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LONG-TIME BEHAVIOR OF FINITE DIFFERENCE SOLUTIONS OF A NONLINEAR SCHRÖDINGER EQUATION WITH WEAKLY DAMPED*1)

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Abstract

A weakly demped Schrödinger equation possessing a global attractor are considered. The dynamical properties of a class of finite difference scheme are analysed. The exsitence of global attractor is proved for the discrete system. The stability of the difference scheme and the error estimate of the difference solution are obtained in the autonomous system case. Finally, long-time stability and convergence of the class of finite difference scheme also are analysed in the nonautonomous system case.

Key words: Global attractor, Nonlinear Schrödinger equation, Finite difference method, Stibility and convergence.

1. Introduction

The nonlinear schrödinger equation with weakly damped

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+g(|u|^2)u+i\alpha u=f \quad x\in\Omega, t>0 \tag{1.1}$$

where $i=\sqrt{-1}, \alpha>0$, together with appropriate boundary and initial conditions, is arised in many physical fields. The existence of an attractor is one of the most important characteristics for a dissipative system. The long-time dynamics is completely determined by the attractor of the system. J.M. Ghidaglia[1] studied the long-time behavior of the nonlinear Schrödinger equation (1.1) and proved the existence of a compact global attractor $\mathcal{A}$ in $H^1(\Omega)$ which has the finite Hausdroff and fractal dimension under the conditions (1.4) and (1.5) in the follows. Guo Boling[6] construct the approximate intertial manifolds for the equation (1.1) and the order of approximation of these manifolds to the global attractor were derived. At the same time, a semidiscrete finite difference method of the equation was discussed by Yin Yan[7]. In this paper, completely discrete scheme is discussed by finite diffenerce method for the equation with initial condition

$$u(x,0)=u_0(x),\ x\in\Omega \tag{1.2}$$

and Dirichlet boundary conditions:

$$u(0,t)=u(L,t)=0, t\in R^+, \tag{1.3}$$

where $\Omega=(0,L), f\in L^2(\Omega), g(s)(0\le s<\infty)$ is a real valued smooth function that satisfies

$$\lim_{s\to+\infty}\frac{G_+(s)}{s^3}=0 \tag{1.4}$$

and

$$\lim_{s\to+\infty} sup\frac{h(s)-\omega G(s)}{s^3}\le 0,\ for\ some\ \omega>0. \tag{1.5}$$

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where $h(s) = sg(s), G(s) = \int_0^s g(\sigma)d\sigma$ and $G_+(s) = max\{G(s), 0\}$.

2. Finite Difference Scheme

First, let us divide the domain $Q_\infty = [0, L] \times [0, \infty)$ into small grids by the parallel lines $x = x_j (j = 0, 1, \cdots, J)$ and $t = t_n (n = 0, 1, \cdots)$, where J is a positive integer, $x_j = jh, Jh = L, t_n = n\triangle t (j = 0, 1, \cdots, J; n = 0, 1, \cdots)$, h and $\triangle t$ are the spatial and temporal mesh lengths respectively. Denote the complex value discrete functions on the grid points $x_0, x_1, \cdots, x_J$ by $\phi, \psi, \cdots$. We employ $\triangle_+, \triangle_-$ and δ_h to denote the forward difference, the backward difference and the forward difference quotient operaters respectively, i.e.

$$\triangle_+\phi_j = \phi_{j+1} - \phi_j, \triangle_-\phi_j = \phi_j - \phi_{j-1}, \delta_h\phi_j = \frac{\triangle_+\phi_j}{h},$$

We introduce the discrete L^2 inner product

$$(\phi, \psi)_h = \sum_{j=0}^{J} \phi_j \overline{\psi}_j h$$

and the discrete H^1 inner product

$$(\phi, \psi)_{1,h} = \sum_{j=0}^{J-1} \delta_h\phi_j \overline{\delta_h\psi}_j h,$$

together with the associated norms

$$\|\phi\|_h = (\phi, \phi)_h^{\frac{1}{2}}, \qquad \|\phi\|_{1,h} = (\phi, \phi)_{1,h}^{\frac{1}{2}}.$$

Finally, let

$$\|\phi\|_\infty = \max_{0 \le j \le J} |\phi_j|.$$

It is convenient to let L_h^2 and H_h^1 be the normed vector space $\{C^{J+1}, \|\bullet\|_h\}$ and $\{C_0^{J+1}, \|\bullet\|_{1,h}\}$ respectively, here $C_0^{J+1} = \{\phi \in C^{J+1}; \phi_0 = \phi_J = 0\}$. We easily obtain by simple calculation

Lemma 2.1. *For any discrete functions $\{\phi_j\}_0^J$ and $\{\psi_j\}_0^J$, there is the relation*

$$\sum_{j=1}^{J-1} \phi_j \triangle_+\triangle_-\psi_j = -\sum_{j=0}^{J-1} (\triangle_+\phi_j)(\triangle_+\psi_j) - \phi_0\triangle_+\psi_0 + \phi_J\triangle_-\psi_J.$$

Lemma 2.2. *For any discrete function $\{\phi_j\}_0^J, \phi_0 = \phi_J = 0$, the following inequality is valid*

$$\|\phi\|_\infty \le \|\phi\|_{1,h}^{\frac{1}{2}} \|\phi\|_h^{\frac{1}{2}}.$$

Proof. From $\phi_0 = \phi_J = 0$ we can see easily the relations

$$\phi_m^2 = \sum_{j=0}^{m-1} (\phi_{j+1} + \phi_j)\delta_h\phi_j h, \ \phi_m^2 = -\sum_{j=m}^{J-1} (\phi_{j+1} + \phi_j)\delta_h\phi_j h,$$

by cauchy inequality, Lemma 2.2 is proved immediately.

Lemma 2.3. *If functions $f(t), f'(t) \in C(R^+) \cap L^2(R^+)$, the following inequality is valid*

$$\triangle t \sum_{k=0}^{\infty} |f(t_k)|^2 \le \triangle t \int_0^\infty |f'(t)|^2 dt + (1 + \triangle t) \int_0^\infty |f(t)|^2 dt.$$

Proof. By the integrating by parts formula, we derive

$$\triangle t |f(t_{k-1})|^2 = \int_{t_{k-1}}^{t_k} 2f(t)f'(t)(t - t_k)dt + \int_{t_{k-1}}^{t_k} f^2(t)dt,$$

then applying Cauchy inequality, summing them up for k from 1 to ∞. This completes the proof.

In order to prove long-time stability and convergence of difference scheme, we need the following discrete Gronwall lemma

Lemma 2.4. *Let* $\{y^n\}, \{g_1^n\}, \{g_2^n\}, \{h^n\}$ *be four nonnegative discrete functions satisfy*

$$\frac{y^{n+1}-y^n}{\triangle t} \le g_2^{n+1}y^{n+1} + g_1^n y^n + h^n, n = 0, 1, \cdots, \tag{2.1}$$

and there exists a constant $\gamma > 0$ *such that for all* $n \ge 0, 1 - \triangle t g_2^n \ge \gamma$. *Then*

$$y^n \le y^0 \exp(\frac{1}{\gamma}\triangle t \sum_{k=1}^{n} g_2^k)\exp(\triangle t \sum_{k=0}^{n-1} g_1^k)$$

$$+ \triangle t \sum_{i=0}^{n-1} h^i \exp(\frac{1}{\gamma}\triangle t \sum_{k=i+1}^{n} g_2^k)\exp(\triangle t \sum_{k=i+1}^{n} g_1^k), n = 0, 1, \cdots.$$

Proof. the relation (2.1) can rewritten as

$$y^{n+1} \le (1-\triangle t g_2^{n+1})^{-1}(1+\triangle t g_1^n)y^n + \triangle t(1-\triangle t g_2^{n+1})^{-1}h^n, n = 0, 1, \cdots.$$

Using frequently the inequality, we derive

$$y^n \le y^0 \prod_{k=1}^{n}(1-\triangle t g_2^k)^{-1}\prod_{k=0}^{n-1}(1+\triangle t g_1^k) + \triangle t \sum_{i=0}^{n-1} h^i \prod_{k=i+1}^{n}(1-\triangle t g_2^k)^{-1}\prod_{k=i+1}^{n}(1+\triangle t g_1^k).$$

On the other hand, since $-ln(1-x) \le \frac{x}{1-x}, \forall x \in [0,1)$, and $1 + x \le e^x, \forall x \in R$, we derive

$$\prod_{k=i+1}^{n+1}(1-\triangle t g_2^k)^{-1} = \exp(-\sum_{k=i+1}^{n+1} ln(1-\triangle t g_2^k))$$

$$\le \exp(\sum_{k=i+1}^{n+1}\frac{\triangle t g_2^k}{1-\triangle t g_2^k}) \le \exp(\frac{1}{\gamma}\triangle t \sum_{k=i+1}^{n+1} g_2^k), n = 0, 1, \cdots,$$

and

$$\prod_{k=i+1}^{n}(1+\triangle t g_1^k) \le \prod_{k=i+1}^{n}\exp(\triangle t g_1^k) = \exp(\triangle t \sum_{k=i+1}^{n} g_1^k), n = 0, 1, \cdots.$$

Therefore

$$y^n \le y^0 \exp(\frac{1}{\gamma}\triangle t \sum_{k=1}^{n} g_2^k)\exp(\triangle t \sum_{k=0}^{n-1} g_1^k)$$

$$+ \triangle t \sum_{i=0}^{n-1} h^i \exp(\frac{1}{\gamma}\triangle t \sum_{k=i+1}^{n} g_2^k)\exp(\triangle t \sum_{k=i+1}^{n} g_1^k), n = 0, 1, \cdots.$$

In this paper we only consider the case

$$g(s) = \kappa s$$

which is physically motivated, and it satisfies the conditions (1.4) and (1.5). We construct the finite difference system

$$i\frac{e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1} - e^{-\frac{\alpha}{2}\triangle t}\phi_j^n}{\triangle t} + \frac{1}{2h^2}\triangle_+\triangle_-(e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)$$

$$+ \frac{\kappa}{4}(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2 + |e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi_j^n) = f_j, \tag{2.2}$$

$$j = 1, 2, \cdots, J-1, n = 0, 1, \cdots.$$

The finite difference boundary conditions are as follows

$$\phi_0^n = \phi_J^n = 0, n = 0, 1, \cdots. \tag{2.3}$$

The initial condition is as

$$\phi_j^0 = u(x_j), j = 0, 1, \cdots, J. \tag{2.4}$$

Now we are going to prove the existence of the solutions $\phi_j^{n+1}(j=0,1,\cdots,J)$ for the finite difference system (2.2) with the boundary conditions (2.3). For any (J+1)-dimensional vector $\phi=(\phi_0,\phi_1,\cdots,\phi_J)$, let us construct (J+1)-dimensional vector $\Phi=(\Phi_0,\Phi_1,\cdots,\Phi_J)$ as follows

$$i(e^{\frac{\alpha}{2}\triangle t}\Phi_j-e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)=\lambda\frac{\triangle t}{2h^2}\triangle_+\triangle_-(e^{\frac{\alpha}{2}\triangle t}\phi_j+e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)$$

$$+\lambda\kappa\frac{\triangle t}{4}(|e^{\frac{\alpha}{2}\triangle t}\phi_j|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\phi_j+e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)=\lambda\triangle tf_j,$$

$$j=1,2,\cdots,J-1.$$

where $\Phi_0=\Phi_J=0, 0\le\lambda\le1$. It defines a mapping $\Phi=T_\lambda\phi$ of C^{J+1} into itself. In order to obtain the existence of the solutions for the finite difference system (2.2) with boundary conditions (2.3), it is sufficient to prove the uniform boundedness for all the possible fixed point of the mapping with respect to the parameter $0\le\lambda\le1$.

Multiplying $(e^{\frac{\alpha}{2}\triangle t}\overline{\Phi}_j+e^{-\frac{\alpha}{2}\triangle t}\overline{\phi}_j^n)h$ and the follow equation

$$i(e^{\frac{\alpha}{2}\triangle t}\Phi_j-e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)=\lambda\frac{\triangle t}{2h^2}\triangle_+\triangle_-(e^{\frac{\alpha}{2}\triangle t}\Phi_j+e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)$$

$$+\lambda\kappa\frac{\triangle t}{4}(|e^{\frac{\alpha}{2}\triangle t}\Phi_j|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\Phi_j+e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)=\lambda\triangle tf_j,$$

and summing them up for j from 1 to $J-1$, then taking the imaginary part, we have

$$\|e^{\frac{\alpha}{2}\triangle t}\Phi\|_h^2-\|e^{-\frac{\alpha}{2}\triangle t}\phi^n\|_h^2=\lambda\triangle tIm(f,e^{\frac{\alpha}{2}\triangle t}\Phi+e^{-\frac{\alpha}{2}\triangle t}\phi^n)_h.$$

By cauchy inequality and ε-inequality, we have

$$\begin{aligned}e^{\alpha\triangle t}\|\Phi\|_h^2&=e^{-\alpha\triangle t}\|\phi^n\|_h^2+\lambda\triangle tIm(f,e^{\frac{\alpha}{2}\triangle t}\Phi+e^{-\frac{\alpha}{2}\triangle t}\phi^n)_h\\&\le e^{-\alpha\triangle t}\|\phi^n\|_h^2+\lambda\triangle t(\|e^{\frac{\alpha}{2}\triangle t}f\|_h\|\Phi\|_h+\|f\|_h\|e^{-\frac{\alpha}{2}\triangle t}\phi^n\|_h)\\&\le\alpha\triangle t\|\Phi\|_h^2+(1+\frac{\triangle t}{2})e^{-\alpha\triangle t}\|\phi^n\|_h^2+\triangle t(\frac{1}{2}+\frac{1}{4\alpha})\|f\|_h^2.\end{aligned}$$

Finally, we obtain

$$\|\Phi\|_h^2\le(1+\frac{\triangle t}{2})e^{-\alpha\triangle t}\|\phi^n\|_h^2+\triangle t(\frac{1}{2}+\frac{1}{4\alpha})\|f\|_h^2.$$

If function $f\in C([0,L])$,then $\sum_{j=1}^{J-1}|\Phi_j|^2$ is uniformly bounded with respect to the parameter $0\le\lambda\le1$. Thus the solution of the finite difference system (2.2) with boundary conditions (2.3) exists. The uniqueness of the solution of the finite difference system is proved directly by theorem 4.1.

3. Long-Time Behavior of Discrete System

In this section, let us put (2.2) with boundary conditions (2.3) in framework of dissipative dynamical systems. For fixed h and $\triangle t$ let us define operator $S_{h,\triangle t}:H_h^1\to H_h^1$ by

$$\phi^1=S_{h,\triangle t}\phi^0,$$

hence for every $n\ge0$, by the theorem 4.1, the family of solution operators $\{(S_{h,\triangle t})^n\}_{n\ge0}$ defined by $\phi^n=(S_{h,\triangle t})^n\phi^0$, forms a continuous semigroup on H_h^1. In the follow, we make mainly t-independent priori estimate for the solutions of the finite difference system (2.2) with boundary conditions (2.3).

Lemma 3.1. *For any initial value $\phi^0\in L_h^2$, and $\phi_0^0=\phi_J^0=0$, $f\in C([0,L])$, there is priori estimate for the solution of the discrete system*

$$\|\phi^n\|_h^2\le(e^{-\frac{\alpha}{2}\triangle t})^n\|\phi^0\|_h^2+\frac{1}{2\alpha^2}e^{\frac{\alpha}{2}\triangle t}(2+e^{\alpha\triangle t})\|f\|_h^2, n=1,2,\cdots. \tag{3.1}$$

Furthermore, there exists a constant $\rho_0>(\frac{1}{2\alpha^2}e^{\frac{\alpha}{2}\triangle t}(2+e^{\alpha\triangle t})\|f\|_h^2)^{\frac{1}{2}}$ such that the ball

$$B_0^h=\{\phi\in L_h^2;\|\phi\|_h\le\rho_0\}$$

is a absorbing set in L_h^2 under the semigroup $(S_{h,\triangle t})^n$.

Proof. Multiplying the relation (2.2) by $\triangle t(e^{\frac{\alpha}{2}\triangle t}\overline{\phi}_j^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\overline{\phi}_j^n)h$, and summing them up for j from 1 to $J-1$, then taking the imaginary part, we have

$$\begin{aligned}&Re(e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} - e^{-\frac{\alpha}{2}\triangle t}\phi^n, e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n)_h\\&+\frac{\triangle t}{2h^2}Im(\triangle_+\triangle_-(e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n), e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n)_h\\&= Im\triangle t(f, e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n)_h.\end{aligned} \tag{3.2}$$

It is easy to see that

$$Re(e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} - e^{-\frac{\alpha}{2}\triangle t}\phi^n, e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n)_h = \|e^{\frac{\alpha}{2}\triangle t}\phi^{n+1}\|_h^2 - \|e^{-\frac{\alpha}{2}\triangle t}\phi^n\|_h^2.$$

By Lemma 2.1

$$\begin{aligned}&Im(\triangle_+\triangle_-(e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n), e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n)_h\\&= -Im(\triangle_+(e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n), \triangle_+(e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n))_h = 0.\end{aligned}$$

Therefore, (3.2) can be rewritten as follows

$$\|e^{\frac{\alpha}{2}\triangle t}\phi^{n+1}\|_h^2 = \|e^{-\frac{\alpha}{2}\triangle t}\phi^n\|_h^2 + Im\triangle t(f, e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n)_h. \tag{3.3}$$

From (3.3), we obtain

$$\begin{aligned}&(1+\alpha\triangle t)\|\phi^{n+1}\|_h^2\\&\le e^{-\alpha\triangle t}\|\phi^n\|_h^2 + \triangle t|(f, e^{\frac{\alpha}{2}\triangle t}\phi^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\phi^n)_h|\\&\le e^{-\alpha\triangle t}\|\phi^n\|_h^2 + \alpha\triangle t\|\phi^{n+1}\|_h^2 + \frac{\triangle t}{4\alpha}e^{\alpha\triangle t}\|f\|_h^2 + \frac{\alpha}{2}\triangle t\|e^{-\frac{\alpha}{2}\triangle t}\phi^n\|_h^2 + \frac{\triangle t}{2\alpha}\|f\|_h^2\\&\le(1+\frac{\alpha}{2}\triangle t)e^{-\alpha\triangle t}\|\phi^n\|_h^2 + \alpha\triangle t\|\phi^{n+1}\|_h^2 + \frac{\triangle t}{4\alpha}(2+e^{\alpha\triangle t})\|f\|_h^2.\end{aligned}$$

Therefore

$$\begin{aligned}\|\phi^{n+1}\|_h^2 &\le e^{-\frac{\alpha}{2}\triangle t}\|\phi^n\|_h^2 + \frac{\triangle t}{4\alpha}(2+e^{\alpha\triangle t})\|f\|_h^2\\&\le \cdots\cdots\\&\le (e^{-\frac{\alpha}{2}\triangle t})^{n+1}\|\phi^0\|_h^2 + \frac{1}{2\alpha^2}e^{\frac{\alpha}{2}\triangle t}(2+e^{\alpha\triangle t})\|f\|_h^2, n = 0, 1, \cdots.\end{aligned}$$

The proof is complete.

Now we prove the existence of an absorbing set in the space H_h^1.

Lemma 3.2. *Under the conditions of Lemma 3.1, there is the estimate*

$$\|\phi^n\|_{1,h}^2 \le 2E^0 + \frac{9\kappa^2}{4}e^{4\alpha\triangle t}K_0^6 + \frac{\kappa^2}{4}K_0^6 + 2(\|f\|_h^2 + K_0^2), n = 1, 2, \cdots.$$

Furthermore, there exists a constant $\rho_1 > (\frac{9\kappa^2}{4}e^{4\alpha\triangle t}\rho_0^6 + \frac{\kappa^2}{4}\rho_0^6 + 2(\|f\|_h^2 + \rho_0^2))^{\frac{1}{2}}$ such that the ball

$$B_1^h = \{\phi \in H_h^1; \|\phi\|_{1,h} \le \rho_1\}$$

is a absorbing set in H_h^1 under the semigroup $(S_{h,\triangle t})^n$. where $K_0^2 = \|\phi^0\|_h^2 + \rho_0^2, E^0 = \|\phi^0\|_{1,h}^2 - \frac{\kappa}{2}e^{-\frac{3\alpha}{2}\triangle t}(|\phi^0|^4, 1)_h + 2Re(f, \phi^0)_h$.

Proof. Multiplying the relation (2.2) by $(e^{\frac{\alpha}{2}\triangle t}\overline{\phi}_j^{n+1} - e^{-\frac{\alpha}{2}\triangle t}\overline{\phi}_j^n)h$, and summing them up for j from 1 to $J-1$, then taking the real part, we have

$$\begin{aligned}&e^{\alpha\triangle t}\|\phi^{n+1}\|_{1,h}^2 - \frac{\kappa}{2}e^{2\alpha\triangle t}(|\phi^{n+1}|^4, 1)_h + 2e^{\frac{\alpha}{2}\triangle t}Re(f, \phi^{n+1})_h\\&= e^{-\alpha\triangle t}\|\phi^n\|_{1,h}^2 - \frac{\kappa}{2}e^{-2\alpha\triangle t}(|\phi^n|^4, 1)_h + 2e^{-\frac{\alpha}{2}\triangle t}Re(f, \phi^n)_h,\end{aligned}$$

or

$$e^{\frac{\alpha}{2}\triangle t}\|\phi^{n+1}\|_{1,h}^2-\frac{\kappa}{2}e^{\frac{3\alpha}{2}\triangle t}(|\phi^{n+1}|^4,1)_h+2Re(f,\phi^{n+1})_h$$
$$=e^{-\alpha\triangle t}(e^{-\frac{\alpha}{2}\triangle t}\|\phi^n\|_{1,h}^2-\frac{\kappa}{2}e^{-\frac{3\alpha}{2}\triangle t}(|\phi^n|^4,1)_h+2Re(f,\phi^n)_h). \quad (3.4)$$

Let

$$E^n=\|\phi^n\|_{1,h}^2-\frac{\kappa}{2}e^{-\frac{3\alpha}{2}\triangle t}(|\phi^n|^4,1)_h+2Re(f,\phi^n)_h.$$

Case 1. $\kappa>0$. From (3.4), using inequality $1+x\le e^x\le 1+xe^x,\ \forall x\in R$, we have

$$\begin{aligned}E^{n+1}+\frac{\alpha}{2}\triangle t\|\phi^{n+1}\|_{1,h}^2&\le e^{-\alpha\triangle t}E^n+\frac{3\kappa\alpha}{2}e^{\frac{3\alpha}{2}\triangle t}\triangle t(|\phi^{n+1}|^4,1)_h\\&\le e^{-\alpha\triangle t}E^n+\frac{3\kappa\alpha}{2}e^{\frac{3\alpha}{2}\triangle t}\triangle t\|\phi^{n+1}\|_\infty^2\|\phi^{n+1}\|_h^2\\&\le e^{-\alpha\triangle t}E^n+\frac{3\kappa\alpha}{2}e^{\frac{3\alpha}{2}\triangle t}\triangle t\|\phi^{n+1}\|_{1,h}\|\phi^{n+1}\|_h^3\\&\le e^{-\alpha\triangle t}E^n+\frac{\alpha}{2}\triangle t\|\phi^{n+1}\|_{1,h}^2+\frac{9\kappa^2\alpha}{8}e^{3\alpha\triangle t}\triangle t\|\phi^{n+1}\|_h^6,\end{aligned}$$

hence

$$\begin{aligned}E^{n+1}&\le e^{-\alpha\triangle t}E^n+\frac{9\kappa^2\alpha}{8}e^{3\alpha\triangle t}\triangle t\|\phi^{n+1}\|_h^6\\&\le (e^{-\alpha\triangle t})^2E^{n-1}+\frac{9\kappa^2\alpha}{8}e^{3\alpha\triangle t}\triangle t(e^{-\alpha\triangle t}\|\phi^n\|_h^6+\|\phi^{n+1}\|_h^6)\\&\le\cdots\ \cdots\\&\le (e^{-\alpha\triangle t})^{n+1-n_0}E^{n_0}+\frac{9\kappa^2\alpha}{8}e^{3\alpha\triangle t}\triangle t\sum_{j=0}^{n-n_0}(e^{-\alpha\triangle t})^j\|\phi^{n+1-j}\|_h^6,\end{aligned}$$

or

$$E^n\le (e^{-\alpha\triangle t})^{n-n_0}E^{n_0}+\frac{9\kappa^2\alpha}{8}e^{3\alpha\triangle t}\triangle t\sum_{j=0}^{n-n_0-1}(e^{-\alpha\triangle t})^j\|\phi^{n-j}\|_h^6. \quad (3.5)$$

If the initial value ϕ^0 satisfies $\|\phi^0\|_h\le R_0$, by Lemma 3.1, for $\forall\rho_0'>\rho_0$, when $n\ge n_0\ge\frac{2}{\alpha\triangle t}ln(\frac{R_0^2}{\rho_0'^2-\rho_0^2})$, we have $\|\phi^n\|_h\le\rho_0'$ and

$$E^n\le (e^{-\alpha\triangle t})^{n-n_0}E^{n_0}+\frac{9\kappa^2}{8}e^{4\alpha\triangle t}(\rho_0')^6.$$

By the definition of E^n, we have

$$\begin{aligned}\|\phi^n\|_{1,h}^2=&E^n+\frac{\kappa}{2}e^{-\frac{3\alpha}{2}\triangle t}(|\phi^n|^4,1)_h-2Re(f,\phi^n)_h\\\le&E^n+\frac{\kappa}{2}\|\phi^n\|_{1,h}\|\phi^n\|_h^3+2\|f\|_h\|\phi^n\|_h\\\le&E^n+\frac{1}{2}\|\phi^n\|_{1,h}^2+\frac{\kappa^2}{8}\|\phi^n\|_h^6+\|f\|_h^2+\|\phi^n\|_h^2. \quad (3.6)\end{aligned}$$

Therefore, when $n\ge n_0\ge\frac{2}{\alpha\triangle t}ln(\frac{R_0^2}{\rho_0'^2-\rho_0^2})$

$$\|\phi^n\|_{1,h}^2\le 2E^n+\frac{\kappa^2}{4}\|\phi^n\|_h^6+2(\|f\|_h^2+\|\phi^n\|_h^2)$$

$$\le 2(e^{-\alpha\triangle t})^{n-n_0}E^{n_0}+\frac{9\kappa^2}{4}e^{4\alpha\triangle t}(\rho_0')^6+\frac{\kappa^2}{4}\|\phi^n\|_h^6+2(\|f\|_h^2+\|\phi^n\|_h^2).$$

Thus

$$\overline{\lim_{n\to\infty}}\|\phi^n\|_{1,h}^2\le\frac{9\kappa^2}{4}e^{4\alpha\triangle t}(\rho_0')^6+\frac{\kappa^2}{4}\rho_0^6+2(\|f\|_h^2+\rho_0^2).$$

Since $\rho_0'>\rho_0$ is arbitrarily, then the follow inequality is also valid

$$\overline{\lim_{n\to\infty}}\|\phi^n\|_{1,h}^2\le\frac{9\kappa^2}{4}e^{4\alpha\triangle t}(\rho_0)^6+\frac{\kappa^2}{4}\rho_0^6+2(\|f\|_h^2+\rho_0^2).$$

Case 2. $\kappa\le 0$. From (3.4) we have

$$E^{n+1}\le e^{-\alpha\triangle t}E^n\le(e^{-\alpha\triangle t})^2E^{n-1}\le\cdots\le(e^{-\alpha\triangle t})^{n+1}E^0.$$

By the definition of the E^n and (3.6), we have

$$\|\phi^n\|_{1,h}^2\le 2E^n+\frac{\kappa^2}{4}\|\phi^n\|_h^6+2(\|f\|_h^2+\|\phi^n\|_h^2)$$

$$\le 2(e^{-\alpha\triangle t})^nE^0+\frac{\kappa^2}{4}\|\phi^n\|_h^6+2(\|f\|_h^2+\|\phi^n\|_h^2). \tag{3.7}$$

Thus

$$\overline{\lim_{n\to\infty}}\|\phi^n\|_{1,h}^2\le\frac{\kappa^2}{4}\rho_0^6+2(\|f\|_h^2+\rho_0^2).$$

By (3.1), we obtain

$$\|\phi^n\|_h^2\le\|\phi^0\|_h^2+\rho_0^2=K_0^2, n=1,2,\cdots.$$

Finally, from (3.5), (3.6) and (3.7), we obtain

$$\|\phi^n\|_{1,h}^2\le 2E^n+\frac{\kappa^2}{4}\|\phi^n\|_h^6+2(\|f\|_h^2+\|\phi^n\|_h^2)$$

$$\le 2E^0+\frac{9\kappa^2}{4}e^{4\alpha\triangle t}K_0^6+\frac{\kappa^2}{4}K_0^6+2(\|f\|_h^2+K_0^2).$$

The Lemma 3.2 is proved.

By Lemma 3.1, Lemma 3.2 and Lemma 2.2, we have

Corollary 3.1. *If $f\in C([0,L])$ and initial value $\|\phi^0\|_{1,h}\le R$, then there exists a constant $C(R)$ independent of h and $\triangle t$ such that*

$$\sup_{n\ge 0}\|\phi^n\|_\infty^2\le C(R).$$

Now we prove the existence of attractor $\mathcal{A}_{h,\triangle t}$ for the discrete system on H_h^1. Obvious, a family operators $(S_{h,\triangle t})^n$ satisfy the semigroup properties

$$(S_{h,\triangle t})^m(S_{h,\triangle t})^n=(S_{h,\triangle t})^{m+n},\forall m,n\ge 0,\quad (S_{h,\triangle t})^0=I.$$

For every $n\ge 0$, $(S_{h,\triangle t})^n$ is a continuous operator from the finite dimensional space H_h^1 into itself. By Lemma 3.2, there exists a bounded set B_1^h which is absorbing in H_h^1 under $(S_{h,\triangle t})^n$. Using theorem 1.1 in [2], we obtain

Theorem 3.1. *If $f\in C([0,L])$, then the discrete system possesses a global attractor $\mathcal{A}_{h,\triangle t}$ on H_h^1, and*

$$\mathcal{A}_{h,\triangle t}=\bigcap_{n\ge 0}\overline{\bigcup_{m\ge n}(S_{h,\triangle t})^mB_1^h}.$$

4. Stability and Convergence of the Difference Scheme

Let $\{\phi^n\}$ and $\{\psi^n\}$ be the two solutions of the difference scheme with initial value $\{\phi^0\}$ and $\{\psi^0\}$ respectively, and initial value satisfy

$$\|\phi^0\|_{1,h}\le R,\ \|\psi^0\|_{1,h}\le R.$$

By the Corollary 3.1, we have

$$\sup_{n\geq 0}\|\phi^n\|_\infty^2\leq C(R),\quad \sup_{n\geq 0}\|\psi^n\|_\infty^2\leq C(R).$$

Let $\epsilon^n=\phi^n-\psi^n$, then $\{\epsilon^n\}$ satisfies

$$\begin{aligned}
&i\frac{e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1}-e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n}{\triangle t}+\frac{1}{2h^2}\triangle_+\triangle_-(e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n)\\
&+\frac{\kappa}{4}(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)\\
&-\frac{\kappa}{4}(|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\psi_j^n)=0, \qquad (4.1)\\
&\qquad j=1,2,\cdots,J-1,n=0,1,\cdots.
\end{aligned}$$

Because

$$\begin{aligned}
&(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)\\
&-(|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\psi_j^n)\\
=&(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n)+(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2\\
&-|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{\frac{\alpha}{2}\triangle t}\phi_j^n|^2-|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\psi_j^n),
\end{aligned}$$

and

$$\begin{aligned}
&(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2-|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2\\
&-|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\psi_j^n)\\
\leq&\{(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|+|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|)|e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1}|+(|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|\\
&+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|)|e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n|\}\cdot(|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|)\\
\leq&|e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1}|(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2+2|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)\\
&+|e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n|(|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2+2|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2+|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2).
\end{aligned}$$

Multiplying the relation (4.1) by $\triangle t(e^{\frac{\alpha}{2}\triangle t}\overline{\epsilon}_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\overline{\epsilon}_j^n)h$, and summing them up for j from 1 to $J-1$, then taking the imaginary part, using the above estimates, we have

$$\begin{aligned}
&\|e^{\frac{\alpha}{2}\triangle t}\epsilon^{n+1}\|_h^2-\|e^{-\frac{\alpha}{2}\triangle t}\epsilon^n\|_h^2\\
\leq&|\kappa|(e^{\alpha\triangle t}+1)C(R)\triangle t(\|e^{\frac{\alpha}{2}\triangle t}\epsilon^{n+1}\|_h^2+\|e^{-\frac{\alpha}{2}\triangle t}\epsilon^n\|_h^2).
\end{aligned}$$

Let $K_2=|\kappa|(e^{\alpha\triangle t}+1)C(R)$. If there exists a constant $\gamma>0$ such that $1-K_2\triangle t\geq\gamma$, then

$$\begin{aligned}
\|\epsilon^{n+1}\|_h^2&\leq\frac{1+K_2\triangle t}{1-K_2\triangle t}\|\epsilon^n\|_h^2\leq(\frac{1+K_2\triangle t}{1-K_2\triangle t})^2\|\epsilon^{n-1}\|_h^2\leq\cdots\\
&\leq(\frac{1+K_2\triangle t}{1-K_2\triangle t})^{n+1}\|\epsilon^0\|_h^2\leq e^{\frac{2K_2}{\gamma}(n+1)\triangle t}\|\epsilon^0\|_h^2, n=0,1,\cdots.
\end{aligned}$$

We obtain the following stability theorem

Theorem 4.1. *Let $\{\phi^n\}$ and $\{\psi^n\}$ be the two solutions of the difference scheme with initial value $\{\phi^0\}$ and $\{\psi^0\}$ respectively, and initial value satisfy*

$$\|\phi^0\|_{1,h}\leq R,\ \|\psi^0\|_{1,h}\leq R.$$

If there exists a constant $\gamma>0$ such that the temporal mesh length $\triangle t$ satisfies

$$1-|\kappa|(e^{\alpha\triangle t}+1)C(R)\triangle t\geq\gamma,$$

$f \in C([0,L])$, then for every $n \geq 0$, we have

$$\|\phi^n - \psi^n\|_h \leq e^{\gamma^{-1}|\kappa|(e^{\alpha\triangle t}+1)C(R)n\triangle t}\|\phi^0 - \psi^0\|_h.$$

By using the same procedures as the proof of theorem 4.1, we can obtain the following theorem

Theorem 4.2. *If the solution of equation (1.1) satisfies*

$$\frac{\partial^3 u}{\partial t^3} \in L^\infty(0,T;H^1(\Omega)),\ \frac{\partial^2 u}{\partial t^2} \in L^\infty(0,T;H^2(\Omega)),$$

$$u \in L^\infty(0,T;H^4(\Omega)\cap H_0^1(\Omega)),$$

and initial value ϕ^0 of the difference scheme (2.2) satisfies

$$\|\phi^0 - u^0\|_h \leq Ch^2.$$

Then for $\forall n \geq 0$ such that $n\triangle t \leq T$, there exists a constant $C(T)$ dependent on T such that

$$\|u^n - \phi^n\|_h \leq C(T)(h^2 + \triangle t^2).$$

Remark 4.1. According to the regularity of solution of Schrödinger equation, if initial value $u_0 \in H^7(\Omega)\cap H_0^1(\Omega)$ and right hand $f \in H^5(\Omega)$ of equation (1.1). Then the conditions of Theorem 4,2 hold.

Remark 4.2. As the parameter $\alpha > 0$, the discrete system can remain well disspitive properties of the original system, and as the parameter $\alpha = 0$,the discrete system can also remain well consevertion properties of the original system.

5. Long-Time Stability and Convergence of the Difference Scheme

First of all, we make t-independent priori estimate for the solutions of the finite difference system (2.2) with boundary conditions (2.3) in the nonautonomous case. Right now let us replace f_j by $f_j^{n+\frac{1}{2}}$ in the right-hand side of (2.2) and suppose that there exists a constant C_0 independent of h and $\triangle t$ such that

$$\Big(\triangle t\sum_{j=0}^{\infty}\|f^{j+\frac{1}{2}}\|_h^2\Big)^{\frac{1}{2}} \leq C_0. \qquad (H1)$$

Using exactly the same technique as in the proof of Lemma 3.1, we can obtain that

Lemma 5.1. *For any initial value $\phi^0 \in L_h^2$, and $\phi_0^0 = \phi_J^0 = 0$, if (H1) is true, then there exists a constant $K'_0 = \|\phi^0\|_h^2 + \frac{(1+2e^{\alpha\triangle t})C_0^2}{4\alpha}$ such that*

$$\sup_{n\geq 0}\|\phi^n\|_h^2 + \frac{\alpha}{2}\triangle t\sum_{j=1}^{\infty}\|\phi^j\|_h^2 \leq K'_0.$$

Now we make the second hypothesis. There exist constants C_1 and C_2 independent of h and $\triangle t$ such that

$$\Big(\int_0^\infty \|f_t\|_h^2 dt\Big)^{\frac{1}{2}} \leq C_1, \quad \sup_{t\geq 0}\|f(\cdot,t)\|_h \leq C_2. \qquad (H2)$$

Lemma 5.2. *Assume that hypothesis (H1) and (H2) are true. Then for any initial value $\phi^0 \in H_h^1$, and $\phi_0^0 = \phi_J^0 = 0$, there exists a constant $K'_1 = \frac{\kappa^2}{4}K'^3_0 + 2C_2^2 + 2K'_0 + 2E^0 + 8\kappa^2 K'^3_0 e^{5\alpha\triangle t} + 2\alpha e^{\alpha\triangle t}C_0^2 + 2C_1^2 + (1+\alpha e^{\alpha\triangle t})\frac{4}{\alpha}K'_0$ such that*

$$\sup_{n\geq 0}\|\phi^n\|_{1,h}^2 + 2\alpha\triangle t\sum_{j=1}^{\infty}\|\phi^j\|_{1,h}^2 \leq K'_1,$$

where $E^0 = \|\phi^0\|_{1,h}^2 - \frac{\kappa}{2}e^{-\frac{3\alpha}{2}\triangle t}(|\phi^0|^4,1)_h + 2Re(f^{\frac{1}{2}},\phi^0)_h$.

Proof. The proof is similar to that of Lemma 3.2, but has some differences. Multiplying the relation (2.2) by $(e^{\frac{\alpha}{2}\triangle t}\overline{\phi}_j^{n+1} - e^{-\frac{\alpha}{2}\triangle t}\overline{\phi}_j^n)h$, and summing them up for j from 1 to $J-1$, then taking the real part, we have

$$e^{\alpha\triangle t}\|\phi^{n+1}\|_{1,h}^2 - \frac{\kappa}{2}e^{2\alpha\triangle t}(|\phi^{n+1}|^4,1)_h + 2e^{\frac{\alpha}{2}\triangle t}Re(f^{n+\frac{1}{2}},\phi^{n+1})_h$$

$$=e^{-\alpha\triangle t}\|\phi^n\|_{1,h}^2-\frac{\kappa}{2}e^{-2\alpha\triangle t}(|\phi^n|^4,1)_h+2e^{-\frac{\alpha}{2}\triangle t}Re(f^{n+\frac{1}{2}},\phi^n)_h,$$

or

$$e^{\frac{3}{2}\alpha\triangle t}\|\phi^{n+1}\|_{1,h}^2-\frac{\kappa}{2}e^{\frac{5\alpha}{2}\triangle t}(|\phi^{n+1}|^4,1)_h+2e^{\alpha\triangle t}Re(f^{n+\frac{1}{2}},\phi^{n+1})_h$$
$$=e^{-\frac{1}{2}\alpha\triangle t}\|\phi^n\|_{1,h}^2-\frac{\kappa}{2}e^{-\frac{3\alpha}{2}\triangle t}(|\phi^n|^4,1)_h+2Re(f^{n+\frac{1}{2}},\phi^n)_h. \tag{5.1}$$

Let

$$E^n=\|\phi^n\|_{1,h}^2-\frac{\kappa}{2}e^{-\frac{3\alpha}{2}\triangle t}(|\phi^n|^4,1)_h+2Re(f^{n+\frac{1}{2}},\phi^n)_h.$$

Case 1. $\kappa>0$. From (5.1) and Lemma 2.2, using inequality $1+x\le e^x\le 1+xe^x, \forall x\in R$, we derive

$$\begin{aligned}
&E^{n+1}+\frac{3\alpha}{2}\triangle t\|\phi^{n+1}\|_{1,h}^2\\
\le& E^n+2\kappa\alpha e^{\frac{5\alpha}{2}\triangle t}\triangle t(|\phi^{n+1}|^4,1)_h-2(e^{\alpha\triangle t}-1)Re(f^{n+\frac{1}{2}},\phi^{n+1})_h\\
&+2Re(f^{n+\frac{3}{2}}-f^{n+\frac{1}{2}},\phi^{n+1})_h\\
\le& E^n+2\kappa\alpha e^{\frac{5\alpha}{2}\triangle t}\triangle t\|\phi^{n+1}\|_{1,h}\|\phi^{n+1}\|_h^3+2\alpha e^{\alpha\triangle t}\triangle t|(f^{n+\frac{1}{2}},\phi^{n+1})_h|\\
&+2|(\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}f_t dt,\phi^{n+1})_h|\\
\le& E^n+\frac{1}{2}\alpha\triangle t\|\phi^{n+1}\|_{1,h}^2+2\kappa^2\alpha e^{5\alpha\triangle t}\triangle t\|\phi^{n+1}\|_h^6\\
&+2\alpha e^{\alpha\triangle t}\triangle t\|f^{n+\frac{1}{2}}\|_h\|\phi^{n+1}\|_h+2\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\|f_t\|_h dt\|\phi^{n+1}\|_h\\
\le& E^n+\frac{1}{2}\alpha\triangle t\|\phi^{n+1}\|_{1,h}^2+2\kappa^2\alpha e^{5\alpha\triangle t}\triangle t\|\phi^{n+1}\|_h^6+\alpha e^{\alpha\triangle t}\triangle t(\|f^{n+\frac{1}{2}}\|_h^2\\
&+\|\phi^{n+1}\|_h^2)+\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\|f_t\|_h^2 dt+\triangle t\|\phi^{n+1}\|_h^2,
\end{aligned}$$

hence

$$\begin{aligned}
E^{n+1}\le& E^n+2\kappa^2\alpha e^{5\alpha\triangle t}\triangle t\|\phi^{n+1}\|_h^6+\alpha e^{\alpha\triangle t}\triangle t\|f^{n+\frac{1}{2}}\|_h^2\\
&+\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\|f_t\|_h^2 dt+(1+\alpha e^{\alpha\triangle t})\triangle t\|\phi^{n+1}\|_h^2-\alpha\triangle t\|\phi^{n+1}\|_{1,h}^2\\
\le&\cdots\ \cdots\\
\le& E^0+2\kappa^2\alpha e^{5\alpha\triangle t}\triangle t\sum_{j=1}^{n+1}\|\phi^j\|_h^6+\alpha e^{\alpha\triangle t}\triangle t\sum_{j=1}^{n+1}\|f^{j-\frac{1}{2}}\|_h^2\\
&+\int_{t_{\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\|f_t\|_h^2 dt+(1+\alpha e^{\alpha\triangle t})\triangle t\sum_{j=1}^{n+1}\|\phi^j\|_h^2-\alpha\triangle t\sum_{j=1}^{n+1}\|\phi^j\|_{1,h}^2,
\end{aligned}$$

or

$$E^n+\alpha\triangle t\sum_{j=1}^{n}\|\phi^j\|_{1,h}^2$$

$$\leq E^0 + 2\kappa^2 \alpha e^{5\alpha\triangle t}\triangle t\sum_{j=1}^{n}\|\phi^j\|_h^6 + \alpha e^{\alpha\triangle t}\triangle t\sum_{j=1}^{n}\|f^{j-\frac{1}{2}}\|_h^2$$

$$+\int_{t_{\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\|f_t\|_h^2 dt + (1+\alpha e^{\alpha\triangle t})\triangle t\sum_{j=1}^{n}\|\phi^j\|_h^2.$$

By the definition of E^n and Lemma 5.1, we have

$$\|\phi^n\|_{1,h}^2 + \alpha\triangle t\sum_{j=1}^{n}\|\phi^j\|_{1,h}^2$$

$$\leq \frac{\kappa}{2}e^{-\frac{3\alpha}{2}\triangle t}(|\phi^n|^4, 1)_h - 2Re(f^{n+\frac{1}{2}}, \phi^n)_h + E^0 + 2\kappa^2\alpha e^{5\alpha\triangle t}\triangle t\sum_{j=1}^{n}\|\phi^j\|_h^6$$

$$+ \alpha e^{\alpha\triangle t}\triangle t\sum_{j=1}^{n}\|f^{j-\frac{1}{2}}\|_h^2 + \int_{t_{\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\|f_t\|_h^2 dt + (1+\alpha e^{\alpha\triangle t})\triangle t\sum_{j=1}^{n}\|\phi^j\|_h^2$$

$$\leq \frac{\kappa}{2}\|\phi^n\|_{1,h}\|\phi^n\|_h^3 + 2\|f^{n+\frac{1}{2}}\|_h\|\phi^n\|_h + E^0 + 4\kappa^2 {K'}_0^3 e^{5\alpha\triangle t}$$

$$+ \alpha e^{\alpha\triangle t}C_0^2 + C_1^2 + (1+\alpha e^{\alpha\triangle t})\frac{2}{\alpha}K'_0$$

$$\leq \frac{1}{2}\|\phi^n\|_{1,h}^2 + \frac{\kappa^2}{8}{K'}_0^3 + C_2^2 + K'_0 + E^0 + 4\kappa^2 {K'}_0^3 e^{5\alpha\triangle t}$$

$$+ \alpha e^{\alpha\triangle t}C_0^2 + C_1^2 + (1+\alpha e^{\alpha\triangle t})\frac{2}{\alpha}K'_0,$$

hence, in the case $\kappa > 0$, we obtain

$$\|\phi^n\|_{1,h}^2 + 2\alpha\triangle t\sum_{k=1}^{n}\|\phi^k\|_{1,h}^2 \leq K'_1, n = 1, 2, \cdots.$$

Case 2. $\kappa \leq 0$. From (5.1) we have

$$E^{n+1} + \frac{3\alpha}{2}\triangle t\|\phi^{n+1}\|_{1,h}^2$$

$$\leq E^n - 2(e^{\alpha\triangle t} - 1)Re(f^{n+\frac{1}{2}}, \phi^{n+1}) + 2Re(f^{n+\frac{3}{2}} - f^{n+\frac{1}{2}}, \phi^{n+1})_h$$

$$\leq E^n + \alpha e^{\alpha\triangle t}\triangle t(\|f^{n+\frac{1}{2}}\|_h^2 + \|\phi^{n+1}\|_h^2) + \int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\|f_t\|_h^2 dt + \triangle t\|\phi^{n+1}\|_h^2$$

hence

$$E^{n+1} \leq E^n + \alpha e^{\alpha\triangle t}\triangle t\|f^{n+\frac{1}{2}}\|_h^2 + \int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\|f_t\|_h^2 dt$$

$$+ (1+\alpha e^{\alpha\triangle t})\triangle t\|\phi^{n+1}\|_h^2 - \frac{3\alpha}{2}\triangle t\|\phi^{n+1}\|_{1,h}^2$$

$$\leq \cdots\ \cdots$$

$$\leq E^0 + \alpha e^{\alpha\triangle t}\triangle t\sum_{j=1}^{n+1}\|f^{j-\frac{1}{2}}\|_h^2 + \int_{t_{\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\|f_t\|_h^2 dt$$

$$+ (1+\alpha e^{\alpha\triangle t})\triangle t\sum_{j=1}^{n+1}\|\phi^j\|_h^2 - \frac{3\alpha}{2}\triangle t\sum_{j=1}^{n+1}\|\phi^j\|_{1,h}^2$$

or

$$E^n + \frac{3\alpha}{2}\triangle t\sum_{j=1}^{n}\|\phi^j\|_{1,h}^2$$

$$\le E^0 + \alpha e^{\alpha\triangle t}\triangle t\sum_{j=1}^{n}\|f^{j-\frac{1}{2}}\|_h^2 + \int_{t_{\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\|f_t\|_h^2 dt$$

$$+ (1+\alpha e^{\alpha\triangle t})\triangle t\sum_{j=1}^{n}\|\phi^j\|_h^2.$$

By the definition of E^n and Lemma 5.1, we have

$$\|\phi^n\|_{1,h}^2 + \frac{3}{2}\alpha\triangle t\sum_{j=1}^{n}\|\phi^j\|_{1,h}^2$$

$$\le \frac{\kappa}{2}e^{-\frac{3\alpha}{2}\triangle t}(|\phi^n|^4,1)_h - 2Re(f^{n+\frac{1}{2}},\phi^n)_h + E^0 + \alpha e^{\alpha\triangle t}\triangle t\sum_{j=1}^{n}\|f^{j-\frac{1}{2}}\|_h^2$$

$$+ \int_{t_{\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\|f_t\|_h^2 dt + (1+\alpha e^{\alpha\triangle t})\triangle t\sum_{k=1}^{n}\|\phi^k\|_h^2$$

$$\le \frac{\kappa}{2}\|\phi^n\|_{1,h}\|\phi^n\|_h^3 + 2\|f^{n+\frac{1}{2}}\|_h\|\phi^n\|_h + E^0 + \alpha e^{\alpha\triangle t}C_0^2 + C_1^2 + (\frac{2}{\alpha} + 2e^{\alpha\triangle t})K'_0$$

$$\le \frac{1}{2}\|\phi^n\|_{1,h}^2 + \frac{\kappa^2}{8}K'^3_0 + C_2^2 + K'_0 + E^0 + \alpha e^{\alpha\triangle t}C_0^2 + C_1^2 + (\frac{2}{\alpha} + 2e^{\alpha\triangle t})K'_0.$$

Therefore, in the case $\kappa \le 0$, we derive

$$\|\phi^n\|_{1,h}^2 + 3\alpha\triangle t\sum_{k=1}^{n}\|\phi^k\|_{1,h}^2 \le K'_1, n = 1,2,\cdots.$$

The Lemma 5.2 is proved.

By Lemma 5.1, Lemma 5.2 and Lemma 2.2, we have

Corollary 5.1. *Under the conditions of Lemma 5.2, if initial value* $\|\phi^0\|_{1,h} \le R$, *then there exists a constant C(R) independent of* h *and* $\triangle t$ *such that*

$$\sup_{n\ge 0}\|\phi^n\|_\infty^2 \le C(R).$$

Let $\{\phi^n\}$ and $\{\psi^n\}$ be the two solutions of the difference scheme with initial value $\{\phi^0\}$ and $\{\psi^0\}$ respectively, and initial value satisfy

$$\|\phi^0\|_{1,h} \le R,\ \|\psi^0\|_{1,h} \le R.$$

By Lemma 5.1, Lemma 5.2 and Corollary 5.1, there exists a constant $C(R)$ independent of h and $\triangle t$ such that

$$\begin{cases} \sup\limits_{n\ge 0}\|\phi^n\|_\infty^2 + \triangle t\sum\limits_{k=1}^{\infty}\|\phi^k\|_\infty^2 \le C(R), \\ \sup\limits_{n\ge 0}\|\psi^n\|_\infty^2 + \triangle t\sum\limits_{k=1}^{\infty}\|\psi^k\|_\infty^2 \le C(R). \end{cases} \tag{5.2}$$

Let $\epsilon^n = \phi^n - \psi^n$, then $\{\epsilon^n\}$ satisfies

$$i\frac{e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1} - e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n}{\triangle t} + \frac{1}{2h^2}\triangle_+\triangle_-(e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1} + e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n)$$

$$+\frac{\kappa}{4}(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)$$

$$-\frac{\kappa}{4}(|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\psi_j^n)=0, \tag{5.3}$$

$$j=1,2,\cdots,J-1, n=0,1,\cdots.$$

Because

$$\begin{aligned}&(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\phi_j^n)\\&-(|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\psi_j^n)\\=&(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n)+(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2\\&-|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{\frac{\alpha}{2}\triangle t}\phi_j^n|^2-|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\psi_j^n),\end{aligned}$$

and

$$\begin{aligned}&(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2-|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2\\&-|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)(e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\psi_j^n)\\\le&\{(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|+|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|)|e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1}|+(|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|\\&+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|)|e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n|\}\cdot(|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|)\\\le&|e^{\frac{\alpha}{2}\triangle t}\epsilon_j^{n+1}|(|e^{\frac{\alpha}{2}\triangle t}\phi_j^{n+1}|^2+2|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2+|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2)\\&+|e^{-\frac{\alpha}{2}\triangle t}\epsilon_j^n|(|e^{-\frac{\alpha}{2}\triangle t}\phi_j^n|^2+2|e^{-\frac{\alpha}{2}\triangle t}\psi_j^n|^2+|e^{\frac{\alpha}{2}\triangle t}\psi_j^{n+1}|^2).\end{aligned}$$

Multiplying the relation (5.3) by $(e^{\frac{\alpha}{2}\triangle t}\overline{\epsilon}_j^{n+1}+e^{-\frac{\alpha}{2}\triangle t}\overline{\epsilon}_j^n)h$, and summing them up for j from 1 to $J-1$, then taking the imaginary part, using the above estimates, we have

$$\begin{aligned}\frac{\|\epsilon^{n+1}\|_h^2-\|\epsilon^n\|_h^2}{\triangle t}\le&\frac{|\kappa|}{8}e^{2\alpha\triangle t}(3\|\phi^{n+1}\|_\infty^2+7\|\psi^{n+1}\|_\infty^2+5\|\psi^n\|_\infty^2+\|\phi^n\|_\infty^2)\|\epsilon^{n+1}\|_h^2\\&+\frac{|\kappa|}{8}e^{2\alpha\triangle t}(3\|\phi^n\|_\infty^2+7\|\psi^n\|_\infty^2+5\|\psi^{n+1}\|_\infty^2+\|\phi^{n+1}\|_\infty^2)\|\epsilon^n\|_h^2,\end{aligned}$$

$$n=0,1,\cdots.$$

Let $y^n=\|\epsilon^n\|_h^2, g_1^n=\frac{|\kappa|}{8}e^{2\alpha\triangle t}(3\|\phi^n\|_\infty^2+7\|\psi^n\|_\infty^2+5\|\psi^{n+1}\|_\infty^2+\|\phi^{n+1}\|_\infty^2), g_2^n=\frac{|\kappa|}{8}e^{2\alpha\triangle t}(3\|\phi^{n+1}\|_\infty^2+7\|\psi^{n+1}\|_\infty^2+5\|\psi^n\|_\infty^2+\|\phi^n\|_\infty^2)$ and $h^n=0$. If there exists a constant $\gamma>0$ such that $\forall n\ge 0, 1-g_2^n\triangle t\ge\gamma$, then by Lemma 2.4 and (5.2), we derive

$$\begin{aligned}\|\epsilon^n\|_h^2&\le\|\epsilon^0\|_h^2\exp(\frac{1}{\gamma}\triangle t\sum_{k=1}^n g_2^k)\exp(\triangle t\sum_{k=0}^{n-1}g_1^k)\\&\le\|\epsilon^0\|_h^2\exp(2\gamma^{-1}|\kappa|C(R)e^{2\alpha\triangle t})\exp(2|\kappa|C(R)e^{2\alpha\triangle t}).\end{aligned}$$

We obtain the following stability theorem

Theorem 5.1. *Assume that hypothesis (H1) and (H2) are true. Let $\{\phi^n\}$ and $\{\psi^n\}$ be the two solutions of the difference scheme with initial value $\{\phi^0\}$ and $\{\psi^0\}$ respectively, and initial value satisfy*

$$\|\phi^0\|_{1,h}\le R,\ \|\psi^0\|_{1,h}\le R.$$

If the temporal mesh length $\triangle t$ small sufficiently such that there exists a constant $\gamma>0$ satisfies

$$1-2|\kappa|C(R)e^{\alpha\triangle t}\triangle t\ge\gamma.$$

Then we have

$$\|\phi^n-\psi^n\|_h\le\exp((\gamma^{-1}+1)|\kappa|C(R)e^{2\alpha\triangle t})\|\phi^0-\psi^0\|_h, n=0,1,2,\cdots.$$

Remark 5.1. If the right term $f(x,t)$ of the equation (1.1) satisfies

$$f, \frac{\partial f}{\partial t} \in L^2(R^+; H^1(\Omega)) \cap L^\infty(R^+; H^1(\Omega)).$$

Then by Lemma 2.3, hypotheses (H1) and (H2) are hold.

Let R denote the local truncation error of the difference scheme (2.2). As the solution $u(x,t)$ of the equation (1.1) with initial condition (1.2) and boundary condition (1.3) smooth sufficiently, the series $\sum_{k=0}^{\infty} \|R^{n+\frac{1}{2}}\|_h^2$ converge, and $\sum_{k=0}^{\infty} \|R^{n+\frac{1}{2}}\|_h^2 = O(h^2 + \triangle t^2)$. Thanks to Lemma (2.4), by using the same procedures as the proof of theorem 5.1, we can obtain that

Theorem 5.2. *The conditions of theorem 5.1 hold, and suppose that the solution of equation (1.1) with boundary condition (1.2) and initial condition (1.3) satisfies*

$$\frac{\partial^3 u}{\partial t^3} \in L^2(R^+; H^1(\Omega)), \frac{\partial^2 u}{\partial t^2} \in L^2(R^+; H^2(\Omega)),$$
$$\frac{\partial u}{\partial t} \in L^2(R^+; H^4(\Omega)) \cap L^\infty(R^+; H^1(\Omega)),$$
$$u \in L^\infty(R^+; H^4(\Omega)) \cap L^2(R^+; H^4(\Omega)),$$

and initial value ϕ^0 of the difference scheme satisfies

$$\|\phi^0 - u^0\|_h = O(h^2).$$

Then there exists a constant C independent of h and $\triangle t$ such that for all $n \geq 0$,

$$\|u^n - \phi^n\|_h \leq C(h^2 + \triangle t^2).$$

Remark 5.2. For the sake of convenience, we have only analysed the Dirichlet boundary condition (1.2) for the equation (1.1). For the Neumann or the periodic boundary condition, some parallel conclusions can also been obtained for the equation (1.1).

In addition, as the initial value u_0 and the right hand $f(x,t)$ of equation (1.1) satisfy

$$u_0 \in H^7(\Omega) \cap H_0^1(\Omega), \ f \in L^2(R^+; H^5(\Omega)),$$

$$\frac{\partial f}{\partial t}, \frac{\partial^2 f}{\partial t^2} \in L^2(R^+; H^1(\Omega)).$$

Then the conditions of Theorem 5.2 hold.

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