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ON THE CELL ENTROPY INEQUALITY FOR THE FULLY DISCRETE RELAXING SCHEMES*1)

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Abstract

In this paper we study the cell entropy inequality for two classes of the fully discrete relaxing schemes approximating scalar conservation laws presented by Jin and Xin in [7], and Tang in [14], which implies convergence for the one-dimensional scalar case.

Key words: The relaxing schemes, Entropy inequality, Conservation laws.

1. Introduction

This paper is interested in studies of the cell entropy inequality for two classes of the fully discrete relaxing schemes approximating the following scalar conservation laws

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{d}\frac{\partial f_i(u)}{\partial x_i}=0, \tag{1.1}$$

with initial data $u(0,x)=u_0(x),\ x=(x_1,...,x_d)$.

It is well known that the above Cauchy problem (1.1) may not always have a smooth global solution even though the initial data u_0 is smooth [8, 9]. Thus, we consider its weak solution so that the problem (1.1) might have a global solution allowing discontinuities (e.g. shock wave etc.). Moreover, the entropy condition must be imposed in order to single out a physically relevant solution(also called the entropy solution) [8, 9, 16, 19].

For the numerical approximation of the equation (1.1), the numerical entropy condition(e.g. the proper cell entropy inequality) must be imposed on it in order that the numerical solution can converge to the entropy solution of the above problem. However, the entropy condition seems difficult to prove for high-order finite difference schemes [13, 19].

Recently, Jin and Xin in [7] constructed a class of the relaxing schemes to approximate nonlinear conservation laws, by using the idea of the local relaxation approximation [1, 2, 3, 10]. The main advantage of their schemes is to use neither nonlinear Riemann solvers spatially nor nonlinear system of algebraic equations solvers temporally. However, the numerical experiments have shown that the implementation of their upwind relaxing schemes for general hyperbolic system is not easy, because of using linear Riemann solvers of a linear hyperbolic system with a stiff source term spatially and the choice of parameters.

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To overcome these drawbacks, we constructed a class of central relaxing schemes for systems of conservation laws in [14], which have the main advantage of the upwind relaxing schemes. In [14, 16] we also studied numerical entropy conditions for the above mentioned two classes of semi–discrete relaxing schemes to 1-D scalar conservation laws.

In this paper we will study the numerical entropy condition for the fully discrete relaxing schemes for scalar conservation laws with general flux. The paper is organized as follows. In section 2, we simply recall the construction of the relaxing system with a stiff source term and the relaxing schemes approximating the equation (1.1), presented by Jin et al.[7] and Tang [14], and establish the relation between the entropy pair for the relaxing system and the entropy pair for the system (1.1). In section 3, we discuss the entropy conditions for the fully discrete upwind relaxing scheme and central relaxing scheme. Finally, we conclude in section 4.

2. Preliminaries

In this section, we will review the construction of the relaxing system with a stiff source term and the relaxing schemes approximating the equation (1.1), which are presented by Jin et al.[7] and Tang [14].

2.1. The Relaxing System For 1-D Scalar Conservation Law

In the following, we will only consider single 1-D scalar conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{2.1}$$

with initial data

$$u(0,x) = u_0(x). \tag{2.2}$$

As in [7], a linear system with a stiff source term (hereafter called the *relaxing system*) can be constructed as follows:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \\ &\frac{\partial v}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{1}{\epsilon}(v - f(u)), \end{aligned} \tag{2.3}$$

where the small positive parameter ϵ is the relaxation rate, and a is a positive constant satisfying

$$| f'(u) | \le \sqrt{a}, \text{ for all } u \in R. \tag{2.4}$$

Remark. We may also consider the more general function $a(x,t)$ instead of the above constant a. The results in this paper are not limited by the above constant a.

In the small relaxation limit $\epsilon \to 0^+$, the relaxing system (2.3) can be approximated to leading order by the following *relaxed* equations

$$v = f(u), \tag{2.5a}$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0. \tag{2.5b}$$

The state satisfying (2.5a) is called the *local equilibrium*. By the Chapman-Enskog expansion [12], we can derive the following first order approximation to (2.3)

$$v = f(u) - \epsilon\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}, \tag{2.6a}$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon \frac{\partial}{\partial x}\{\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}\}. \tag{2.6b}$$

It is clear that the above second equation (2.6b) is dissipative under condition (2.4) (which is called as the *sub–characteristic condition* by Liu in [10]).

2.2. Entropy

Assume $(\eta,\ q)$ represent the entropy pairs for models (2.3) [3,4], where the convex function $\eta(u,v) \in C^2(R^2)$, then the following consistency condition holds:

$$(\eta_u, \eta_v)\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} = (q_u, q_v), \tag{2.7}$$

where η_u represent the partial differentiation with respect to u. Furthermore, we have

$$\begin{aligned} \eta_{uu} - a\eta_{vv} &= 0, \\ q_{uu} - aq_{vv} &= 0. \end{aligned} \tag{2.8}$$

Thus the general representation of the entropy pairs for (2.3) is

$$\begin{aligned} \eta &= G(v + \sqrt{a}u) + H(v - \sqrt{a}u), \\ q &= \sqrt{a}(G(v + \sqrt{a}u) - H(v - \sqrt{a}u)) \end{aligned} \tag{2.9}$$

for any functions G and H in $C^2(R)$.

It is easy to verify that η is convex function if and only if $H''G'' \geq 0$. At the equilibrium state $v = f(u)$, we have

$$\begin{aligned} \eta \mid_{v=f(u)} &= G(f(u) + \sqrt{a}u) + H(f(u) - \sqrt{a}u) \equiv \bar{\eta}(u), \\ q \mid_{v=f(u)} &= \sqrt{a}(G(f(u) + \sqrt{a}u) - H(f(u) + \sqrt{a}u)) \equiv \bar{q}(u), \end{aligned} \tag{2.10}$$

and expect to have

$$\eta_v \mid_{v=f(u)} = 0, \tag{2.11}$$

which means that η_v vanishes at the equilibrium state $v = f(u)$.

Then the pair $(\bar{\eta}(u), \bar{q}(u))$ forms an entropy pair for scalar conservation laws (2.1), that is

$$\bar{\eta}'(u)f'(u) = \bar{q}'(u), \text{ if } H'' \geq 0 \text{ and } G'' \geq 0.$$

At the equilibrium state $v = f(u)$, we expect to have the following entropy condition for equation (2.1)

$$\frac{\partial \bar{\eta}}{\partial t} + \frac{\partial \bar{q}}{\partial x} \leq 0. \tag{2.12}$$

and the corresponding numerical entropy condition for the numerical method for solving scalar conservation laws (2.1).

2.3. The Relaxing Schemes For 1-D Scalar Conservation Law

Now, we introduce the spatial grid points $x_j,\ j \in Z$ with the uniform mesh width $\Delta x = x_{j+1} - x_j$, i.e. $\Delta x, \lambda = \Delta t/\Delta x$ is a constant, and denote by $w_j(t)$ the approximate point value of $w(x,t)$ at $x = x_j$. As in [7, 14], the relaxing scheme is easily obtained by approximating numerically the system (2.3).

In this section and next section, we limit to consider the following first order semi-implicit relaxing schemes

$$\begin{aligned} &u_j^{n+1} - u_j^n + \lambda(v_{j+\frac{1}{2}}^n - v_{j-\frac{1}{2}}^n) = 0, \\ &v_j^{n+1} - v_j^n + \lambda a(u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n) = -\frac{\Delta t}{\epsilon}(v_j^{n+1} - f(u_j^{n+1})), \end{aligned} \tag{2.13}$$

where the numerical flux $u_{j+1/2}$ and $v_{j+1/2}$ will be defined in two ways specified below.

For the sake of simplicity in the presentation, define $\hat{w} = v + \sqrt{a}u$ and $\check{w} = v - \sqrt{a}u$, which imply $v = \frac{1}{2}(\hat{w} + \check{w})$ and $u = \frac{1}{2\sqrt{a}}(\hat{w} - \check{w})$.

Algorithm I. (the upwind relaxing scheme) The numerical flux in (2.13) is defined as:

$$\begin{aligned} v_{j+1/2} &= \frac{1}{2}(v_{j+1} + v_j) - \frac{\sqrt{a}}{2}(u_{j+1} - u_j), \\ u_{j+1/2} &= \frac{1}{2}(u_{j+1} + u_j) - \frac{1}{2\sqrt{a}}(v_{j+1} - v_j), \end{aligned} \tag{2.14a}$$

i.e.

$$\hat{w}_{j+1/2} = \hat{w}_j, \quad \check{w}_{j+1/2} = \check{w}_{j+1}. \tag{2.14b}$$

Algorithm II. (the central relaxing scheme) The numerical flux in (2.7) is defined as:

$$\begin{aligned} v_{j+1/2} &= \frac{1}{2}(v_{j+1} + v_j) - \frac{1}{2\lambda}(u_{j+1} - u_j), \\ u_{j+1/2} &= \frac{1}{2}(u_{j+1} + u_j) - \frac{1}{2a\lambda}(v_{j+1} - v_j), \end{aligned} \tag{2.15a}$$

i.e.

$$\begin{aligned} \hat{w}_{j+1/2} &= \frac{1}{2}(\hat{w}_j + \hat{w}_{j+1}) - \frac{1}{2\lambda\sqrt{a}}(\hat{w}_{j+1} - \hat{w}_j), \\ \check{w}_{j+1/2} &= \frac{1}{2}(\check{w}_j + \check{w}_{j+1}) + \frac{1}{2\lambda\sqrt{a}}(\check{w}_{j+1} - \check{w}_j). \end{aligned} \tag{2.15b}$$

Moreover, we often choose the special initial condition for the relaxing system (2.3):

$$\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x) \equiv f(u_0(x)). \end{aligned} \tag{2.16}$$

In doing so the state is already in equilibrium initially. On the other hand, to avoid any new boundary layers in solving boundary value problems, we can also impose the boundary conditions for v that are consistent to the local equilibrium.

3. The Cell Entropy Inequality For The Relaxing Schemes

For the relaxing schemes (2.13), we want to have the following numerical entropy inequality to guarantee convergence of numerical solution to the entropy solution.

$$\eta_j^{n+1} - \eta_j^n + \lambda(q_{j+\frac{1}{2}}^n - q_{j-\frac{1}{2}}^n) + \eta_v|_j^{n+1}\frac{\Delta t}{\epsilon}(v_j^{n+1} - f(u_j^{n+1})) \le 0. \tag{3.1}$$

Multiplying both sides of equation (2.13) by $(\eta_u, \eta_v)|_j^n$, we have

$$\begin{aligned} &\eta_u|_j^n(u_j^{n+1} - u_j^n) + \eta_v|_j^n(v_j^{n+1} - v_j^n) + \lambda\eta_u|_j^n(v_{j+\frac{1}{2}}^n - v_{j-\frac{1}{2}}^n) \\ &\quad + a\lambda\eta_v|_j^n(u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n) + \eta_v|_j^n\frac{\Delta t}{\epsilon}(v_j^{n+1} - f(u_j^{n+1})) = 0. \end{aligned} \tag{3.2}$$

In the following, we will consider the upwind scheme (2.14) and central scheme (2.15), respectively. Moreover, we will use the simple identity:

$$U'(a)(g(b)-g(a)) = G(b)-G(a)-\int_a^b (g(b)-g(\xi))U''(\xi)d\xi, \tag{3.3}$$

where

$$G(b) = \int^b U'(\xi)g'(\xi)d\xi, \tag{3.4}$$

and denote the left hand side of equation (3.1) and (3.2) by *LHS(3.1)* and *LHS(3.2)*, respectively.

3.1. The Upwind Relaxing Schemes

Now, substituting (2.9) into (3.2), then we get

$$\begin{aligned} LHS(3.2) = {} & G'(\hat{w}_j^n)(\hat{w}_j^{n+1}-\hat{w}_j^n) + H'(\check{w}_j^n)(\check{w}_j^{n+1}-\check{w}_j^n) \\ & +\lambda\sqrt{a}\big[G'(\hat{w}_j^n)(\hat{w}_{j+\frac{1}{2}}^n - \hat{w}_{j-\frac{1}{2}}^n) - H'(\check{w}_j^n)(\check{w}_{j+\frac{1}{2}}^n - \check{w}_{j-\frac{1}{2}}^n)\big] \\ & +\frac{\Delta t}{\epsilon}[G'(\hat{w}_j^n)+H'(\check{w}_j^n)](v_j^{n+1}-f(u_j^{n+1})). \end{aligned} \tag{3.5}$$

Using identity (3.3),

$$\begin{aligned} LHS(3.2) = {} & G(\hat{w}_j^{n+1}) - G(\hat{w}_j^n) - \int_{\hat{w}_j^n}^{\hat{w}_j^{n+1}} (\hat{w}_j^{n+1}-\xi)G''(\xi)d\xi \\ & +H(\check{w}_j^{n+1}) - H(\check{w}_j^n) - \int_{\check{w}_j^n}^{\check{w}_j^{n+1}} (\check{w}_j^{n+1}-\eta)H''(\eta)d\eta \\ & +\lambda\sqrt{a}\big[G(\hat{w}_{j+\frac{1}{2}}^n) - G(\hat{w}_j^n) - \int_{\hat{w}_j^n}^{\hat{w}_{j+\frac{1}{2}}^n} (\hat{w}_{j+\frac{1}{2}}^n-\xi)G''(\xi)d\xi \\ & +G(\hat{w}_j^n) - G(\hat{w}_{j-\frac{1}{2}}^n) + \int_{\hat{w}_j^n}^{\hat{w}_{j-\frac{1}{2}}^n} (\hat{w}_{j-\frac{1}{2}}^n-\xi)G''(\xi)d\xi \\ & -H(\check{w}_{j+\frac{1}{2}}^n) + H(\check{w}_j^n) + \int_{\check{w}_j^n}^{\check{w}_{j+\frac{1}{2}}^n} (\check{w}_{j+\frac{1}{2}}^n-\eta)H''(\eta)d\eta \\ & -H(\check{w}_j^n) + H(\check{w}_{j-\frac{1}{2}}^n) - \int_{\check{w}_j^n}^{\check{w}_{j-\frac{1}{2}}^n} (\check{w}_{j-\frac{1}{2}}^n-\eta)H''(\eta)d\eta\big] \\ & +\frac{\Delta t}{\epsilon}[G'(\hat{w}_j^{n+1}) - \int_{\hat{w}_j^n}^{\hat{w}_j^{n+1}} G''(\xi)d\xi](v_j^{n+1}-f(u_j^{n+1})) \\ & +\frac{\Delta t}{\epsilon}[H'(\check{w}_j^{n+1}) - \int_{\check{w}_j^n}^{\check{w}_j^{n+1}} H''(\eta)d\eta](v_j^{n+1}-f(u_j^{n+1})). \end{aligned} \tag{3.6}$$

Thus, we have

$$LHS(3.2) = LHS(3.1) + A, \tag{3.7a}$$

where

$$q_{j+\frac{1}{2}}^n = q(u_{j+\frac{1}{2}}^n, v_{j+\frac{1}{2}}^n) = \sqrt{a}[G(\hat{w}_{j+\frac{1}{2}}^n) - H(\check{w}_{j+\frac{1}{2}}^n)], \tag{3.7b}$$

and

$$
\begin{aligned}
A = & -\int_{\hat{w}_j^n}^{\hat{w}_j^{n+1}} [\hat{w}_j^{n+1} - \xi + \frac{\Delta t}{\epsilon}(v_j^{n+1} - f(u_j^{n+1}))]G''(\xi)d\xi \\
& - \int_{\check{w}_j^n}^{\check{w}_j^{n+1}} [\check{w}_j^{n+1} - \eta + \frac{\Delta t}{\epsilon}(v_j^{n+1} - f(u_j^{n+1}))]H''(\eta)d\eta \\
& - \lambda\sqrt{a}\big[\int_{\hat{w}_j^n}^{\hat{w}_{j+\frac{1}{2}}^n} (\hat{w}_{j+\frac{1}{2}}^n - \xi)G''(\xi)d\xi - \int_{\hat{w}_j^n}^{\hat{w}_{j-\frac{1}{2}}^n} (\hat{w}_{j-\frac{1}{2}}^n - \xi)G''(\xi)d\xi \big] \\
& + \lambda\sqrt{a}\big[\int_{\check{w}_j^n}^{\check{w}_{j+\frac{1}{2}}^n} (\check{w}_{j+\frac{1}{2}}^n - \eta)H''(\eta)d\eta - \int_{\check{w}_j^n}^{\check{w}_{j-\frac{1}{2}}^n} (\check{w}_{j-\frac{1}{2}}^n - \eta)H''(\eta)d\eta \big].
\end{aligned} \tag{3.7c}
$$

Using the definitions of u_j^{n+1}, v_j^{n+1}, $\hat{w}$ and $\check{w}$ in (2.7) and (2.8), we get

$$
\begin{aligned}
A = & -\int_{\hat{w}_j^n}^{\hat{w}_j^{n+1}} [\hat{w}_j^n - \lambda\sqrt{a}(\hat{w}_j^n - \hat{w}_{j-1}^n) - \xi]G''(\xi)d\xi + \lambda\sqrt{a}\int_{\hat{w}_j^n}^{\hat{w}_{j-1}^n} [\hat{w}_{j-1}^n - \xi]G''(\xi)d\xi \\
& - \int_{\check{w}_j^n}^{\check{w}_j^{n+1}} [\check{w}_j^n + \lambda\sqrt{a}(\check{w}_{j+1}^n - \check{w}_j^n) - \eta]H''(\eta)d\eta + \lambda\sqrt{a}\int_{\check{w}_j^n}^{\check{w}_{j+1}^n} [\check{w}_{j+1}^n - \eta]H''(\eta)d\eta \\
= & -\int_{\hat{w}_j^n}^{\hat{w}_j^{n+1}} (1-\lambda\sqrt{a})(\hat{w}_j^n - \xi)G''(\xi)d\xi + \lambda\sqrt{a}\int_{\hat{w}_j^{n+1}}^{\hat{w}_{j-1}^n} [\hat{w}_{j-1}^n - \xi]G''(\xi)d\xi \\
& - \int_{\check{w}_j^n}^{\check{w}_j^{n+1}} (1-\lambda\sqrt{a})(\check{w}_j^n - \eta)H''(\eta)d\eta + \lambda\sqrt{a}\int_{\check{w}_j^{n+1}}^{\check{w}_{j+1}^n} [\check{w}_{j+1}^n - \eta]H''(\eta)d\eta.
\end{aligned} \tag{3.8}
$$

By the inequality:

$$
\int^b (b-s)f(s)ds \geq 0,\ if\ f(s) \geq 0,
$$

it is easily verified if $\lambda\sqrt{a} \leq 1$, $A \geq 0$. Therefore, we have

Theorem 3.1. *For the first order upwind scheme (2.13) and (2.14), the entropy inequality (3.1) is valid, if CFL condition*

$$
\lambda\sqrt{a} \leq 1 \tag{3.9}
$$

is satisfied.

3.2. The Central Relaxing Schemes

Using the definitions of $\hat{w}_{j+\frac{1}{2}}^n$, $\hat{w}_{j-\frac{1}{2}}^n$, $\check{w}_{j+\frac{1}{2}}^n$ and $\check{w}_{j-\frac{1}{2}}^n$ in (2.15b), we have

$$
\begin{aligned}
LHS(3.2) = & G'(\hat{w}_j^n)(\hat{w}_j^{n+1} - \hat{w}_j^n) + H'(\check{w}_j^n)(\check{w}_j^{n+1} - \check{w}_j^n) \\
& + \lambda\sqrt{a}\{G'(\hat{w}_j^n)[\frac{1}{2}(\hat{w}_{j+1}^n - \hat{w}_{j-1}^n) - \frac{1}{2\lambda\sqrt{a}}(\hat{w}_{j+1}^n - \hat{w}_j^n) + \frac{1}{2\lambda\sqrt{a}}(\hat{w}_j^n - \hat{w}_{j-1}^n)]\} \\
& - \lambda\sqrt{a}\{H'(\check{w}_j^n)[\frac{1}{2}(\check{w}_{j+1}^n - \check{w}_{j-1}^n) + \frac{1}{2\lambda\sqrt{a}}(\check{w}_{j+1}^n - \check{w}_j^n) - \frac{1}{2\lambda\sqrt{a}}(\check{w}_j^n - \check{w}_{j-1}^n)]\} \\
& + \frac{\Delta t}{\epsilon}[G'(\hat{w}_j^n) + H'(\check{w}_j^n)](v_j^{n+1} - f(u_j^{n+1})).
\end{aligned} \tag{3.10}
$$

Again using identify(3.3), we get

$$\begin{aligned}LHS(3.2) = & G(\hat{w}_j^{n+1}) - G(\hat{w}_j^n) - \int_{\hat{w}_j^n}^{\hat{w}_j^{n+1}} (\hat{w}_j^{n+1} - \xi)G''(\xi)d\xi \\ & + H(\check{w}_j^{n+1}) - H(\check{w}_j^n) - \int_{\check{w}_j^n}^{\check{w}_j^{n+1}} (\check{w}_j^{n+1} - \eta)H''(\eta)d\eta \\ & + \lambda\sqrt{a}\{(\frac{1}{2} - \frac{1}{2\lambda\sqrt{a}})[G(\hat{w}_{j+1}^n) - G(\hat{w}_j^n) - \int_{\hat{w}_j^n}^{\hat{w}_{j+1}^n} (\hat{w}_{j+1}^n - \xi)G''(\xi)d\xi] \\ & + (\frac{1}{2} + \frac{1}{2\lambda\sqrt{a}})[G(\hat{w}_j^n) - G(\hat{w}_{j-1}^n) + \int_{\hat{w}_j^n}^{\hat{w}_{j-1}^n} (\hat{w}_{j-1}^n - \xi)G''(\xi)d\xi] \\ & - (\frac{1}{2} + \frac{1}{2\lambda\sqrt{a}})[H(\check{w}_{j+1}^n) - H(\check{w}_j^n) - \int_{\check{w}_j^n}^{\check{w}_{j+1}^n} (\check{w}_{j+1}^n - \eta)H''(\eta)d\eta] \\ & - (\frac{1}{2} - \frac{1}{2\lambda\sqrt{a}})[H(\check{w}_j^n) - H(\check{w}_{j-1}^n) + \int_{\check{w}_j^n}^{\check{w}_{j-1}^n} (\check{w}_{j-1}^n - \eta)H''(\eta)d\eta]\} \\ & + \frac{\Delta t}{\epsilon}[G'(\hat{w}_j^{n+1}) - \int_{\hat{w}_j^n}^{\hat{w}_j^{n+1}} G''(\xi)d\xi](v_j^{n+1} - f(u_j^{n+1})) \\ & + \frac{\Delta t}{\epsilon}[H'(\check{w}_j^{n+1}) - \int_{\check{w}_j^n}^{\check{w}_j^{n+1}} H''(\eta)d\eta](v_j^{n+1} - f(u_j^{n+1})).\end{aligned} \tag{3.11}$$

Thus, we have

$$LHS(3.2) = LHS(3.1) + B, \tag{3.12a}$$

where

$$\begin{aligned}q_{j+\frac{1}{2}}^n = & \frac{\sqrt{a}}{2}\{G(\hat{w}_{j+1}^n) + G(\hat{w}_j^n) - \frac{1}{\lambda\sqrt{a}}[G(\hat{w}_{j+1}^n - G(\hat{w}_j^n)] \\ & - H(\check{w}_{j+1}^n) - H(\check{w}_j^n) - \frac{1}{\lambda\sqrt{a}}[H(\check{w}_{j+1}^n) - H(\check{w}_j^n)]\},\end{aligned} \tag{3.12b}$$

and

$$\begin{aligned}B = & -\int_{\hat{w}_j^n}^{\hat{w}_j^{n+1}} [\hat{w}_j^{n+1} + \frac{\Delta t}{\epsilon}(v_j^{n+1} - f(u_j^{n+1})) - \xi]G''(\xi)d\xi \\ & - \int_{\check{w}_j^n}^{\check{w}_j^{n+1}} [\check{w}_j^{n+1} + \frac{\Delta t}{\epsilon}(v_j^{n+1} - f(u_j^{n+1})) - \eta]H''(\eta)d\eta \\ & + \frac{1-\lambda\sqrt{a}}{2}\int_{\hat{w}_j^n}^{\hat{w}_{j+1}^n} [\hat{w}_{j+1}^n - \xi]G''(\xi)d\xi + \frac{1+\lambda\sqrt{a}}{2}\int_{\hat{w}_j^n}^{\hat{w}_{j-1}^n} [\hat{w}_{j-1}^n - \xi]G''(\xi)d\xi \\ & + \frac{1+\lambda\sqrt{a}}{2}\int_{\check{w}_j^n}^{\check{w}_{j+1}^n} [\check{w}_{j+1}^n - \eta]H''(\eta)d\eta + \frac{1-\lambda\sqrt{a}}{2}\int_{\check{w}_j^n}^{\check{w}_{j-1}^n} [\check{w}_{j-1}^n - \eta]H''(\eta)d\eta.\end{aligned} \tag{3.12c}$$

Using the definitions of u_j^{n+1}, v_j^{n+1}, $\hat{w}$ and $\check{w}$ in (2.7) and (2.9), we get

$$\begin{aligned}B = & \frac{1-\lambda\sqrt{a}}{2}\int_{\hat{w}_j^{n+1}}^{\hat{w}_{j+1}^n} [\hat{w}_{j+1}^n - \xi]G''(\xi)d\xi + \frac{1+\lambda\sqrt{a}}{2}\int_{\hat{w}_j^{n+1}}^{\hat{w}_{j-1}^n} [\hat{w}_{j-1}^n - \xi]G''(\xi)d\xi \\ & + \frac{1+\lambda\sqrt{a}}{2}\int_{\check{w}_j^{n+1}}^{\check{w}_{j+1}^n} [\check{w}_{j+1}^n - \eta]H''(\eta)d\eta + \frac{1-\lambda\sqrt{a}}{2}\int_{\check{w}_j^{n+1}}^{\check{w}_{j-1}^n} [\check{w}_{j-1}^n - \eta]H''(\eta)d\eta.\end{aligned} \tag{3.13}$$

Therefore, we have

Theorem 3.2. *For the first order central scheme (2.13) and (2.15), the entropy inequality (3.1) is valid, if CFL condition (3.9) is satisfied.*

4. Conclusions

In this paper we have studied the numerical entropy conditions for two classes of the fully discrete relaxing schemes approximating scalar conservation laws, which implies convergence for one dimensional case. To our knowledge, the cell entropy inequality for the fully discrete high-resolution relaxing scheme seems difficult to be obtained.

In future, theoretical studies such as the convergence of the fully discrete relaxing schemes need to be considered.

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