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THE NUMERICAL SOLUTION OF FIRST KIND INTEGRAL EQUATION FOR THE HELMHOLTZ EQUATION ON SMOOTH OPEN ARCS*1)

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Abstract

Consider solving the Dirichlet problem of Helmholtz equation on unbounded region $R^2\backslash\Gamma$ with Γ a smooth open curve in the plane. We use simple-layer potential to construct a solution. This leads to the solution of a logarithmic integral equation of the first kind for the Helmholtz equation. This equation is reformulated using a special change of variable, leading to a new first kind equation with a smooth solution function. This new equation is split into three parts. Then a quadrature method that takes special advantage of the splitting of the integral equation is used to solve the equation numerically. An error analysis in a Sobolev space setting is given. And numerical results show that fast convergence is clearly exhibited.

Key words: Helmholtz equation, Quadrature method.

1. Introduction

The mathematical tratement of the scattering of time-harmonic acoustic or electromagnetic waves by an infinitely long semi-cylindrical obstacle with a smooth open contour cross-section $\Gamma \subset R^2$ leads to unbounded boundary value problems for the Helmholtz equation [3]

$$\begin{cases} \Delta w + k^2 w &= 0, \text{ in } R^2\backslash\Gamma, \\ w &= g, \text{ on } \Gamma, \\ \frac{\partial w}{\partial r} - ikw &= o(\frac{1}{\sqrt{r}}), \qquad r = |x| \to \infty, \end{cases} \tag{1.1}$$

with wave number $k > 0$.

In the single-layer approach one seeks the solution in the form

$$w(x) = \int_\Gamma K_0(|x-y|)\varphi(y)ds_y, y \in R^2\backslash\Gamma, \tag{1.2}$$

where ds_y is the element of arc length, and the fundamental solution to the Helmholtz equation is given by

$$K_0(|x-y|) := \frac{1}{2i} H_0^{(1)}(k|x-y|), x \neq y, \tag{1.3}$$

in terms of the Hankel function $H_0^{(1)}$ of order zero and of the first kind. It is known that

$$H_0^{(1)} = J_0 + iN_0, \tag{1.4}$$

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with Bessel function of order zero J_0 and Neumann function of order zero N_0

$$\begin{cases} J_0(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}(\frac{z}{2})^{2n} \\ N_0(z) &= \frac{2}{\pi}(\ln\frac{z}{2}+C)J_0(z)+\frac{2}{\pi}\sum_{n=1}^{\infty}\{\sum_{m=1}^{\infty}\frac{1}{m}\}\frac{(-1)^{n+1}}{(n!)^2}(\frac{z}{2})^{2n} \end{cases} \tag{1.5}$$

where $C = 0.57721\cdots$ is the Euler's constant.

The single-layer potential (1.2) solves the unbounded Dirichlet problem provided the density φ is a solution of the integral equation

$$\int_\Gamma K_0(|x-y|)\varphi(y)ds_y = g(x), x \in \Gamma, \tag{1.6}$$

This integral equation can be shown to be uniquely solvable provided the homogeneous Dirichlet problem for the case of domain bounded by open arc Γ admits only the trivial solution, that is, if the wave number k is not a Dirichlet eigenvalue for the negative Laplacian for the domain bounded by Γ. These eigenvalues are discrete and accumulate only at infinitely [3].

Let Γ have a parametrization

$$r(x) = (\xi(x), \eta(x)), -1 \le x \le 1, \tag{1.7}$$

with

$$|r'(x)| := \{[\xi'(x)]^2 + [\eta'(x)]\}^{\frac{1}{2}} \neq 0, -1 \le x \le 1, \tag{1.8}$$

To simplify the analysis, assume $r(x)$ is C^∞. Following [2, 14], we make the additional change of variable

$$t = \arccos(x), -1 \le x \le 1. \tag{1.9}$$

The equation (1.6) can now be written as

$$-\frac{1}{\pi}\int_0^\pi u(\tau)K(t,\tau)d\tau = f(\tau), 0 \le t \le \pi, \tag{1.10}$$

with

$$\begin{cases} a(t) = r(\cos t), \\ u(t) = \varphi(a(t))|r'(\cos t)|\sin t, \\ f(t) = g(a(t)), \\ K(t,\tau) = -\pi K_0(|a(t)-a(\tau)|). \end{cases} \tag{1.11}$$

Note that $a \in C^\infty$. From the expansions (1.5) we see that the kernel $K(t,\tau)$ can be written in the form

$$K(t,\tau) = (1 + K_1(t,\tau)|\cos t - \cos\tau|^2)\ln(\frac{2}{e}|\cos t - \cos\tau|) + K_2(t,\tau), \tag{1.12}$$

where

$$K_1(t,\tau) = -\frac{J_0(k|a(t)-a(\tau)|)-1}{|\cos t - \cos\tau|^2}, t \neq \tau, \tag{1.13}$$

$$K_2(t,\tau) = K(t,\tau) - (1 + K_1(t,\tau)|\cos t - \cos\tau|^2)\ln(\frac{2}{e}|\cos t - \cos\tau|), t \neq \tau. \tag{1.14}$$

With the assumption on $r(x)$, it can be shown that $K_1(t,\tau), K_2(t,\tau)$ are infinitely differentiable on t and τ and also 2π-periodic and even with respect to each variable. Furthermore, we have the diagonal terms

$$\begin{cases} K_1(t,t) = \frac{k^2}{4}|r'(\cos t)|, \\ K_2(t,t) = -\frac{\pi}{2i} - C - \ln(\frac{ke}{4}|r'(\cos t)|). \end{cases} \tag{1.15}$$

An advantage of the formulation (1.10), (1.11) is that the singularities in φ in the original problem now become explicit: for if the solution u of (1.10) is smooth, then φ automatically has the expected $x^{-\frac{1}{2}}$ type singularities at the two ends of the contour, because of the factor $\sin t$ in (1.11). And a smooth solution u of (1.10) arises naturally if g in (1.1) is smooth. This follows from the observation that the function a and f above are even, 2π-periodic functions; and if $g \in C^\infty(\Gamma)$ then $f \in C^\infty(R)$. If we define u to be an 2π-periodic and even function, then the resulting equation (1.10) can now be replaced by one half of the integral over a full period.

The equation (1.10) is split as

$$(L_e + A_e + B_e)u = f, \tag{1.16}$$

where

$$(L_e u)(t) := -\frac{1}{\pi}\int_0^\pi u(\tau)\ln(\frac{2}{e}|\cos t - \cos\tau|)d\tau, \tag{1.17}$$

$$(A_e u)(t) := -\frac{1}{\pi}\int_0^\pi u(\tau)K_1(t,\tau)|\cos t - \cos\tau|^2\ln(\frac{2}{e}|\cos t - \cos\tau|)d\tau, \tag{1.18}$$

$$(B_e u)(t) := -\frac{1}{\pi}\int_0^\pi u(\tau)K_2(t,\tau)d\tau. \tag{1.19}$$

Atkinson and Sloan [2] presented a discrete Galerkin method for Laplace equation with Γ an open curve. The logarithmic single-layer integral equation for the Laplace equation has the basic property that it can be split into two parts. But the situation changes if one wants to apply similar ideas for the Helmholtz equation. Here due to the more complicated structure of the fundamental solution to the Helmholtz equation in R^2 given by the Hankel function of order zero, after splitting off the integral equation similar with [2] one is still left with the remaining part containing a logarithmic singularity. Hence, both the setting up of an approximation method and its error analysis have to take into account this fact.

For the integral equation (1.6) for Helmholtz equation with Γ a smooth closed curve, a numerical method is presented and analyzed in [6]. For other results on the numerical solution of integral equation of the first kind for Laplace equation with Γ an open curve or smooth closed curve, see [1, 4, 5, 7, 8, 11]

Our approximation method will be a quadrature method (see [6, 9, 10] based on interpolatory trigonometric numerical integration rules which take proper care of the logarithmic singularities. In the case $K_1 = 0$ the method is similar to one proposed by [1]. In this paper, we shall set up our error and convergence analysis in a Sobolev space setting.

2. Prelimilaries

For our discussion of the operators L_e, we quote freely from [14]. Let H^p denote the Sobolev space of 2π-periodic functions

$$u = \sum_{m=-\infty}^{\infty}\hat{u}_m f_m \tag{2.1}$$

$$\hat{u}_m = \frac{1}{2\pi}\int_0^{2\pi} u(t)e^{-imt}dt, m = 0, \pm1, \pm2, \cdots, \tag{2.2}$$

whose Fourier coefficients $\hat{u}_m$ satisfy

$$||u||_p^2 := \sum_{m=-\infty}^{\infty}\max\{1, |m|^{2p}\}|\hat{u}_m|^2 < \infty \tag{2.3}$$

It is well-known that if $p > \frac{1}{2}$, then $H^p \subset C_p(2\pi)$, the space of 2π-periodic continuous functions. We define

$$H_e^p = \{u \in H^p | u(-\tau) = u(\tau)\}, \tag{2.4}$$

$$C_{p,e} = \{f \in C_p(2\pi) | f(-t) = f(t)\}. \tag{2.5}$$

Then (2.1)-(2.3) become, respectively, for $u \in H_e^p$:

$$u(t) = \hat{u}_0 + 2\sum_{m=1}^{\infty} \hat{u}_m \cos(mt) \tag{2.6}$$

$$\hat{u}_m = \frac{1}{\pi}\int_0^{\pi} u(t)\cos(mt)dt, m \geq 0 \tag{2.7}$$

$$||u||_p^2 := |\hat{u}_0|^2 + 2\sum_{m=0}^{\infty} m^{2p}|\hat{u}_m|^2 < \infty \tag{2.8}$$

Although (2.2) and (2.7) are different formulas, they give the same result when $u \in H_e^p$.

Define the following integral operators

$$Lu(t) := -\frac{1}{2\pi}\int_0^{2\pi} u(\tau)\ln|\frac{4}{e}\sin^2(\frac{t-\tau}{2})|d\tau \tag{2.9}$$

$$Au(t) := -\frac{1}{2\pi}\int_0^{2\pi} \sin^2(\frac{t-\tau}{2})\ln(\frac{4}{e}\sin^2(\frac{t-\tau}{2}))K_1(t,\tau)4\sin^2(\frac{t+\tau}{2})u(\tau)d\tau, \tag{2.10}$$

$$Bu(t) = -\frac{1}{2\pi}\int_0^{2\pi} K_2(t,\tau)u(\tau)d\tau, \tag{2.11}$$

for $u \in H_e^p$.

We know that operator L maps H_e^p onto H_e^{p+1} [1, 14]. L can also be written as the following Fourier series

$$Lu(t) = \hat{u}_0 + 2\sum_{m=1}^{\infty} c_m\hat{u}_m \cos(mt) \tag{2.12}$$

where

$$c_m = \frac{1}{\max\{1, m\}}, m \geq 0, \tag{2.13}$$

It is easily follows that

$$L : H_e^p \longrightarrow (^{1-1}_{onto}) H_e^{p+1}, \tag{2.14}$$

and

$$||Lu||_{p+1} = ||u||_p . u \in H_e^p \tag{2.15}$$

A Fourier series representation of L^{-1} follows easily from (2.9), (2.12), that is,

$$L^{-1}u(t) = \hat{u}_0 + 2\sum_{m=1}^{\infty} m\hat{u}_m \cos(mt). \tag{2.16}$$

We can show that the operator $A + B$ is bounded from H_e^p to H_e^{p+2} for all $p > 0$. Actually, it can be shown that $A + B$ is bounded from H_e^p to H_e^{p+3} and for all p.

Lemma 2.1. *The operator $A + B$ given by (2.10), (2.11) is bounded from H_e^p to H_e^{p+2} for all p.*

Proof. For the operator with the infinitely differentiable kernel this is trivial since it maps H_e^p boundedly into H_e^q for any pair p and q. Hence we need only to be concerned with the operator A. We can write

$$(Au)(t) = \gamma_0 a_0(t) + 2\sum_{m=1}^{\infty} \gamma_m a_m(t)\cos(mt) \tag{2.17}$$

where

$$\begin{aligned} &\gamma_0 = 0, \\ &\gamma_m = \tfrac{1}{4}(c_{m+1} - 2c_m + c_{m-1}).m > 0, \end{aligned} \tag{2.18}$$

$$a_m(t) = \frac{1}{2\pi}\int_0^{2\pi} K_1(t,\tau)4\sin^2(\frac{t+\tau}{2})u(\tau)e^{-imt}d\tau, \tag{2.19}$$

denote the Fourier coefficients of $K_1(t,\cdot)u$. This implies

$$|(Au)(t)| \leq 2\sum_{m=1}^{\infty} |\gamma_m||a_m(t)|, \tag{2.20}$$

and with the aid of the Schwarz inequality, using the fact that

$$\gamma_m = \frac{1}{2(m^2-1)m}, m \geq 2, \tag{2.21}$$

we can estimate

$$|(Au)(t)|^2 \leq 2\sum_{m=1}^{\infty} |\gamma_m|^2 m^6 \frac{|a_m(t)|^2}{m^6} = c||K_1(t,\cdot)4\sin^2(\frac{t+\cdot}{2})||_{-3}^2. \tag{2.22}$$

From this we obtain

$$||Au||_0 \leq c \sup_{0\leq t\leq 2\pi} ||K_1(t,\cdot)4\sin^2(\frac{t+\cdot}{2})||_{-3} \tag{2.23}$$

whence the boundedness of A from H^{-3} into H^0 follows.

By partial integration it can be seen that the derivative of Au is given by

$$\begin{aligned} |(Au)'(t)|^2 = \; & \int_0^{2\pi} F(t-\tau)\{\tfrac{\partial}{\partial t}\{K_1(t,\tau)4\sin^2(\tfrac{t+\tau}{2})\}u(\tau) \\ & +\tfrac{\partial}{\partial \tau}\{K_1(t,\tau)4\sin^2(\tfrac{t+\tau}{2})u(\tau)\}\}d\tau \end{aligned} \tag{2.24}$$

where

$$F(t) := \frac{1}{2\pi}\sin^2\frac{t}{2}\ln(\frac{4}{e}\sin^2\frac{t}{2}), 0 < t < 2\pi. \tag{2.25}$$

Hence $(Au)'$ is of the same structure as Au and therefore, analogously to (2.24), we have

$$\begin{aligned} ||(Au)'||_0 \leq \; & c \sup_{0\leq t\leq 2\pi} ||K_1(t,\cdot)4\sin^2(\tfrac{t+\cdot}{2})u'||_{-3} + c \sup_{0\leq t\leq 2\pi} \\ & \{||\tfrac{\partial}{\partial t}\{K_1(t,\cdot)4\sin^2(\tfrac{t+\cdot}{2})\}u||_{-3} + ||\tfrac{\partial}{\partial \tau}\{K_1(t,\cdot)4\sin^2(\tfrac{t+\cdot}{2})u\}||_{-3}\} \end{aligned} \tag{2.26}$$

This imples that A is bounded from H^{-1} into H^1. Repeating this argument, by induction, it follows that A: $H^{p-3} \to H^p$ is bounded for all integer values $p = 0, 1, \cdots$, and from this boundedness for all positive p follows by interpolation. Since same property holds for the adjoint of A, the boundedness for negative p follows by duality.

The main use of L, A, B in this paper lies in the fact that

$$L_e u = Lu, u \in H_e^p, \tag{2.27}$$

$$A_e u = Au, u \in H_e^p, \tag{2.28}$$

$$B_e u = Bu, u \in H_e^p, \tag{2.29}$$

Thus results on L, A, B can be used in investigating L_e, A_e, B_e. The integral equation (1.6) can be written as the following form

$$(L + A + B)u = f, u \in H_e^p, \tag{2.30}$$

3. Quadrature Method

It is known that L is an even pseudo-differential operator of order -1 and an isomorphism from H_e^p onto H_e^{p+1} for all $p > 0$, and that $A + B$ is bounded operator from H_e^p to H_e^{p+2} for $p > 0$ from section 2. From section 1 and (2.30) we also know $L + A + B$ has a trivial nullspace in H_e^p for $p > 0$. Since $L^{-1}(A + B) : H_e^p \to H_e^p$ is compact, by the Riesz-Fredholm theory the operator $L + A + B : H_e^p \to H_e^{p+1}$ also is an isomorphism.

We choose $n \in N$ and

$$t_j^{(n)} := jh, j = 0, 1, \cdots, n \tag{3.1}$$

with

$$h = \frac{\pi}{n}. \tag{3.2}$$

Let T_n denote the $(n+1)$-dimensional space of even trigonometric polynomials with the Lagrange basis functions given by

$$l_s^{(n)}(t) = \frac{1}{2n}\{1 + 2\sum_{m=1}^{n-1} \cos m(t - t_s^{(n)}) + \cos n(t - t_s^{(n)})\}, s = 0, n, \tag{3.3}$$

$$\begin{aligned} l_k^{(n)}(t) = \quad & \tfrac{1}{2n}\{2 + 2\textstyle\sum_{m=1}^{n-1}[\cos m(t - t_k^{(n)}) + \cos m(t + t_k^{(n)})] \\ & +[\cos n(t - t_k^{(n)}) + \cos n(t + t_k^{(n)})]\}, 1 \le k \le n - 1, \end{aligned} \tag{3.4}$$

Then we define following interpolation operator

$$P_n : C_{p,e} \to T_n$$

with the explicit expression

$$P_n u(t) = \sum_{k=0}^{n} u(t_k^{(n)}) l_k^{(n)}(t). \tag{3.5}$$

$p > \frac{1}{2}$ will be the least restriction since it makes sure that the nodal values of u exist and hence that $P_n u$ exists. Later we will give further restriction for p.

We have following Lemma

Lemma 3.1. *For the trigonometric interpolation P_n we have the error estimate*

$$||P_n u - u||_q \le ch^{p-q}||u||_p, \quad 0 \le q \le p, \frac{1}{2} < p, \tag{3.6}$$

for all $u \in H_e^p$ and some constant c (depending on p and q).

The proof is similar to the proof of an analogous result for all 2π-periodic funtions [6]. Hence this proof is omitted here.

Consider a fully discrete approximation method for the solution of (2.30) as follows.

$$P_n(Lu_n + Au_n + Bu_n) = P_n f \tag{3.7}$$

Since $Lv_n \in T_n$ if $v_n \in T_n$, it follows that

$$P_n L v_n = L v_n, v_n \in T_n. \tag{3.8}$$

Thus an equivalent formulation of (3.7) is given by: find $u_n \in T_n$ satisfying

$$L u_n + P_n(A u_n + B u_n) = P_n f \tag{3.9}$$

The following Lemma states the simple observation that our approximation scheme convergence for the unperturbed equation where $A + B = 0$. [6]

Lemma 3.2. *For each $u \in H_e^p$ with $f = Lu \in C_{p,e}$ there exist a unique solution $u_n \in T_n$ of*

$$(L u_n)(t_j^{(n)}) = f(t_j^{(n)}), j = 0, 1, \cdots, n \tag{3.10}$$

and we have the asymptotic error estimate

$$||u - u_n||_q \leq c h^{p-q} ||u||_p, \quad -1 \leq q \leq p, -\frac{1}{2} < p, \tag{3.11}$$

holds for some constant c (depending on p and q).

We now need to set up finite dimensional approximation for the operator L, A, B. We use the following interpolation quadrature rules by repalcing $g \in H_e^p$ by its trigonometric interpolation polynomial $P_n g \in T_n$ and then integrating exactly.

$$-\frac{1}{2\pi}\int_0^{2\pi} g(\tau)\ln(\frac{4}{e}\sin^2(\frac{t-\tau}{2}))d\tau \approx \sum_{k=0}^{n} R_k^{(n)}(t) g(t_k^{(n)}), \tag{3.12}$$

$$-\frac{1}{2\pi}\int_0^{2\pi} g(\tau)\sin^2(\frac{t-\tau}{2})\ln(\frac{4}{e}\sin^2(\frac{t-\tau}{2}))d\tau \approx \sum_{k=0}^{n} F_k^{(n)}(t) g(t_k^{(n)}), \tag{3.13}$$

$$-\frac{1}{2\pi}\int_0^{2\pi} g(\tau)d\tau \approx \frac{1}{2n}\sum_{k=0}^{n} d_k g(t_k^{(n)}), \tag{3.14}$$

with the weights

$$R_k^{(n)}(t) = -\frac{1}{2\pi}\int_0^{2\pi} l_k^{(n)}(\tau)\ln(\frac{4}{e}\sin^2(\frac{t-\tau}{2}))d\tau, 0 \leq k \leq n, \tag{3.15}$$

$$F_k^{(n)}(t) = -\frac{1}{2\pi}\int_0^{\pi} l_k^{(n)}(\tau) 2\sin^2(\frac{t-\tau}{2})\ln(\frac{4}{e}\sin^2(\frac{t-\tau}{2}))d\tau. 0 \leq k \leq n, \tag{3.16}$$

$$\begin{aligned} d_s &= -1, s = 0, n \\ d_k &= -2, 1 \leq k \leq n-1. \end{aligned} \tag{3.17}$$

From (3.4), (2.9), (2.10), (2.12), (2.17) we have the explicit formulas

$$R_s^{(n)}(t) = \frac{1}{2n}\{1 + 2\sum_{m=1}^{n-1} c_m \cos m(t - t_s^{(n)}) + c_n \cos n(t - t_s^{(n)})\}, s = 0, n, \tag{3.18}$$

$$\begin{aligned} R_k^{(n)}(t) = \quad & \tfrac{1}{2n}\{2 + 2\textstyle\sum_{m=1}^{n-1} c_m[\cos m(t - t_k^{(n)}) + \cos m(t + t_k^{(n)})] \\ & + c_n[\cos n(t - t_k^{(n)})) + \cos n(t + t_k^{(n)})]\}, 1 \leq k \leq n-1, \end{aligned} \tag{3.19}$$

$$F_s^{(n)}(t) = \frac{1}{2n}\{\gamma_0 + 2\sum_{m=1}^{n-1} \gamma_m \cos m(t - t_s^{(n)}) + \gamma_n \cos n(t - t_s^{(n)})\}, s = 0, n, \tag{3.20}$$

$$F_k^{(n)}(t) = \frac{1}{2n}\{2\gamma_0 + 2\sum_{m=1}^{n-1}\gamma_m[\cos m(t-t_k^{(n)}) + \cos m(t+t_k^{(n)})] + \gamma_n[\cos n(t-t_k^{(n)})) + \cos n(t+t_k^{(n)})]\}, 1 \le k \le n-1, \tag{3.21}$$

where the c_m, γ_m are given by (2.13), (2.18).

By (2.27)-(2.29), we apply the quadrature rules (3.12)-(3.14) to the integral equation (2.30), i.e., we apply (3.12) to $g = u$, (3.13) to $g = K_1(t,\cdot)4\sin^2(\frac{t+\cdot}{2})u$ and (3.14) to $g = K_2(t,\cdot)u$. Then we obtain the linear system

$$\sum_{k=0}^{n} u_n(t_k^{(n)})\{R_k^{(n)}(t_j^{(n)}) + F_k^{(n)}(t_j^{(n)})K_1(t_j^{(n)},t_k^{(n)})4\sin^2(\frac{t_j^{(n)}+t_k^{(n)}}{2}) + \frac{1}{2n}d_k K_2(t_j^{(n)},t_k^{(n)})\} = f(t_j^{(n)}), j = 0,\cdots,n. \tag{3.22}$$

With the numerical quadrature operators

$$(A_n u)(t) := \sum_{k=0}^{n} F_k^{(n)}(t)K_1(t,t_k^{(n)})4\sin^2(\frac{t+t_k^{(n)}}{2})u(t_k^{(n)}), \tag{3.23}$$

$$(B_n u)(t) := \frac{1}{2n}\sum_{k=0}^{n} d_k K_2(t,t_k^{(n)})u(t_k^{(n)}) \tag{3.24}$$

as approximation for A and B our scheme (3.9) is of the form

$$(Lu_n + A_n u_n + B_n u_n)(t_j^{(n)}) = f(t_j^{(n)}), j = 0,1,\cdots,n. \tag{3.25}$$

We have the following error estimate for the approximation $A_n + B_n$ for $A + B$ [6].

Lemma 3.3. *The estimate*

$$||(A+B)u - (A_n+B_n)u_n||_{q+1} \le ch^{p-q}||u||_{p-1}, \ \ 1 \le q \le p, \frac{3}{2} < p, \tag{3.26}$$

is valid for all $u \in T_n$ and some constant c (depending on p and q).

Now we will established our main convergence result. The proof is modelled after a corresponding error analysis in [12, 6]. However, it differs from [12] through the incorporation of discrete approximations $A_n + B_n$ for $A + B$. It also differs from [6] through the restriction for p, the weaker restriction for p given by [6] is invalid (see [13]).

Theorem 3.1. *For sufficiently small h and for each $u \in H_e^p$ with $f \in C_{p,e}$ there exists a unique solution $u_n \in T_n$ of (3.25) and we have the following error estimate*

$$||u - u_n||_q \le ch^{p-q}||u||_p, \ \ 1 \le q \le p, 2 < p, \tag{3.27}$$

for some constant c (depending on p and q).

Proof. Assume first that (3.25) has a solution, we have

$$Lu_n = P_n L(u + L^{-1}((A+B)u - (A_n+B_n)u_n)). \tag{3.28}$$

Then by Lemma 2.2 and the triangle inequality we can conclude that

$$\begin{aligned} & ||u + L^{-1}((A+B)u - (A_n+B_n)u_n) - u_n||_q \\ \le\ & ch^{p-q}||u + L^{-1}((A+B)u - (A_n+B_n)u_n)||_p \\ \le\ & ch^{p-q}\{||u||_p + ||L^{-1}(A+B)(u-u_n)||_p \\ & + ||L^{-1}((A+B)-(A_n+B_n))u_n||_p\}. \end{aligned} \tag{3.29}$$

Since $I + L^{-1}(A+B) : H^q \to H^q$ is an isomorphism (here I is an identity operator) we have

$$\begin{aligned} ||u-u_n||_q \leq\ & c||(I+L^{-1}(A+B))(u-u_n)||_q \\ \leq\ & c||u-u_n+L^{-1}((A+B)u-(A_n+B_n)u_n)||_q \\ & +c||L^{-1}((A+B)-(A_n+B_n))u_n||_q \end{aligned} \tag{3.30}$$

by (3.29) (3.30) we get

$$\begin{aligned} ||u-u_n||_q \leq\ & ch^{p-q}\{||u||_p+||L^{-1}(A+B)(u-u_n)||_p \\ & +||L^{-1}((A+B)-(A_n+B_n))u_n||_p\} \\ & +c||L^{-1}((A+B)-(A_n+B_n))u_n||_q. \end{aligned} \tag{3.31}$$

It is known that the operator $L^{-1}(A+B) : H^{p-1} \to H^p$ is bounded, then we have

$$||L^{-1}(A+B)(u-u_n)||_p \leq c||u-u_n||_{p-1}. \tag{3.32}$$

Using Lemma 3.3 and the mapping properties of L^{-1} we deduce that

$$\begin{aligned} & h^{p-q}||L^{-1}((A+B)-(A_n+B_n))u_n||_p+||L^{-1}((A+B)-(A_n+B_n))u_n||_q \\ \leq\ & ch^{p-q}||u_n||_{p-1} \leq ch^{p-q}(||u||_p+||u-u_n||_{p-1}) \end{aligned} \tag{3.33}$$

Combining (3.31)-(3.33) we obtain

$$||u-u_n||_q \leq ch^{p-q}(||u||_p+||(u-u_n)||_{p-1}) \tag{3.34}$$

for all p and q with $1 \leq q \leq p$ and $2 < p$. From this, choosing $q = p - \gamma$ with $0 < \gamma \leq 1$ we conclude that

$$||u-u_n||_{p-1} \leq ch^{\gamma}||u||_p \tag{3.35}$$

for sufficiently small h. Inserting this back into (3.34) concludes the proof of the error estimate (3.27).

From above we have assumed that (3.25) has a solution. But (3.27) with u repalced zero establishes the uniqueness of the solution of (3.25) for sufficiently small h, and from this the existence of a solution to the inhomogeneous equation (3.25) follows immediately by the Riesz-Fredholm alternative theory.

To construct the solution w of (1.1), we use the quadrature method of this paper to solve (1.6). Denote the approximation solution by φ_n.

Define

$$w_n(x) = \int_\Gamma K_0(|x-y|)\varphi_n(y)ds_y, \tag{3.36}$$

Easily,

$$\begin{aligned} w(x)-w_n(x) &= \int_\Gamma K_0(|x-y|)[\varphi(y)-\varphi_n(y)]ds_y \\ &= \int_0^\pi K(t,\tau)[u(\tau)-u_n(\tau)]ds_\tau \end{aligned} \tag{3.37}$$

where

$$u(t) = \varphi(a(t))|r'(\cos t)|\sin t, \tag{3.38}$$

$$u_n(t) = \varphi_n(a(t))|r'(\cos t)|\sin t, \tag{3.39}$$

We further approximate (3.36) by evaluating the integral numerically, using the trapezoidal rule with q nodes. Denote the resulting approximation by $w_{n,q}$. Thus by using Sobolev inbedding theorem and (3.27), it can be shown that for each x, and for all sufficiently large n,

$$\begin{aligned} |w(x)-w_{n,q}(x)| &\leq ||u-u_n||_\infty \int_0^\pi |K(t,\tau)|ds_\tau \\ &\leq d_q(x)h^{p-1}. \end{aligned} \tag{3.40}$$

The constant $d_q(x)$ approaches ∞ as x approaches Γ, because the integrand in (3.36) becomes singular in this case. For x near Γ, it is better to use $w_{n,q}$ with q much larger than n.

The far-field pattern w_∞ is defined by the asymptotic behaviour of the scattered wave

$$w(x) = \frac{e^{ix|x|}}{\sqrt{|x|}}\{w_\infty(\hat{x} + O(\frac{1}{|x|})\}, |x| \to \infty, \tag{3.41}$$

uniformly in all direction $\hat{x} = \frac{x}{|x|}$ [3].

From the asymptotics

$$H_0^1(x) = \sqrt{\frac{2}{\pi x}} e^{i(x-\frac{\pi}{4})}\{1 + O(\frac{1}{x}\}, x \to \infty, \tag{3.42}$$

for the Hankel function, we see that the far-field pattern of the single-layer potential (1.2) is given by

$$w_\infty(\hat{x}) = c\int_\Gamma e^{-ikx\cdot y}\varphi(y)ds_y \tag{3.43}$$

where $c = -\frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi k}}$.

The integral (3.43) may be evaluated by the trapezoidel rule as above, hence it can be shown that for each x, and for all sufficiently large n,

$$|w_\infty(\hat{x}) - w_{\infty,n,q}(\hat{x})| \leq ch^{p-1}. \tag{3.44}$$

4. Numerical Examples

We consider the scattering of a plane wave w^i by a sound-soft semi-cylinder with following smooth open contour cross section with boundary Γ. Here we give two examples, to illustrate the numerical method studied in section 3. The forward directions are $d_1 = (0, 1)$ and $d_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(a) Let Γ be a straight-line segment of length 2, e.g.,

$$\Gamma = \{(s, 0), -1 \leq s \leq 1\}.$$

Table 1 gives some approximate values for the far-field pattern $w_\infty(d_1)$ and $w_\infty(d_2)$

(b) Let Γ be the upper half of the unit circle

$$x^2 + y^2 = 1,$$

with $y > 0$. For a parameterization, use

$$r(x) = (-\sin(\frac{\pi}{2}x), \cos(\frac{\pi}{2}x)), -1 \leq s \leq 1.$$

In table 2, we give some approximate values for the far-field pattern $w_\infty(d_1)$ and $w_\infty(d_2)$. Note that the fast convergence is clearly exhibited.

Table 1

	N	Re$w_\infty(d_1)$	Im$w_\infty(d_1)$	Re$w_\infty(d_2)$	Im$w_\infty(d_2)$
k=1	8	.367770D-1	.667942	.144435	.839851
	16	.367231D-1	.668025	.144399	.840132
	32	.367100D-1	.668044	.144390	.840200
	64	.367068D-1	.668049	.144388	.840217
	128	.367061D-1	.668050	.144388	.840221
	256	.367059D-1	.668050	.144388	.840222
k=3	8	.212585	.389334	.146669	.541306
	16	.212777	.390049	.146945	.540683
	32	.212817	.390211	.147025	.540505
	64	.212826	.390247	.147049	.540452
	128	.212828	.390256	.147057	.540437
	256	.212828	.390258	.147059	.540432
k=5	8	.174779	.304319	.181852	.450536
	16	.175597	.305401	.181486	.448180
	32	.175791	.305586	.181404	.447525
	64	.175841	.305612	.181384	.447335
	128	.175854	.305614	.181380	.447280
	256	.175857	.305613	.181379	.447265

Table 2

	N	Re$w_\infty(d_1)$	Im$w_\infty(d_1)$	Re$w_\infty(d_2)$	Im$w_\infty(d_2)$
k=1	8	.278608	.732781	.291279	.860374
	16	.278383	.732553	.291089	.860609
	32	.278329	.732497	.291043	.860665
	64	.278316	.732483	.291032	.860678
	128	.278312	.732479	.291029	.860680
	256	.278312	.732478	.291028	.860681
k=3	8	.314541	.668848	.320227	.676005
	16	.312063	.663895	.320640	.674064
	32	.311583	.662600	.320703	.673130
	64	.311458	.662247	.320722	.672865
	128	.311425	.662150	.320728	.672790
	256	.311417	.662124	.320730	.672769
k=5	8	.297184	.666579	.301187	.669639
	16	.309291	.655751	.333977	.691056
	32	.309379	.651606	.334170	.688028
	64	.309411	.650430	.334216	.687122
	128	.309422	.650097	.334227	.686855
	256	.309426	.650004	.334230	.686778

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