]^2$ and $\nu_{m,n}=(B_\tau u_m,u_n)_\tau$, where

$$u_k(t)=\frac{1}{\sqrt{\tau}}e^{2\pi ikt/\tau},\quad k\in\mathbb{Z}. \tag{8}$$

Then the kernel function of its optimal preconditioner $\mathcal{P}(B_\tau)$ is given by [3]

$$p_{B_\tau}(t-s)=\sum_{k=-\infty}^{\infty}\nu_{k,k}u_k(t)\bar{u}_k(s). \tag{9}$$

It follows that the eigenvalues of $\mathcal{P}(B_\tau)$ are $\{v_{k,k}\}_{k=-\infty}^{\infty}$, see [6, p.106].

Before we give the solution of (6), we prove the following lemma which is about $(\sigma I+H_\tau)^{1/2}$ and $(\sigma I+H_\tau)^{-1/2}$.

Lemma 1. *Let $\sigma>0$ and H_τ be any circulant integral operator with kernel function $h_\tau\in L_2[-\tau,\tau]$ such that $(\sigma I+H_\tau)$ is positive definite. Let $\lambda_k=\sqrt{\tau}(h_\tau,u_k)_\tau=\sqrt{\tau}\int_0^\tau h_\tau(t)\overline{u_k(t)}dt$, $k\in\mathbb{Z}$ (the kth-eigenvalue of the operator H_τ), then*

$$(\sigma I+H_\tau)^{1/2}=\sqrt{\sigma}I+\tilde{H}_\tau, \tag{10}$$

where $\tilde{H}_\tau$ is a circulant operator with kernel function given by

$$\tilde{h}_\tau(t-s)=\sum_{k=-\infty}^{\infty}(\sqrt{\sigma+\lambda_k}-\sqrt{\sigma})u_k(t)\bar{u}_k(s). \tag{11}$$

Similarly,

$$(\sigma I + H_\tau)^{-1/2} = \frac{1}{\sqrt{\sigma}} I - \tilde{Q}_\tau, \tag{12}$$

where $\tilde{Q}_\tau$ is a circulant operator with kernel function given by

$$\tilde{q}_\tau(t-s) = \sum_{k=-\infty}^{\infty} \frac{\sqrt{\lambda_k + \sigma} - \sqrt{\sigma}}{\sqrt{\sigma(\sigma + \lambda_k)}} u_k(t)\overline{u_k(s)}. \tag{13}$$

Proof. Because $h_\tau \in L_2[-\tau, \tau]$, we have $\sum_{k=-\infty}^{\infty} \lambda_k^2 < \infty$. Note that $\sigma I + H_\tau$ is positive and $\lim_{|k|\to\infty} \lambda_k = 0$, we see that there exists $\delta > 0$ such that $\sigma + \lambda_k \geq \delta$ for all k. Thus

$$0 \leq \sqrt{\lambda_k + \sigma} - \sqrt{\sigma} \leq \frac{\lambda_k}{\sqrt{\sigma}} \quad \text{and} \quad 0 \leq \frac{\sqrt{\lambda_k + \sigma} - \sqrt{\sigma}}{\sqrt{\sigma(\sigma + \lambda_k)}} \leq \frac{\lambda_k}{\sigma\sqrt{\delta}}.$$

Hence $\tilde{h}_\tau$ and $\tilde{q}_\tau$ are functions in $L_2[-\tau, \tau]$. It is easily seen that $2\sqrt{\sigma}\tilde{H}_\tau + \tilde{H}_\tau\tilde{H}_\tau$ is a circulant operator. Moreover, its kernel function at the point (t,s) is given by

$$\begin{aligned} & 2\sqrt{\sigma} \sum_{k=-\infty}^{\infty} (\sqrt{\sigma + \lambda_k} - \sqrt{\sigma}) u_k(t)\bar{u}_k(s) + \sum_{k=-\infty}^{\infty} (\sqrt{\sigma + \lambda_k} - \sqrt{\sigma})^2 u_k(t)\bar{u}_k(s) \\ = & \sum_{k=-\infty}^{\infty} \lambda_k u_k(t)\bar{u}_k(s). \end{aligned}$$

Therefore, $2\sqrt{\sigma}\tilde{H}_\tau + \tilde{H}_\tau\tilde{H}_\tau = H_\tau$. Thus, $(\sqrt{\sigma}I + \tilde{H}_\tau)^2 = \sigma I + 2\sqrt{\sigma}\tilde{H}_\tau + \tilde{H}_\tau\tilde{H}_\tau = \sigma I + H_\tau$ and (10) follows. We prove (12) by proving that

$$(\sqrt{\sigma}I + \tilde{H}_\tau)^{-1} = \frac{1}{\sqrt{\sigma}} I - \tilde{Q}_\tau. \tag{14}$$

In fact, from (7), (11) and (13), it is easy to check that $-\sqrt{\sigma}\tilde{Q}_\tau + \frac{1}{\sqrt{\sigma}}\tilde{H}_\tau - \tilde{Q}_\tau\tilde{H}_\tau = 0$. Therefore,

$$(\frac{1}{\sqrt{\sigma}} I - \tilde{Q}_\tau)(\sqrt{\sigma}I + \tilde{H}_\tau) = I - \sqrt{\sigma}\tilde{Q}_\tau + \frac{1}{\sqrt{\sigma}}\tilde{H}_\tau - \tilde{Q}_\tau\tilde{H}_\tau = I.$$

Hence, (14) follows.

The following lemma gives the solution of (6).

Lemma 2. *Let $\nu_{m,n} = (A_\tau u_m, u_n)_\tau$. Suppose that (6) has a solution V_τ. Then the kernel function v_τ of V_τ is given by*

$$v_\tau(t-s) = \sum_{m=-\infty}^{\infty} \theta_m u_m(t)\overline{u_m(s)},$$

where $\{\theta_m + \sigma\}_{m=-\infty}^{\infty}$ satisfy

$$\frac{2(\sigma + \nu_{m,m})^2}{\theta_m + \sigma} + \sum_{\substack{n=-\infty \\ n\neq m}}^{\infty} \frac{|\nu_{m,n}|^2}{\theta_n + \sigma} = 2(\sigma + \nu_{m,m}), \quad m \in \mathbb{Z}. \tag{15}$$

Remark. Let $y_m = \frac{1}{\theta_m + \sigma}$, then $\{y_m\}_{m=-\infty}^{\infty}$ is the solution of the following linear system

$$2(\sigma + \nu_{m,m})^2 y_m + \sum_{\substack{n=-\infty \\ n\neq m}}^{\infty} |\nu_{m,n}|^2 y_n = 2(\sigma + \nu_{m,m}), \quad m \in \mathbb{Z}.$$

Proof. By Lemma 1, we have that $(\sigma I + H_\tau)^{-1/2} = \frac{1}{\sqrt{\sigma}} I - \tilde{Q}_\tau$. Therefore we consider minimizing the Hilbert-Schmidt norm of the operator

$$\begin{aligned} & I - (\frac{1}{\sqrt{\sigma}} I - \tilde{Q}_\tau)(\sigma I + A_\tau)(\frac{1}{\sqrt{\sigma}} I - \tilde{Q}_\tau) \\ = \quad & 2\sqrt{\sigma}\tilde{Q}_\tau - \sigma \tilde{Q}_\tau^2 - \frac{1}{\sigma} A_\tau + \frac{1}{\sqrt{\sigma}} A_\tau \tilde{Q}_\tau + \frac{1}{\sqrt{\sigma}} \tilde{Q}_\tau A_\tau - \tilde{Q}_\tau A_\tau \tilde{Q}_\tau \end{aligned} \tag{16}$$

over all circulant operators $\tilde{Q}_\tau$. Let $\tilde{q}_\tau(t-s) = \sum_{m=-\infty}^{\infty} \alpha_m u_m(t)\overline{u_m(s)}$. Note that $a_\tau(t-s) = \sum_{m,n=-\infty}^{\infty} \nu_{m,n} u_m(t)\overline{u_n(s)}$, by using (7) we have that the kernel function of $A_\tau \tilde{Q}_\tau$ at the point (t,s) is

$$\int_0^\tau a(t-\eta)\tilde{q}_\tau(\eta - s)d\eta = \sum_{m,n=-\infty}^{\infty} \nu_{m,n}\alpha_n u_m(t)\overline{u_n(s)}.$$

After simplifying, the kernel function of (16) at the point (t,s) is found to be

$$\sum_{m=-\infty}^{\infty} (1 - (\sigma + \nu_{m,m})(\tilde{\alpha}_m)^2) u_m(t)\overline{u_m(s)} - \sum_{\substack{m,n=-\infty \\ n\neq m}}^{\infty} \nu_{m,n}\tilde{\alpha}_m\tilde{\alpha}_n u_m(t)\overline{u_n(s)},$$

where $\tilde{\alpha}_m = \alpha_m - \frac{1}{\sqrt{\sigma}}$. From Lemma 1, we see that $y_m = (\theta_m + \sigma)^{-1} = (\alpha_m - \frac{1}{\sqrt{\sigma}})^2 = \tilde{\alpha}_m^2$. It easily follows from the above formula that

$$\begin{aligned} & |||I - (\sigma I + H_\tau)^{-1/2}(\sigma I + A_\tau)(\sigma I + H_\tau)^{-1/2}|||^2 \\ = \quad & \sum_{m=-\infty}^{\infty} ((\sigma + \nu_{m,m})y_m - 1)^2 + \sum_{\substack{m,n=-\infty \\ n\neq m}}^{\infty} y_m y_n |\nu_{m,n}|^2. \end{aligned}$$

It is clear that the above expression is minimized if and only if

$$2(\sigma + \nu_{m,m})^2 y_m - 2(\sigma + \nu_{m,m}) + \sum_{\substack{n=-\infty \\ n\neq m}}^{\infty} |\nu_{m,n}|^2 y_n = 0, \quad m \in \mathbb{Z}.$$

That is,

$$2(\sigma + \nu_{m,m})^2 y_m + \sum_{\substack{n=-\infty \\ n\neq m}}^{\infty} |\nu_{m,n}|^2 y_n = 2(\sigma + \nu_{m,m}), \quad m \in \mathbb{Z}.$$

Lemma 3. *[7, 3] Let $a(t) \in L_1(\mathbb{R}) \bigcap L_2(\mathbb{R})$, A_τ be defined as in (2) and $\mathcal{P}(A_\tau)$ be the optimal circulant preconditioner of A_τ. Then for each $\epsilon > 0$, there exist a $\tau^* > 0$ and an integer $\rho > 0$ such that for $\tau > \tau^*$,*

$$\mathcal{P}(A_\tau) - A_\tau = S_\tau + R_\tau,$$

where the self-adjoint operators S_τ and R_τ satisfy

$$||S_\tau|| < \epsilon \text{ and } \operatorname{rank}(R_\tau) \leq \rho.$$

Moreover, let $\nu_{m,n} = (Au_m, u_n)_\tau$, we have

$$\lim_{\tau\to\infty} \sup_{m\in\mathbb{Z}} \sum_{\substack{n=-\infty \\ n\neq m}}^{\infty} |\nu_{m,n}|^2 = 0. \tag{17}$$

Now, we prove that the genuine-optimal preconditioners V_τ produce clustered spectra.

Theorem 1. *Let $a(t) \in L_1(\mathbb{R}) \bigcap L_2(\mathbb{R})$ and A_τ be defined as in (2). If there exists $c > 0$ such that $||(\sigma I + \mathcal{P}(A_\tau))^{-1}|| \leq c$ for all τ, then there exists $\tau_1^* > 0$, such that for $\tau > \tau_1^*$, the genuine-optimal preconditioner V_τ for (1) is the unique solution of (15). Furthermore, for each $\epsilon > 0$, there exist a positive integer ρ and a $\tau^* > 0$ such that for each $\tau > \tau^*$, there exists a decomposition*

$$A_\tau - V_\tau = S_\tau + T_\tau,$$

where the self-adjoint operators S_τ and T_τ satisfy

$$||S_\tau||_2 < \epsilon \quad \textit{and} \quad \mathrm{rank}(T_\tau) \leq \rho.$$

It follows that at most ρ eigenvalues of $(\sigma I + V_\tau)^{-1/2}(\sigma I + A_\tau)(\sigma I + V_\tau)^{-1/2}$ lie outside the interval $(1 - \epsilon/\sigma, 1 + \epsilon/\sigma)$.

Proof. We first prove that the equation (15) has a unique solution. Let $e_{m,n} = \dfrac{|\nu_{m,n}|^2}{2(\sigma + \nu_{m,m})^2}$ for $m \neq n$ and $p_m = \dfrac{1}{\sigma + \nu_{m,m}}$, then (15) becomes

$$\mathbf{y} + \mathbf{E}\mathbf{y} = \mathbf{p}, \tag{18}$$

where $\mathbf{y} = \{y_m\}_{m=-\infty}^{\infty}$, $\mathbf{E} = \{e_{m,n}\}_{m,n=-\infty}^{\infty}$ ($e_{m,m} = 0$ for $m \in \mathbb{Z}$) and $\mathbf{p} = \{p_m\}_{m=-\infty}^{\infty}$. From $||(\sigma I + \mathcal{P}(A_\tau))^{-1}|| \leq c$, we see that $||\mathbf{E}||_\infty = \sup\limits_{m \in \mathbb{Z}} \sum\limits_{n=-\infty}^{\infty} |e_{m,n}| < \infty$. Moreover, by using (17) we get that there exists τ_1^* such that for $\tau > \tau_1^*$, $||\mathbf{E}||_\infty < 1/2$. It follows that the solution of (18) is given by

$$\mathbf{y} = \mathbf{p} - \sum_{n=1}^{\infty} \mathbf{E}^n \mathbf{p}. \tag{19}$$

Furthermore, $||\mathbf{y} - \mathbf{p}||_\infty \leq ||\mathbf{p}||_\infty / (1 - 1/2) = 2||\mathbf{p}||_\infty$.

Now, we prove the rest of the theorem by using Lemma 3 and proving that $\lim\limits_{\tau \to \infty} ||V_\tau - \mathcal{P}(A_\tau)||_2 = 0$, i.e., $\lim\limits_{\tau \to \infty} \sup\limits_m |\theta_m - \nu_{m,m}| = 0$. In fact, by (17) again, $\lim\limits_{\tau \to \infty} ||\mathbf{E}||_\infty = 0$. That is, for each $\epsilon > 0$, there exists $\tau_2^* > 0$ such that for $\tau > \tau_2^*$, we have $||\mathbf{E}||_\infty < \epsilon$. Therefore, from (19), we see that for $\tau > \max(\tau_1^*, \tau_2^*)$,

$$|y_m - p_m| < \frac{\epsilon}{1 - \epsilon} ||\mathbf{p}||_\infty, \quad m \in \mathbb{Z}.$$

This implies that $\lim\limits_{\tau \to \infty} y_m = p_m$ uniformly for m, i.e., $\lim\limits_{\tau \to \infty} \sup\limits_m |\theta_m - \nu_{m,m}| = 0$. Hence the result follows.

To end this section, we would like to emphasize that the genuine-optimal circulant preconditioner is different from the super-optimal circulant preconditioner. We recall that the eigenvalues of the genuine-optimal circulant preconditioner are given by θ_m, $m \in \mathbb{Z}$, where θ_m is the solution of (15). In the other hand, from Lemma 5 of [3], we have that the mth-eigenvalue of the super-optimal circulant preconditioner is given by $\left(\sum_{n=-\infty}^{\infty} |\nu_{m,n}|^2 + \sigma \nu_{m,m}\right) / (\sigma + \nu_{m,m})$.

3. Numerical Examples

In this section, two examples are tested. In both cases, we set $\sigma = 0.01$, $g(t) = 1$ for $t \leq 1024$ and $g(t) = 0$ for $t > 1024$, and the kernel functions are

1. $a_1(t) = \dfrac{1}{1 + e^{|t|}}$;

2. $a_2(t) = \dfrac{1}{1+t^2}$.

We discretize the integral operators by the rectangular rule with different step-size $\iota = \tau/N$ ($\iota \leq 0.25$). We recall that the discretized equation is given by (4) with $\mathbf{D}_p = \mathbf{I}_p$. The numerical results are shown in Table 1. All computations are done by MATLAB. We select the same random vectors as initial guesses. The stopping criterion is $||\mathbf{g} - (\sigma\mathbf{I}_p + \mathbf{A}_p)\mathbf{x}_p^{(k)}||_2 < 10^{-6}$, where $\mathbf{x}_p^{(k)}$ is the approximation solution of the kth iteration. In Table 1, the symbols O, W, S, G and I denote that the optimal, wrap-around, super-optimal, genuine-optimal preconditioner and no preconditioner are used respectively. We note that all the above preconditioners can be obtained within $O(p \log p)$ operations, see [10].

		$a_1(t) = 1/(1+e^{\|t\|})$				$a_2(t) = 1/(1+t^2)$			
τ	N	512	1024	2048	4096	512	1024	2048	4096
64	O	9	9	9	9	6	7	7	7
	W	9	9	9	9	6	6	6	6
	S	16	16	17	17	8	8	9	9
	G	10	10	10	10	7	7	7	7
	I	96	97	100	99	62	67	73	71
256	O	*	9	9	9	*	6	6	6
	W	*	8	9	9	*	6	6	6
	S	*	11	11	11	*	6	7	6
	G	*	9	9	9	*	6	6	6
	I	*	117	120	120	*	63	77	84
1024	O	*	*	*	9	*	*	*	6
	W	*	*	*	9	*	*	*	6
	S	*	*	*	10	*	*	*	6
	G	*	*	*	9	*	*	*	6
	I	*	*	*	128	*	*	*	66

Table1. The number of iterations for different preconditioners.

We see from Table 1 that the convergence of the PCG method of different preconditioner is the same when τ is large and that the number of iterations of the PCG method is independent of τ while the CG method without preconditioner is very slow especially for large τ.

4. Numerical Treatments for High Order Quadrature Rules

In [12], we constructed matrix preconditioners for

$$(\sigma\mathbf{I} + \mathbf{A}_p\mathbf{D}_p)\mathbf{x}_p = \mathbf{g}_p \tag{20}$$

based on operator preconditioners, where $\mathbf{D}_p \neq \mathbf{I}_p$. Our idea was obtaining the kernel function of M_τ from $\frac{1}{\sigma}I - M_\tau = (\sigma I + C_\tau)^{-1}$ and discretizing the operators M_τ and A_τ by the same quadrature rule. More precisely, we solved the following preconditioned system

$$(\frac{1}{\sigma}\mathbf{I}_p - \mathbf{D}_p^{\frac{1}{2}}\mathbf{M}_p\mathbf{D}_p^{\frac{1}{2}})(\sigma\mathbf{I}_p + \mathbf{D}_p^{\frac{1}{2}}\mathbf{A}_p\mathbf{D}_p^{\frac{1}{2}})\tilde{\mathbf{x}}_p = (\frac{1}{\sigma}\mathbf{I}_p - \mathbf{D}_p^{\frac{1}{2}}\mathbf{M}_p\mathbf{D}_p^{\frac{1}{2}})\tilde{\mathbf{g}}, \tag{21}$$

where $\tilde{\mathbf{x}}_p = \mathbf{D}_p^{\frac{1}{2}}\mathbf{x}_p$ and $\tilde{\mathbf{g}}_p = \mathbf{D}_p^{\frac{1}{2}}\mathbf{g}_p$.

Numerical results showed that (21) can be solved by the PCG method efficiently provided that the step-size ι is small enough, see the numbers of iterations in Tables 2–3. However, the PCG method may converge very slow if the step-size is not small enough. For example, let $\sigma = 0.01$ and $a(t) = e^{-0.1|t|}$. We observe that for the case $\tau = 256$ and $N = 1024$, the PCG

method converges slower than the CG method (the numbers of iterations of the PCG method and the CG method are 954 and 172 respectively, cf. Table 3). Recently, we found that for Simpson's rule, the matrix preconditioner $\frac{1}{\sigma}\mathbf{I}_p - \mathbf{D}_p^{\frac{1}{2}}\mathbf{M}_p\mathbf{D}_p^{\frac{1}{2}}$ may be *not* positive if the step-size is not small enough. For example, for the kernel function $a(t) = e^{-0.1|t|}$ with $\sigma = 0.01$, the minimal eigenvalue for the case τ=16 and N=256 is about -33.

In the following, we discuss how to solve (20) efficiently. We only discuss the use of Simpson's rule here. Our method can be modified to handle other high order rules. Let $N = 2n$ and $\mathbf{e}_i$ $(i = 0, 1, \cdots, 2n)$ be the $(2n+1)$-dimensional (i.e. p-dimensional) vectors defined as

$$[\mathbf{e}_i]_j = \begin{cases} 1, & j = i+1, \\ 0, & j \neq i+1, \end{cases}$$

and $\mathbf{P}$ be the permutation matrix defined as

$$\mathbf{P} = (\mathbf{e}_0, \mathbf{e}_2, \cdots, \mathbf{e}_{2n-2}, \mathbf{e}_1, \mathbf{e}_3, \cdots, \mathbf{e}_{2n-1}, \mathbf{e}_{2n}).$$

Let $t_j = \iota a(j\iota)$, $j = 0, \pm 1, \cdots, \pm N$. It can be easily proved that

$$\mathbf{P}^T\mathbf{A}_p\mathbf{P} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{u}_1 \\ \mathbf{A}_2^* & \mathbf{A}_1 & \mathbf{u}_2 \\ \mathbf{u}_1^* & \mathbf{u}_2^* & t_0 \end{pmatrix} \quad \text{and} \quad \mathbf{P}^T\mathbf{D}_p\mathbf{P} = \begin{pmatrix} \frac{1}{3} & & & \\ & \frac{2}{3}\mathbf{I}_{n-1} & & \\ & & \frac{4}{3}\mathbf{I}_n & \\ & & & \frac{1}{3} \end{pmatrix}, \tag{22}$$

where $\mathbf{u}_1 = (t_{2n}, t_{2n-2}, \cdots, t_2)^*$, $\mathbf{u}_2 = (t_{2n-1}, t_{2n-3}, \cdots, t_1)^*$, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are Toeplitz matrices given by

$$\mathbf{A}_1 = \begin{pmatrix} t_0 & t_{-2} & \cdots & t_{-(2n-2)} \\ t_2 & t_0 & \cdots & t_{-(2n-4)} \\ \vdots & & \ddots & \\ t_{2n-2} & t_{2n-4} & \cdots & t_0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{pmatrix} t_{-1} & t_{-3} & \cdots & t_{-(2n-3)} \\ t_1 & t_{-1} & \cdots & t_{-(2n-5)} \\ \vdots & & \ddots & \\ t_{2n-3} & t_{2n-5} & \cdots & t_{-1} \end{pmatrix}$$

respectively. Similarly, let $c_j = \iota h_\tau(j\iota)$, $j = 0, \pm 1, \cdots, \pm N$, we have

$$\mathbf{P}^T\mathbf{C}_p\mathbf{P} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{v}_1 \\ \mathbf{C}_2^* & \mathbf{C}_1 & \mathbf{v}_2 \\ \mathbf{v}_1^* & \mathbf{v}_2^* & c_0 \end{pmatrix}, \tag{23}$$

where $\mathbf{v}_1 = (c_{2n}, c_{2n-2}, \cdots, c_2)^*$, $\mathbf{v}_2 = (c_{2n-1}, c_{2n-3}, \cdots, c_1)^*$, and $\mathbf{C}_1$ and $\mathbf{C}_2$ are circulant matrices with the first columns given by $(c_0, c_2, \cdots, c_{2n-2})^T$ and $(c_{-1}, c_1, \cdots, c_{2n-2})^T$ respectively.

On the other hand, we have that the preconditioned system (4) is equivalent to

$$\mathbf{P}^T(\sigma\mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}\mathbf{P}\mathbf{P}^T(\sigma\mathbf{I}_p + \mathbf{A}_p\mathbf{D}_p)\mathbf{P}\mathbf{P}^T\mathbf{x}_p = \mathbf{P}^T(\sigma\mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}\mathbf{P}\mathbf{P}^T\mathbf{g}_p,$$

i.e.

$$\begin{aligned} & \left(\sigma\mathbf{I}_p + (\mathbf{P}^T\mathbf{C}_p\mathbf{P})(\mathbf{P}^T\mathbf{D}_p\mathbf{P})\right)^{-1}\left(\sigma\mathbf{I}_p + (\mathbf{P}^T\mathbf{A}_p\mathbf{P})(\mathbf{P}^T\mathbf{D}_p\mathbf{P})\right)\left(\mathbf{P}^T\mathbf{x}_p\right) \\ = \; & \left(\sigma\mathbf{I}_p + (\mathbf{P}^T\mathbf{C}_p\mathbf{P})(\mathbf{P}^T\mathbf{D}_p\mathbf{P})\right)^{-1}\left(\mathbf{P}^T\mathbf{g}_p\right). \end{aligned}$$

Let $\mathbf{E} = \left(\mathbf{P}^T\mathbf{D}_p\mathbf{P}\right)^{\frac{1}{2}} = \operatorname{diag}\left(\sqrt{\frac{1}{3}}, \sqrt{\frac{2}{3}}\mathbf{I}_{n-1}, \frac{2}{\sqrt{3}}\mathbf{I}_n, \sqrt{\frac{1}{3}}\right)$, $\mathbf{A} = \mathbf{P}^T\mathbf{A}_p\mathbf{P}$ (cf. (22)), and $\mathbf{C} = \mathbf{P}^T\mathbf{C}_p\mathbf{P}$ (cf. (23)). The above equation is equivalent to

$$(\sigma\mathbf{I}_p + \mathbf{E}\mathbf{C}\mathbf{E})^{-1}(\sigma\mathbf{I}_p + \mathbf{E}\mathbf{A}\mathbf{E})\left(\mathbf{E}\mathbf{P}^T\mathbf{x}_p\right) = (\sigma\mathbf{I}_p + \mathbf{E}\mathbf{C}\mathbf{E})^{-1}\left(\mathbf{E}\mathbf{P}^T\mathbf{g}_p\right). \tag{24}$$

It is still expensive to get the inverse of $(\sigma\mathbf{I}_p + \mathbf{E}\mathbf{C}\mathbf{E})$. In order that the cost per iteration of the PCG method is minimized, we replace the matrix $\mathbf{E}\mathbf{C}\mathbf{E}$ in (24) by

$$\tilde{\mathbf{C}} = \begin{pmatrix} \sqrt{\frac{2}{3}}\mathbf{I}_n & & \\ & \frac{2}{\sqrt{3}}\mathbf{I}_n & \\ & & 1 \end{pmatrix}\begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 & 0 \\ \mathbf{C}_2^* & \mathbf{C}_1 & 0 \\ 0 & 0 & c_0 \end{pmatrix}\begin{pmatrix} \sqrt{\frac{2}{3}}\mathbf{I}_n & & \\ & \frac{2}{\sqrt{3}}\mathbf{I}_n & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3}\mathbf{C}_1 & \frac{2\sqrt{2}}{3}\mathbf{C}_2 & 0 \\ \frac{2\sqrt{2}}{3}\mathbf{C}_2^* & \frac{4}{3}\mathbf{C}_1 & 0 \\ 0 & 0 & c_0 \end{pmatrix}.$$

That is, we replace $\mathbf{E}$ by $\operatorname{diag}\left(\sqrt{\frac{2}{3}}\mathbf{I}_n, \frac{2}{\sqrt{3}}\mathbf{I}_n, \sqrt{\frac{1}{3}}\right)$ and set the vectors $\mathbf{v}_i$ $(i = 1, 2)$ in (23) to zero vectors. Obviously, $\tilde{\mathbf{C}}$ is the sum of $\mathbf{ECE}$ and a matrix with rank less than 4. Thus, the preconditioned systems are

$$(\sigma\mathbf{I}_p + \tilde{\mathbf{C}})^{-1}(\sigma\mathbf{I}_p + \mathbf{EAE})\left(\mathbf{EP}^T\mathbf{x}_p\right) = (\sigma\mathbf{I}_p + \tilde{\mathbf{C}})^{-1}\left(\mathbf{EP}^T\mathbf{g}_p\right). \tag{25}$$

The inverse of $\sigma\mathbf{I}_p + \tilde{\mathbf{C}}$ can be obtained efficiently by using the well known property of circulant matrices that an $n \times n$ circulant matrix can be diagonalized by the Fourier matrix $\mathbf{F}_n$ of order n [5]. Let $\mathbf{C}_i = \mathbf{F}_n^*\Lambda_i\mathbf{F}_n$, $i = 1, 2$, where Λ_i is the diagonal matrix with diagonals given by $\mathbf{F}_n\mathbf{C}_i\mathbf{e}_0$. We note that $\mathbf{F}_n\mathbf{C}_i\mathbf{e}_0$ is obtained in $O(n\log n)$ operations by using the FFT to the first column of $\mathbf{C}_i$. Then

$$\begin{aligned} \sigma\mathbf{I}_p + \tilde{\mathbf{C}} &= \sigma\mathbf{I}_p + \begin{pmatrix} \frac{2}{3}\mathbf{F}_n^*\Lambda_1\mathbf{F}_n & \frac{2\sqrt{2}}{3}\mathbf{F}_n^*\Lambda_2\mathbf{F}_n & 0 \\ \frac{2\sqrt{2}}{3}\mathbf{F}_n^*\overline{\Lambda}_2\mathbf{F}_n & \frac{4}{3}\mathbf{F}_n^*\Lambda_1\mathbf{F}_n & 0 \\ 0 & 0 & c_0 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{F}_n^* & & \\ & \mathbf{F}_n^* & \\ & & 1 \end{pmatrix} \begin{pmatrix} \sigma + \frac{2}{3}\Lambda_1 & \frac{2\sqrt{2}}{3}\Lambda_2 & 0 \\ \frac{2\sqrt{2}}{3}\overline{\Lambda}_2 & \sigma + \frac{4}{3}\Lambda_1 & 0 \\ 0 & 0 & \sigma + c_0 \end{pmatrix} \begin{pmatrix} \mathbf{F}_n & & \\ & \mathbf{F}_n & \\ & & 1 \end{pmatrix}. \end{aligned}$$

Let $\Lambda_i = \operatorname{diag}(\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in})$, $i = 1, 2$ and

$$\begin{pmatrix} \sigma + \frac{2}{3}\lambda_{1j} & \frac{2\sqrt{2}}{3}\lambda_{2j} \\ \frac{2\sqrt{2}}{3}\overline{\lambda}_{2j} & \sigma + \frac{4}{3}\lambda_{1j} \end{pmatrix}^{-1} = \begin{pmatrix} \omega_{11}^{(j)} & \omega_{12}^{(j)} \\ \overline{\omega}_{12}^{(j)} & \omega_{22}^{(j)} \end{pmatrix}, \quad j = 1, 2, \cdots, n.$$

Let $\Omega_{ij} = \operatorname{diag}(\omega_{ij}^{(1)}, \cdots, \omega_{ij}^{(n)})$, $i, j = 1, 2$, then

$$(\sigma\mathbf{I}_p + \tilde{\mathbf{C}})^{-1} = \begin{pmatrix} \mathbf{F}_n^* & & \\ & \mathbf{F}_n^* & \\ & & 1 \end{pmatrix} \begin{pmatrix} \Omega_{11} & \Omega_{12} & 0 \\ \overline{\Omega}_{12} & \Omega_{22} & 0 \\ 0 & 0 & \frac{1}{\sigma + c_0} \end{pmatrix} \begin{pmatrix} \mathbf{F}_n & & \\ & \mathbf{F}_n & \\ & & 1 \end{pmatrix}.$$

From the above discussions, we see that $(\sigma\mathbf{I}_p + \tilde{\mathbf{C}})^{-1}$ can be obtained from $(\sigma\mathbf{I}_p + \tilde{\mathbf{C}})$ by two FFTs of n-dimensional vectors and the inverses of n matrices of size 2×2. The total cost is $O(n\log n)$ operations. Moreover, for any $(2n+1)$-dimensional vector $\mathbf{x}$, the main cost of the matrix-vector multiplication $(\sigma\mathbf{I}_p + \tilde{\mathbf{C}})^{-1}\mathbf{x}$ is 4 FFTs of n-dimensional vectors.

Finally, we show numerical results to compare the performance of the method introduced in [12] and our new method. The numerical results are shown in Tables 2–3. In the tables, the symbol M denotes that we use the PCG method to solve (21) and the symbol C denotes that we solve (25) by the PCG method. As in [12], we select two kernel functions $a_3(t) = 1/(1+t^2)$ and $a_4(t) = \mathrm{e}^{-0.1|t|}$. The right-hand side functions are chosen such that the corresponding solution for the Wiener-Hopf equation is

$$s(t) = \begin{cases} (16-t)^2, & 0 \le t \le 16, \\ 0, & t > 16. \end{cases}$$

In the test, we set $\sigma = 0.01$ and use the same random vectors as our initial guesses for all methods. The stopping criterion is

$$\left|\left|\left(\mathbf{P}^T\mathbf{g}_p\right) - \left(\sigma\mathbf{I}_p + (\mathbf{P}^T\mathbf{A}_p\mathbf{P})(\mathbf{P}^T\mathbf{D}_p\mathbf{P})\right)\left(\mathbf{P}^T\mathbf{x}_p\right)\right|\right|_2 = ||\mathbf{g}_p - (\sigma\mathbf{I}_p + \mathbf{A}_p\mathbf{D}_p)\mathbf{x}_p^{(k)}||_2 < 10^{-6},$$

where $\mathbf{x}_p^{(k)}$ is the approximation solution of the kth iteration. In the tables, the symbol "***" denotes that the number of iterations exceeds 1000.

From Tables 2–3, we observe that the numbers of iterations for C keep small for different τ and $\iota = \tau/N$. The PCG method for solving (21) is very efficient for small step-size ι but it may converge very slow if the step-size is not small enough. We conclude that our new method is more stable than the method given in [12].

	τ=64			τ=256			τ=1024		
N	M	C	I	M	C	I	M	C	I
512	10	13	101	57	11	98	*	*	*
1024	10	13	106	11	11	129	*	*	*
2048	10	12	106	10	12	132	58	10	102
4096	10	12	109	10	12	134	11	11	138

Table 2. Number of iterations for $a_3(t) = \frac{1}{1+t^2}$ and $\sigma = 0.01$ with Simpson's rule.

	τ=64			τ=256			τ=1024		
N	M	C	I	M	C	I	M	C	I
512	47	13	84	***	12	139	*	*	*
1024	24	12	86	954	11	172	*	*	*
2048	16	12	89	226	11	190	***	10	216
4096	11	12	89	80	11	199	***	10	271

Table 3. Number of iterations for $a_4(t) = \mathrm{e}^{-0.1|t|}$ and $\sigma = 0.01$ with Simpson's rule.

Acknowledgment. We would like to thank the referees for their valuable suggestions.

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