Journal of Computational Mathematics, Vol.19, No.4, 2001, 347–356.

APPLICATION OF NEWTON'S AND CHEBYSHEV'S METHODS TO PARALLEL FACTORIZATION OF POLYNOMIALS*[1)]

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Abstract

In this paper it is shown in two different ways that one of the family of parallel iterations to determine all real quadratic factors of polynomials presented in [12] is Newton's method applied to the special equation (1.7) below. Furthermore, we apply Chebyshev's method to (1.7) and obtain a new parallel iteration for factorization of polynomials. Finally, some properties of the parallel iterations are discussed.

Key words: Newton's method, Chebyshev's method, Parallel iteration, Factorization of polynomial.

1. Introduction

Let $F : R^N \to R^N$ be a nonlinear map. Newton's method

$$x^+ = x - F'(x)^{-1}F(x) \tag{1.1}$$

and Chebyshev's method

$$\begin{aligned} \widehat{x} &= x - [I + \tfrac{1}{2}F'(x)^{-1}F''(x)F'(x)^{-1}F(x)]^{-1}F'(x)^{-1}F(x) \\ &= x - F'(x)^{-1}F(x) - \tfrac{1}{2}F'(x)^{-1}F''(x)(F'(x)^{-1}F(x))^2 \end{aligned} \tag{1.2}$$

are well known for solving nonlinear equation

$$F(x) = 0, \tag{1.3}$$

where I is the unit matrix of order N, x is an approximation of the solution x^* of (1.3), x^+ and $\widehat{x}$ are new approximations of x^* produced by Newton's and Chebyshev's methods, respectively. It is well known that the order of convergence for Newton's and Chebyshev's methods is 2 and 3, respectively, if $F'(x^*)$ is nonsingular.

Let

$$p(t) = \sum_{\nu=0}^{N} a_\nu t^{N-\nu},\ a_0 = 1 \tag{1.4}$$

be a monic polynomial of degree $N = 2n$. Then the convergence is quadratic or cubic, respectively, if Newton's or Chebyshev's method is used to find *one* simple zero of (1.4). Many parallel iterations have been proposed and studied to determine *all* zeros of (1.4) simultaneously (see [1]-[3], [5]-[11]).

* Received January 18, 1997.
1) The Project Supported by National Natural Science Foundation of China and by Natural Science Foundation of Zhejiang Province.

In the following it is assumed that $p(t)$ in (1.4) is a monic polynomial of degree $N = 2n$ with real coefficients. Then it can be factorized as

$$p(t) = \prod_{j=1}^{n} (t^2 - p_j t - q_j), \tag{1.5}$$

where $p_j, q_j (j = 1, 2, \cdots, n)$ are real.

Bairstow's method is a well known iteration to determine *one* real quadratic factor of $p(t)$ (see [4]). Its advantages are that the computational program is simple and that the convergence is quadratic if there are only simple or real double zeros of $p(t)$.

From the viewpoint of linear interpolation Zheng[12] proposed a family of parallel iterations to determine *all* real quadratic factors of polynomials, which keeps the advantages of Bairstow's method.

Let

$$g(t) = \prod_{j=1}^{n} (t^2 - u_j t - v_j) = \sum_{\nu=0}^{2n} b_\nu t^{2n-\nu}, b_0 = 1, \tag{1.6}$$

where $b_\nu = b_\nu(x)$ is the function of $x = (u_1, v_1, \cdots, u_n, v_n)^T \in R^{2n}$. It is clear that $(p_1, q_1, \cdots, p_n, q_n)^T$ of (1.5) is the solution of the system of nonlinear equations

$$F(x) = (f_1(x), \cdots, f_{2n}(x))^T = (b_1(x) - a_1, \cdots, b_{2n}(x) - a_{2n})^T = 0 \tag{1.7}$$

In section 4 of this paper it is shown in two different ways that one of the family in [12] is Newton's method applied to (1.7). In section 5 we apply Chebyshev's method to (1.7) and obtain a new parallel iteration for factorization of polynomials. For purpose of convenience the linear interpolation operators and their properties are introduced in section 2. A simple condition for nonsingulity of $F'(x)$ in (1.7) is given in section 3. Finally, some properties of the parallel iterations for factorization of polynomials are discussed in section 6.

2. Linear Interpolation Operators and Their Properties

For purposes of brevity, all formulas, sums and products involving indices i, j, k will assume the range $1, 2, \cdots, n$ and $\nu = 0, 1, \cdots, 2n$, unless explicit stated otherwise. We denote I and E the unit matrix of order $2n$ and 2, respectively. And 0 will denote real zero $0 \in R^1$, zero vector $(0, 0)^T \in R^2$ or null matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, respectively, when it may not be mixed from context.

Definition [12]**.** *Let*

$$x_i = (u_i, v_i)^T \in R^2, \tag{2.1}$$

α_i, β_i be the zeros of

$$Q_i(t) = t^2 - u_i t - v_i. \tag{2.2}$$

Suppose that

$$L(f) = L(f; x_i, c; t) = l_1(f; x_i, c)(t - c) + l_2(f; x_i, c) \tag{2.3}$$

is the linear interpolation of $f(t)$ with nodes α_i, β_i, where $c \in R^1$ is a number independent of f and t. Denote

$$l(f; x_i, c) = (l_1(f; x_i, c), l_2(f; x_i, c))^T \in R^2, \tag{2.4}$$

$$A(f; x_i, c) = \begin{pmatrix} (u_i - 2c)l_1(f; x_i, c) + l_2(f; x_i, c) & l_1(f; x_i, c) \\ (v_i + u_i c - c^2)l_1(f; x_i, c) & l_2(f; x_i, c) \end{pmatrix}. \tag{2.5}$$

Particularly, we denote

$$L(f; x_i; t) = l(f; x_i, 0; t),$$

$$l(f;x_i)=l(f;x_i,0)=(l_1(f;x_i),l_2(f;x_i))^T, \tag{2.6}$$

$$A(f;x_i)=A(f;x_i,0)=\begin{pmatrix} u_il_1(f;x_i,c)+l_2(f;x_i)) & l_1(f;x_i) \\ v_il_1(f;x_i) & l_2(f;x_i)\end{pmatrix}. \tag{2.7}$$

$L(f)$ is determined uniquely in spite of c, but a suitable choice of c may reduce the computations (see [12]). Clearly,

$$L(f;x_i,0;t)=L(f;x_i,c;t)-cl_1(f;x_i,c).$$

Therefore we always assume $c=0$ in the following.

It is enough to determine $l(f;x_i)$ for finding $L(f;x_i;t)$. Zheng[12] showed the following properties for the operators of linear interpolation defined above:

$$l(Q_i;x_i)=0=(0,0)^T, \tag{2.8}$$

$$l(Q_k;x_i)=(u_i-u_k,v_i-v_k)^T, \tag{2.9}$$

$$l(af+bg;x_i)=al(f;x_i)+bl(g;x_i), A(af+bg;x_i)=aA(f;x_i)+bA(g;x_i), \forall a,b\in R^1, \tag{2.10}$$

$$l(fg;x_i)=A(f;x_i)l(g;x_i)=A(g;x_i)l(f;x_i), \tag{2.11}$$

$$det(A(f;x_i))=f(\alpha_i)f(\beta_i), \tag{2.12}$$

$$l(f/g;x_i)=A(g;x_i)^{-1}l(f;x_i). \tag{2.13}$$

Moreover, from definition and above properties it is easy to prove the followings:

$$A(Q_i;x_i)=0=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \tag{2.14}$$

$$A(fg;x_i)=A(f;x_i)A(g;x_i)=A(g;x_i)A(f;x_i), \tag{2.15}$$

$$A(f/g;x_i)=A(f;x_i)A(g;x_i)^{-1}=A(g;x_i)^{-1}A(f;x_i), \tag{2.16}$$

$$l(1;x_i)=(0,1)^T,\ l(t;x_i)=(1,0)^T,\ l(t^2;x_i)=(u_i,v_i)^T, \tag{2.17}$$

$$l(f;x_i)=A(f;x_i)(0,1)^T,\ l(tf;x_i)=A(f;x_i)(1,0)^T, \tag{2.18}$$

$$A(1;x_i)=E, \tag{2.19}$$

$$A(t^2;x_i)=\begin{pmatrix}u_i^2+v_i & u_i\\ u_iv_i & v_i\end{pmatrix}. \tag{2.20}$$

Notations. Some notations are often used through the paper. For convenience we list them here and will not explain them repeatedly. Let

$$x_i=(u_i,v_i)^T\in R^2,$$

$$x=(u_1,v_1,\cdots,u_n,v_n)^T\in R^{2n},$$

α_i,β_i are the zeros of

$$Q_i(t)=t^2-u_it-v_i,$$

$$g(t)=\prod_{j=1}^{n}Q_j(t)=\prod_{j=1}^{n}(t^2-u_jt-v_j)=\sum_{\nu=0}^{2n}b_\nu(x)t^{2n-\nu}, b_0=1, \tag{2.21}$$

$$g_i(t)=\prod_{j\neq i}Q_j(t)=\prod_{j\neq i}(t^2-u_jt-v_j)=\sum_{\nu=0}^{2n-2}b_\nu^i(x)t^{2n-\nu-2}, b_0^i=1, \tag{2.22}$$

$$z_i = (\xi_i, \eta_i)^T = l(\frac{p}{g_i}; x_i) \in R^2,$$

$$z = z(x) = (\xi_1, \eta_1, \cdots, \xi_n, \eta_n)^T \in R^{2n}.$$

3. Nonsingulity of $F'(x)$ of (1.7)

In order to apply Newton's and Chebyshev's methods to solving the equation (1.7) it is necessary to give the condition for nonsingulity of the derivative $F'(x)$ of (1.7). We have

Theorem 1. *The derivative $F'(x)$ of (1.7) is nonsingular if $Q_i(\alpha_j)Q_i(\beta_j) \neq 0$ for $j \neq i$. Furthermore,*

$$F'(x)^{-1} = -D_n^{-1} C_n, \tag{3.1}$$

where

$$C_n = \begin{pmatrix} l(t^{2n-1}; x_1) & l(t^{2n-2} : x_1) & \cdots & l(t; x_1) & l(1; x_1) \\ l(t^{2n-1}; x_2) & l(t^{2n-2} : x_2) & \cdots & l(t; x_2) & l(1; x_2) \\ \cdots & & & & \\ l(t^{2n-1}; x_n) & l(t^{2n-2} : x_n) & \cdots & l(t; x_n) & l(1; x_n) \end{pmatrix}, \tag{3.2}$$

$$D_n = \begin{pmatrix} A(g_1; x_1) & & & 0 \\ & A(g_1; x_2) & & \\ 0 & & \ddots & \\ & & & A(g_1; x_n) \end{pmatrix}. \tag{3.3}$$

Proof. By differentiation with respect to u_i and v_i in (2.21), respectively, we have

$$\sum_{\nu=1}^{2n} \frac{\partial f_\nu}{\partial u_i} t^{2n-\nu} = \sum_{\nu=1}^{2n} \frac{\partial b_\nu}{\partial u_i} t^{2n-\nu} = -t g_i(t), \tag{3.4}$$

$$\sum_{\nu=1}^{2n} \frac{\partial f_\nu}{\partial v_i} t^{2n-\nu} = \sum_{\nu=1}^{2n} \frac{\partial b_\nu}{\partial v_i} t^{2n-\nu} = -g_i(t). \tag{3.5}$$

Observing (2.8),(2.11),(2.18),(1.7) it yields $l(g_i; x_k) = 0$ for $k \neq i$ and

$$\sum_{\nu=1}^{2n} \frac{\partial f_\nu}{\partial u_i} l(t^{2n-\nu}; x_k) = \begin{cases} -A(g_i; x_i)(1,0)^T, & k = i, \\ 0, & k \neq i, \end{cases}$$

$$\sum_{\nu=1}^{2n} \frac{\partial f_\nu}{\partial v_i} l(t^{2n-\nu}; x_k) = \begin{cases} -A(g_i; x_i)(0,1)^T, & k = i, \\ 0, & k \neq i, \end{cases}$$

i. e.,

$$C_n F'(x) = -D_n, \tag{3.6}$$

where C_n, D_n are defined by (3.2) and (3.3), respectively. By (2.12), (2.15), (2.20) we see that

$$det(D_n) = \prod_{i=1}^{n} g_i(\alpha_i) g_i(\beta_i) = \prod_{i=1}^{n} \prod_{j \neq i} Q_j(\alpha_i) Q_j(\beta_i) \neq 0. \tag{3.7}$$

In the other hand it can be proved by mathematical induction that

$$det(C_n) \neq 0. \tag{3.8}$$

In fact, (3.8) is true for $n = 1$ because $C_1 = E$. Suppose that (3.8) holds for $n-1$. We take the following elementary transformation for C_n in (3.2) as follows. Subtracting the $k+2$-th column left multiplied by $A(t^2; x_n)$ from the k-th column, $k = 1, 2, \cdots, 2n-2$, by

$$l(t^{m+2}; x_i) = A(t^2; x_i) l(t^m; x_i), \quad (l(t; x_i)\ l(1; x_i)) = E$$

the matrix C_n is transformed to

$$C_n' = \begin{pmatrix} B_1 l(t^{2n-3}; x_1) & B_1 l(t^{2n-4}; x_1) & \vdots & B_1 l(1; x_1) & l(t; x_1) & l(1; x_1) \\ B_2 l(t^{2n-3}; x_2) & B_2 l(t^{2n-4}; x_2) & \vdots & B_2 l(1; x_2) & l(t; x_2) & l(1; x_2) \\ \cdots\cdots & & & & & \\ B_{n-1} l(t^{2n-3}; x_{n-1}) & B_{n-1} l(t^{2n-4}; x_{n-1}) & \vdots & B_{n-1} l(1; x_{n-1}) & l(t; x_{n-1}) & l(1; x_{n-1}) \\ 0 & 0 & \vdots & 0 & l(t; x_n) & l(1; x_n) \end{pmatrix},$$

where

$$B_i = A(t^2; x_i) - A(t^2; x_n).$$

So

$$det(C_n) = det(C_n') = [\prod_{i=1}^{n-1} det(B_i)] det(C_{n-1}).$$

By (2.20) and $\alpha_i + \beta_i = u_i, \alpha_i \beta_i = -v_i, \alpha_i^2 + \beta_i^2 = u_i^2 + 2v_i$, it can be verified that

$$det(B_i) = det(A(t^2; x_i) - A(t^2; x_n)) = Q_n(\alpha_i) Q_n(\beta_i).$$

Therefore (3.8) is true for n. Then the nonsingulity of $F'(x)$ and (3.1) are obtained by (3.6), (3.7) and (3.8). The proof of the theorem is completed.

We always assume $Q_i(\alpha_j) Q_i(\beta_j) \neq 0$ for $j \neq i$ in the following because of Theorem 1.

4. Newton's Method Applied to (1.7)

Suppose that $x_i = (u_i, v_i)^T$ is an approximation of $(p_i, q_i)^T$ in (1.5). By (1.5) and (2.9) it yields

$$\begin{aligned} (u_i, v_i)^T - (p_i, q_i)^T &= (u_i - p_i, v_i - q_i)^T \\ &= l(t^2 - p_i t - q_i; x_i) = l(\frac{p(t)}{\prod_{j \neq i} (t^2 - p_j t - q_j)}; x_i). \end{aligned}$$

By replacing $\prod_{j \neq i} (t^2 - p_j t - q_j)$ with some approximation functions Zheng[12] proposed a family of parallel iterations $P(q)$ with parameter $q = 1, 2, \cdots$ to determine all real factors of (1.5). One of them is

$$x_i^+ = x_i - z_i = x_i - l(\frac{p}{g_i}; x_i) \tag{4.1}$$

corresponding to $q = 1$, where $x_i^+ = (u_i^+, v_i^+)^T$ denotes new approximation of $(p_i, q_i)^T$. We now show the following theorem.

Theorem 2. *Iteration (4.1) is Newton's method applied to (1.7).*

Proof. By (1.7), (3.1), (3.2) and (3.3) we see that

$$F'(x)^{-1} F(x) = -D_n^{-1} C_n F(x)$$

$$= -\begin{pmatrix} A(g_1;x_1)^{-1}\sum\limits_{\nu=1}^{2n}(b_\nu(x)-a_\nu)l(t^{2n-\nu};x_1) \\ A(g_2;x_2)^{-1}\sum\limits_{\nu=1}^{2n}(b_\nu(x)-a_\nu)l(t^{2n-\nu};x_2) \\ \cdots\cdots \\ A(g_n;x_n)^{-1}\sum\limits_{\nu=1}^{2n}(b_\nu(x)-a_\nu)l(t^{2n-\nu};x_n) \end{pmatrix}.$$

Observing $b_0 = a_0 = 1$ and $l(g;x_i) = 0$, the i-th subvector above is

$$A(g_i;x_i)^{-1}\sum_{\nu=1}^{2n}(b_\nu(x)-a_\nu)l(t^{2n-\nu};x_i) = A(g_i;x_i)^{-1}\sum_{\nu=0}^{2n}(b_\nu(x)-a_\nu)l(t^{2n-\nu};x_i)$$

$$= A(g_i;x_i)^{-1}(l(g;x_i)-l(p;x_i)) = -A(g_i;x_i)^{-1}l(p;x_i) = -l(\frac{p}{g_i};x_i).$$

Therefore (4.1) is obtained once again by applying Newton's method to (1.7). Theorem 2 is proved.

It is interest that Theorem 2 can be proved in a different way. And we show it as follows because some conclusions obtained in the next proof are useful in the following.

Proof of Theorem 2 in another way

Let

$$p_{2n-1}(t) = \sum_{i=1}^{n}(\xi_i t + \eta_i)g_i(t).$$

By

$$\xi_j\alpha_j + \eta_j = \frac{p(\alpha_j)}{g_j(\alpha_j)},\ \xi_j\beta_j + \eta_j = \frac{p(\beta_j)}{g_j(\beta_j)}$$

and $g_i(\alpha_j) = g_i(\beta_j) = 0$ for $i \neq j$, we see that

$$p_{2n-1}(\alpha_j) = p(\alpha_j),\ p_{2n-a}(\beta_j) = p(\beta_j).$$

Therefore $p_{2n-1}(t)$ is the interpolation polynomial of $p(t)$ with nodes $\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n$ and

$$p(t) = \sum_{i=1}^{n}(\xi_i t + \eta_i)g_i(t) + g(t). \tag{4.2}$$

Substituting (2.22) into (3.4) and (3.5), it yields

$$\sum_{\nu=1}^{2n}\frac{\partial f_\nu}{\partial u_i}t^{2n-\nu} = -\sum_{\nu=0}^{2n-2}b_\nu^i(x)t^{2n-\nu-1} = -\sum_{\nu=1}^{2n-1}b_{\nu-1}^i(x)t^{2n-\nu},$$

$$\sum_{\nu=1}^{2n}\frac{\partial f_\nu}{\partial v_i}t^{2n-\nu} = -\sum_{\nu=0}^{2n-2}b_\nu^i(x)t^{2n-\nu-2} = -\sum_{\nu=2}^{2n-1}b_{\nu-2}^i(x)t^{2n-\nu}.$$

Comparing the coefficients of $t^{2n-\nu}$, we have

$$\frac{\partial f_\nu}{\partial u_i} = -b_{\nu-1}(x),\ \frac{\partial f_\nu}{\partial v_i} = -b_{\nu-2}(x),\ \nu = 1, 2, \cdots, 2n, \tag{4.3}$$

where $b_0^i(x) = 1, b_\nu^i(x) = 0$ for $\nu < 0$ or $\nu \geq 2n-1$. Substituting (1.4), (2.21) and (2.22) into (4.2), it is obtained that

$$\sum_{\nu=0}^{2n}a_\nu t^{2n-\nu} = \sum_{\nu=0}^{2n}t^{2n-\nu}[\sum_{i=1}^{n}(b_{\nu-1}^i(x)\xi_i + b_{\nu-2}^i(x)\eta_i) + b_\nu(x)].$$

Comparing the coefficients of $t^{2n-\nu}$, by (4.3) and (1.7) we see that

$$-\sum_{i=1}^{n}(\frac{\partial f_\nu}{\partial u_i}\xi_i+\frac{\partial f_\nu}{\partial v_i}\eta_i)+f_\nu(x)=0,\ \nu=1,2,\cdots,2n,$$

i. e.,

$$-F'(x)z+F(x)=0, \tag{4.4}$$

$$z=F'(x)^{-1}F(x). \tag{4.5}$$

Therefore (4.1) is Newton's method applied to (1.7)

$$x^+=x-z=x-F'(x)^{-1}F(x).$$

Theorem 2 is proved once again.

5. Chebyshev's Method Applied to (1.7)

In this section we applied Chebyshev's method to (1.7) and obtain a new parallel iteration for factorization of polynomials. We have

Theorem 3. *Chebyshev's method applied to (1.7) is parallel iteration*

$$\widehat{x}_i=x_i-z_i+w_i, \tag{5.1}$$

where x_i *is an approximation of* $(p_i,q_i)^T$ *in (1.5),* $\widehat{x}_i$ *is the new one and*

$$w_i=A(\frac{p}{g_i};x_i)\sum_{j\neq i}A(Q_j;x_i)^{-1}z_j. \tag{5.2}$$

Proof. By differentiation with respect to u_k in (4.2) it yields

$$\begin{aligned}0&=\sum_{i=1}^{n}(\frac{\partial\xi_i}{\partial u_k}t+\frac{\partial\eta_i}{\partial u_k})g_i(t)+\sum_{i\neq k}(\xi_it+\eta_i)(-t)\prod_{j\neq i,k}Q_j(t)-tg_k(t)\\&=\sum_{i=1}^{n}(\frac{\partial\xi_i}{\partial u_k}t+\frac{\partial\eta_i}{\partial u_k})g_i(t)-t[\sum_{i\neq k}\frac{\xi_it+\eta_i}{Q_i(t)}+1]g_k(t).\end{aligned} \tag{5.3}$$

Similarly,

$$0=\sum_{i=1}^{n}(\frac{\partial\xi_i}{\partial v_k}t+\frac{\partial\eta_i}{\partial v_k})g_i(t)-[\sum_{i\neq k}\frac{\xi_it+\eta_i}{Q_i(t)}+1]g_k(t). \tag{5.4}$$

Observing $A(g_i;x_k)=0$ for $i\neq k, l(at+b;x_k)=(a,b)^T$ and nonsingulity of $A(g_k;x_k)$, by (2.18), (2.19), (5.3) and (5.4) we have

$$(\frac{\partial\xi_k}{\partial u_k},\frac{\partial\eta_k}{\partial u_k})^T=[E+\sum_{i\neq k}A(Q_i;x_k)^{-1}A(\xi_it+\eta_i;x_k)](1,0)^T,$$

$$(\frac{\partial\xi_k}{\partial v_k},\frac{\partial\eta_k}{\partial v_k})^T=[E+\sum_{i\neq k}A(Q_i;x_k)^{-1}A(\xi_it+\eta_i;x_k)](0,1)^T,$$

and

$$\begin{pmatrix}\frac{\partial\xi_k}{\partial u_k}&\frac{\partial\xi_k}{\partial v_k}\\\frac{\partial\eta_k}{\partial u_k}&\frac{\partial\eta_k}{\partial v_k}\end{pmatrix}=E+\sum_{i\neq k}A(Q_i;x_k)^{-1}A(\xi_it+\eta_i;x_k). \tag{5.5}$$

For $i \neq k$ by (2.15), (2.16) and (2.18)

$$(\frac{\partial \xi_k}{\partial u_i}, \frac{\partial \eta_k}{\partial u_i})^T = \frac{\partial l(p(t)/g_k(t); x_k)}{\partial u_i} = l(\frac{\partial (p/g_k)}{\partial u_i}; x_k)$$

$$= l(\frac{t}{Q_i(t)} \frac{p(t)}{g_k(t)}; x_k) = A(Q_i; x_k)^{-1} A(\frac{p}{g_k}; x_k)(1,0)^T.$$

Similarly,

$$(\frac{\partial \xi_k}{\partial v_i}, \frac{\partial \eta_k}{\partial v_i})^T = A(Q_i; x_k)^{-1} A(\frac{p}{g_k}; x_k)(0,1)^T.$$

Therefore

$$\begin{pmatrix} \frac{\partial \xi_k}{\partial u_i} & \frac{\partial \xi_k}{\partial v_i} \\ \frac{\partial \eta_k}{\partial u_i} & \frac{\partial \eta_k}{\partial v_i} \end{pmatrix} = A(Q_i; x_k)^{-1} A(\frac{p}{g_k}; x_k), \ i \neq k. \tag{5.6}$$

Then we obtain from (5.5), (5.6)

$$z'(x) = I + S,$$

where

$$S =$$

$$\begin{pmatrix} \sum_{j\neq 1} A(Q_j; x_1)^{-1} A(L_j(t); x_1) & A(Q_2; x_1)^{-1} A(\frac{p}{g_1}; x_1) & \vdots & A(Q_n; x_1)^{-1} A(\frac{p}{g_1}; x_1) \\ A(Q_1; x_2)^{-1} A(\frac{p}{g_2}; x_2) & \sum_{j\neq 2} A(Q_j; x_2)^{-1} A(L_j(t); x_2) & \vdots & A(Q_n; x_2)^{-1} A(\frac{p}{g_2}; x_2) \\ \cdots\cdots & & & \\ A(Q_1; x_n)^{-1} A(\frac{p}{g_n}; x_n) & A(Q_2; x_n)^{-1} A(\frac{p}{g_n}; x_n) & \vdots & \sum_{j\neq n} A(Q_j; x_n)^{-1} A(L_j(t); x_n) \end{pmatrix} \tag{5.7}$$

with $L_j(t) = \xi_j t + \eta_j$.

By differentiation with respect to x in (4.4) we have

$$F''(x)z + F'(x)z'(x) = F'(x),$$

$$F'(x)^{-1}F''(x)F'(x)^{-1}F(x) = F'(x)^{-1}F''(x)z = I - z'(x) = -S,$$

$$F'(x)^{-1}F''(x)(F'(x)^{-1}F(x))^2 = -Sz. \tag{5.8}$$

From (5.7) we see that the i-th subvector in R^2 of Sz is

$$(Sz)_i = \sum_{j\neq 1} A(Q_j; x_i)^{-1} A(\xi_j t + \eta_j; x_i) z_i + \sum_{j\neq i} A(Q_j; x_i)^{-1} A(\frac{p}{g_j}; x_j) z_j. \tag{5.9}$$

By (2.11)

$$A(\xi_j t + \eta_j; x_i) z_i = A(\xi_j t + \eta_j; x_i) l(\frac{p}{g_i}; x_i) = A(\frac{p}{g_i}; x_i) l(\xi_j t + \eta_j; x_i)$$

$$= A(\frac{p}{g_i}; x_i)(\xi_j, \eta_j)^T = A(\frac{p}{g_i}; x_i) z_j. \tag{5.10}$$

Then we have by (2.16), (5.8) and (5.9)

$$(Sz)_i = 2w_i, \tag{5.11}$$

where w_i is defined by (5.2). Substituting (4.5), (5.8) and (5.11) into (1.2), (5.1) is obtained. The proof of Theorem 3 is completed.

6. Convergence of Parallel Iterations (4.1) and (5.1)

By Theorem 1, 2, 3 and the well known results about the convergence of Newton's and Chebyshev's methods we immediately have

Theorem 4. *The convergence of parallel iterations (4.1) and (5.1) is quadratic and cubic, respectively, if the zeros of* $t^2 - p_i t - q_i$ *are not those of* $t^2 - p_j t - q_j$ *in (1.5) for* $j \neq i$.

The following theorem shows that the arithmetic mean of all $2n$ zeros of the approximation quadratic factors after one iteration step by (4.1) or (5.1) is equal to that of the exact zeros, no matter how to choose the initial approximations $(u_i^0, v_i^0)^T$.

Theorem 5. *For any initial approximations* $(u_i^0, v_i^0)^T$ *the sequence* $x_i^k = (u_i^k, v_i^k)^T$ *produced by (4.1) or (5.1) satisfies the following relation*

$$\sum_{i=1}^{n} u_i^{k+1} = \sum_{i=1}^{n} p_i = -a_1, \forall k \geq 0. \tag{6.1}$$

Proof. It is clear that the first component of $F(x)$ defined by (1.7) is

$$f_1(x) = b_1(x) - a_1 = -\sum_{i=1}^{n} u_i - a_1$$

and that

$$\frac{\partial f_1}{\partial u_i} = -1, \frac{\partial^2 f_1}{\partial u_i \partial u_k} = \frac{\partial^2 f_1}{\partial u_i \partial v_k} = \frac{\partial^2 f_1}{\partial v_i \partial v_k} = 0.$$

In the Newton's and Chebyshev's iterations

$$F(x) + F'(x)(x^+ - x) = 0,$$

$$F(x) + F'(x)(\widehat{x} - x) + \tfrac{1}{2} F''(x)(F'(x)^{-1} F(x))^2 = 0$$

we compare the first component of both sides to see

$$-\sum_{i=1}^{n} u_i - a_1 - \sum_{i=1}^{n} (u_i^+ - u_i) = 0,$$

$$-\sum_{i=1}^{n} u_i - a_1 - \sum_{i=1}^{n} (\widehat{u}_i - u_i) = 0.$$

These imply

$$\sum_{i=1}^{n} u_i^+ = \sum_{i=1}^{n} \widehat{u}_i = -a_1 = \sum_{i=1}^{n} p_i.$$

The theorem is proved.

Remark. The conclusion about (4.1) in Theorem 5 has been proved in [12] in a different way.

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