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A HOMOTOPY METHOD OF SWITCHING SOLUTION BRANCHES AT THE PITCHFORK BIFURCATION POINT*[1)]

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Abstract

By introducing proper parameters in the original nonlinear system, a continuation method for switching solution branches at a pitchfork point is proposed and their theories have been established in this paper. It is sufficient to implement this method that a standard continuation procedure is only used. Some numerical examples are given to illustrate the effectiveness of this method.

Key words: Homotopy method, Switching solution branches, Pitchfork point.

1. Introduction

In order to compute all solutions of a nonlinear system of equations, a numerical method is needed for changing solution branches at bifurcation points. Suppose that a bifurcation point (x^*, λ^*) hase been located. H.B.Keller [4] proposed a method of switching solution branches at (x^*, λ^*) by means of distinct roots of a homogeneously quadratic system of equations, W.C.Rheinboldt [5] gave a method for switching solution branches at a simple bifurcation point, using a singular chord method, which does not require the bifurcation point to be known accurately. The methods mentioned above depend on the asymptotic expressions of bifurcation solutions near a bifurcation point in principle. In this paper we present a new method for switching the solution branches which does not depend on the asymptotic expressions of bifurcation solutions. In our method, firstly we make an unfolding of the original system by introducing proper parameters. Then by using the standard continuation method in [1] [4] we track a solution curve of the unfolding problem to realize the switching of solution branches. Our method like the method described in [5] also does not require the bifurcation point to be known and any information of the second order derivatives. Because it is sufficient to implement our method that the standard continuation software is only used, our method is more effective and simpler than the methods mentioned above. In addition, as our method independs on the the asymptotic expressions of bifurcation solutions, one may expect that this method is valid for many kinds of bifurcation points.

In the second section, we discuss carefully the numerical method for switching solution branches for the standard pitchfork problem. Although the solutions of this problem are obtained straightforwards, yet it is necessary to understand the principle of our method and to analyse the general cases. In the third section, we discuss the case of system of equations and show the validity of the method of this paper. Finally some numerical examples are given to demonstrate the effectiveness of our method in the fourth section.

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2. The Case of Standard Pitchfork

In this section, the problem of standard pitchfork:

$$f(\xi,\lambda) := \xi^3 - \lambda\xi = 0 \tag{1}$$

will be examined in detail. Obviously, the bifurcation diagram of (1),namely, the solution set of (1) consists of two curves $\Gamma_1 : \xi = 0$ and $\Gamma_2 : \xi^2 = \lambda$ which intersect at the pitchfork bifurcation point $(\xi, \lambda) = (0,0)$ (see Fig.1.)

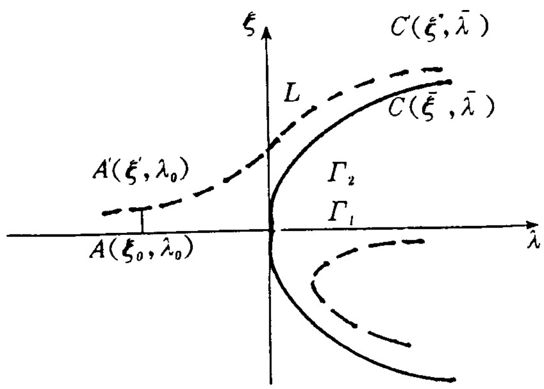

Fig.1. The bifurcation diagram of (1) and (2).

Now assume that by tracking the solution curve Γ_1, we get a point A on Γ_1 near the pitchfork point $(0,0)$, the coordinate of A is denoted by (ξ_0, λ_0),where $\lambda_0 < 0$. To obtain a point on Γ_2, we make a one-parameter unfolding of $f(\xi, \lambda)$[3] as follows:

$$g(\xi,\lambda;\beta) := \xi^3 - \lambda\xi - \beta = 0. \tag{2}$$

The bifurcation diagram of (2) with $\beta \neq 0$ is shown by the broken lines in Fig1. The original idea for realizing the switching of solution branches is represented as follows. In the first stage, we choose any ξ' near ξ_0 such that $\beta' = f(\xi', \lambda_0) \neq 0$, the point (ξ', λ_0) is denoted by A' in the Fig1. Obviously, we have $g(\xi', \lambda_0; \beta') = 0$. As 0 is the regular value of $g(\bullet, \bullet; \beta')$[2] , we track solution curve L of the equation $g(\bullet, \bullet; \beta') = 0$ passing A' by using the standard continuation method until obtain a point $C'(\xi^{"}, \bar{\lambda})$ on L in which $\bar{\lambda} > 0$ in the second stage. As 0 is also the regular value of $g(\bullet, \bar{\lambda}; \bullet)$, by using the continuation method once again, track the solution curve of the equation $g(\bullet, \bar{\lambda}; \bullet) = 0$ passing point $(\xi^{"}, \beta')$ until get the solution $(\bar{\xi}, 0)$ in the third stage. Finally, we will show that $(\bar{\xi}, \bar{\lambda})$ denoted by C in Fig.1 is just a desired point on Γ_2. According to above frame, two continuation procedures are needed for switching solution branches. However, if we allow to change the value of β' in the second stage, the three stage in above frame can be combined into one continuation procedure. The following describe the details of our method mentioned above, consider the system of equations:

$$H(\xi,\lambda;s) := \begin{pmatrix} \xi^3 - \lambda\xi - \alpha\theta(\xi,\lambda;\alpha) \\ \lambda - \lambda_0 - \eta s \end{pmatrix} = 0, \tag{3}$$

in which $\alpha = \alpha(s) := \epsilon s(1-s)$, θ is a smooth function and $\theta(0,0;0) \neq 0$,η and ϵ are given parameters. Next we shall show that if (ξ_0, λ_0) is sufficiently close to pitchfork point $(0,0)$ and the parameters ϵ and η are chosen properly, for any given neighborhood Ω of $(0,0)$, there exists unique smooth solution curve $\Gamma : (\xi(s), \lambda(s)) \in \bar{\Omega}, (0 \le s \le 1)$ of (3) satisfying $\xi(0) = \xi_0, \lambda(0) = \lambda_0, \lambda(1) > 0$, and that $\frac{\partial H}{\partial(\xi,\lambda)}$ at any point on Γ are invertible, namely, all of the points on Γ are regular points of H. Hence we can track the solution curve Γ of (3) by means of the continuation method with initial value $(\xi, \lambda; s) = (\xi_0, \lambda_0, 0)$ until get a point $(\bar{\xi}, \bar{\lambda}) := (\xi(1), \lambda(1))$. Finally we shall prove that the point $(\bar{\xi}, \bar{\lambda})$ is just one we expect on Γ_2.

For convenience we shall assume that $\theta(0,0;0) > 0$ in the latter contex, in the opposite case the discussion is fairly similar. From the expression of g and $\theta(0,0;0) > 0$, it is easy to see that there exist positive numbers $a, b, \delta, \delta', c_0, c_1$ and c_2 so that

$$\begin{cases} g(a, \lambda; \beta) > 0, \\ g(-a, \lambda; \beta) < 0. \end{cases} \qquad as \qquad |\lambda| \le b, \qquad |\beta| \le \delta, \tag{4}$$

$$\begin{cases} 0 < c_0 \le \theta(\xi, \lambda; \alpha) \le c_1, \\ 0 < c_2 \le \theta(\xi, \lambda; \alpha) - \xi\theta_\xi(\xi, \lambda; \alpha), \end{cases} \qquad as \qquad |\xi| \le a, \qquad |\lambda| \le b, \qquad |\alpha| \le \delta'. \tag{5}$$

And the positive numbers a and b may be chosen so small that $(-a, a) \times (-b, b) \subset \Omega$. Hence we may regard $(-a, a) \times (-b, b)$ as Ω without generality of loss. As the known point (ξ_0, λ_0) on Γ_1 is close to the pitchfork point $(0,0)$, we may assume that

$$-b < \lambda_0 < 0. \tag{6}$$

Now we chose the parameters ϵ and η to satisfy conditions:

$$\begin{cases} 0 < \epsilon \le \epsilon_0, \\ -\lambda_0 < \eta < b - \lambda_0, \end{cases} \tag{7}$$

where $\epsilon_0 := min(\frac{4\delta}{c_1}, 4\delta')$.

Lemma 2.1. *Suppose that (4)-(7) are hold, there exists unique smooth solution* $\Gamma : (\xi(s), \lambda(s)) \in \bar{\Omega}$ $(0 \le s \le 1)$ *of (3) such that* $\xi(0) = \xi_0, \lambda(0) = \lambda_0$ *and the Jacobian matrices* $\frac{\partial H}{\partial(\xi,\lambda)}$ *at any points on* Γ *are all nonsingular.*

Proof. From (4) (5) and (7) it is obvious that

$$|\alpha(s)| \le \frac{\epsilon}{4}, \qquad |\alpha(s)\theta(\xi, \lambda; \alpha(s))| \le \delta, \qquad 0 \le s \le 1, \qquad (\xi, \lambda) \in \bar{\Omega},$$

$$\begin{cases} g(a, \lambda; \alpha(s)\theta(a, \lambda; \alpha(s)) > 0, \\ g(-a, \lambda; \alpha(s)\theta(-a, \lambda; \alpha(s)) < 0, \end{cases}$$

where $0 \le s \le 1, |\lambda| \le b$. And from (6) (7) we have

$$-b - \lambda_0 - \eta s < 0, \qquad b - \lambda_0 - \eta s > 0, \qquad 0 \le s \le 1.$$

Thus we assert

$$H(\xi, \lambda; s) \ne 0, \qquad 0 \le s \le 1, \qquad (x, \lambda) \in \partial\Omega. \tag{8}$$

Next, it is easy to see that the equation (3) with $s = 0$ has only one solution $(0, \lambda_0)$ and the equation (3) with $s = 1$ has three solutions $(0, \eta + \lambda_0)$ and $(\pm(\eta + \lambda_0)^{\frac{1}{2}}, \eta + \lambda_0)$. Moreover, the Jacobian matrices $\frac{\partial H}{\partial(\xi,\lambda)}$ at these solutions are all nonsingular. Finally, if $(\xi, \lambda, s) \in \Omega \times (0,1)$ is a solution of (3), since we track the solution curve L (see Fig. 1) which satisfies $\xi > 0$ (also to see the conclusion (a) (b) in the proof of Lemma 2.2), from (5) and $0 < \alpha(s) \le \delta'$ we have

$$(3\xi^2 - \lambda - \alpha\theta_\xi)\xi = 2\xi^3 + \alpha(s)(\theta - \xi\theta_\xi) > 0.$$

Thus the Jacobian matrix

$$\frac{\partial H}{\partial(\xi,\lambda)} = \begin{pmatrix} 3\xi^2 - \lambda - \alpha\theta_\xi & -(\xi + \alpha\theta_\lambda) \\ 0 & 1 \end{pmatrix} \tag{9}$$

at the solution (ξ, λ, s) of (3) is invertible. By means of the implicit function theorem or standard continuation theorem in [1,2], the conclusions of the Lemma are valid. The proof is complete.

Let

$$(\bar{\xi}, \bar{\lambda}) := (\xi(1), \lambda(1)), \tag{10}$$

Obviously, $(\bar{\xi}, \bar{\lambda})$ is a solution of (1) and we have

$$\bar{\lambda} := \lambda_0 + \eta > 0. \tag{11}$$

Lemma 2.2. *If the conditions of Lemma 2.1 are true, we have $\bar{\xi} > 0$ and*

$$\bar{\xi}^2 = \bar{\lambda}, \tag{12}$$

where $(\bar{\xi}, \bar{\lambda})$ is defined by (10).

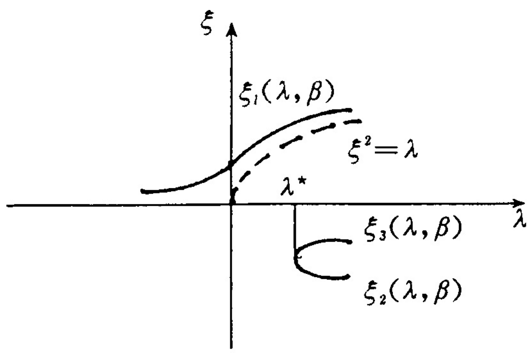

Fig.2. The bifurcation diagram of (2).

Proof. Through elementary operations for (2), it is verified that

1) the equation (2) has only one solution denoted by $\xi_1(\lambda, \beta)$ as $\lambda < \lambda^* := \frac{1}{3}(\frac{\beta}{2})^{\frac{2}{3}}$,
2) the equation (2) has three solutions denoted by $\xi_i(\lambda, \beta), i = 1, 2, 3$ as $\lambda > \lambda^*$,
3) there are following relations between these solutions

$$\xi_1(\lambda, \beta) > 0; \quad \lambda^{\frac{1}{2}} < \xi_1(\lambda, \beta), \quad as \quad \lambda > 0, \tag{13}$$

$$-\lambda^{\frac{1}{2}} < \xi_2(\lambda, \beta) < -(\frac{\lambda}{3})^{\frac{1}{2}} < \xi_3(\lambda, \beta) < 0, \quad as \quad \lambda > \lambda^*. \tag{14}$$

The bifurcation diagram of (2) is shown in Fig.2. as $\beta > 0$

Now suppose that (ξ, λ, β) is a solution of (2), where $\beta > 0$. From (13) (14) we assert that
(a) if $\lambda < 0$, then $\xi > 0$
(b) if $\lambda > 0$ and $\xi > 0$ then $\xi = \xi_1(\lambda, \beta) > \lambda^{\frac{1}{2}}$.

Let

$$\beta(s) := \alpha(s)\theta(\xi(s), \lambda(s); \alpha(s)), \tag{15}$$

where $(\xi(s), \lambda(s))$ is defined in Lemma 2.1, we see that

$$\begin{cases} \xi^3(s) - \lambda(s)\xi(s) - \beta(s) = 0, \\ \lambda(s) = \lambda_0 + \eta s. \end{cases} \qquad 0 \le s \le 1, \tag{16}$$

From (15) and (16) we get $\beta(s) > 0$ and $\xi(s) \neq 0$ as $s \in (0,1)$ .Further, we are sure of

$$\xi(s) > 0, \qquad s \in (0,1), \tag{17}$$

from $\lambda(0) = \lambda_0 < 0$ and the conclusion (a) mentioned above . As $\lambda(1) = \lambda_0 + \eta > 0$, it follows that there exist $\bar{s} \in (0,1)$ and $0 < \eta_0 < \lambda_0 + \eta$ such that $\lambda(s) \ge \eta_0$ as $s \in (\bar{s}, 1)$. Thus the conclusion (b) mentioned above implies

$$\xi(s) > \lambda^{\frac{1}{2}}(s) \ge \eta_0{}^{\frac{1}{2}}.$$

Let $s \to 1$ in the above inequalities, it yields that

$$\bar{\xi} = \xi(1) \ge \eta_0{}^{\frac{1}{2}} > 0.$$

Finally, because $(\bar{\xi}, \bar{\lambda})$ is a solution of (1), we get $\bar{\xi}^2 = \bar{\lambda}$. The proof is completed.

Summarizing the above two Lemmas, we get the following theorem.

Theorem 2.3. *Assume that $\theta(\xi, \lambda; \alpha)$ is a smooth function with $\theta(0,0;0) > 0(< 0)$ and Ω is a neighborhood of $(\xi, \lambda) = (0,0)$. If choose ϵ and η such that (6) and (7) are hold, there exists unique smooth solution curve $\Gamma : (\xi(s), \lambda(s)) \in \bar{\Omega}, 0 \le s \le 1$ of (3) such that the Jacobian matrices $\frac{\partial H}{\partial(\xi,\lambda)}$ at any points on Γ are all invertible and $\xi(s)$ and $\lambda(s)$ satisfy*

$$\xi(0) = \xi_0, \qquad \lambda(0) = \lambda_0, \tag{18}$$

$$\bar{\xi} := \xi(1) > 0(< 0), \qquad \bar{\lambda} := \lambda(1) = \bar{\xi}^2 > 0. \tag{19}$$

3. The Case of System of Equations

Let us consider the system of equations

$$F(x, \lambda) = 0, \tag{20}$$

where $F : R^n \times R \to R^n$ is a smooth function. If a solution (x^*, λ^*) of (20) satisfies the following hypothesis:

$H_1\rangle \quad Null(F_x{}^*) = span\{\phi\}, \quad Null(F_x{}^*)^T = span\{\psi\}, \quad \|\phi\|_2 = \|\psi\|_2 = 1,$
$H_2\rangle \quad \psi^T F_\lambda{}^* = 0, \quad \psi^T F_{xx}{}^* \phi\phi = 0,$
$H_3\rangle \quad a := \psi^T(F_{x\lambda}^* \phi + F_{xx}^* \phi v) < 0, \quad b := \psi^T(F_{xxx}^* \phi\phi\phi + 3F_{xx}^* \phi u) > 0,$

then we call (x^*, λ^*) a pitchfork bifurcation point of (20) (in [3]), in which, u and v are uniquely defined by the following equations

$$F_x^* v + F_\lambda^* = 0, \qquad \phi^T v = 0,$$
$$F_x^* u + F_{xx}^* \phi\phi = 0, \qquad \phi^T u = 0,$$

respectively, here we have used the notations $F_x^* := F_x(x^*, \lambda^*)$, $F_{x\lambda}^* := F_{x\lambda}(x^*, \lambda^*)$ and so on.

Now assume that (x^*, λ^*) is a pitchfork bifucation point of (20). To switch solution branches of (20) at the point (x^*, λ^*) we construct a system of equations as follows:

$$H(x, \lambda; s) := \begin{pmatrix} F(x, \lambda) - \alpha\phi_0 \\ \lambda - \lambda_0 - \eta s \end{pmatrix}, \tag{21}$$

where $\phi_0 \notin RangF_x^*, \alpha = \alpha(s) := \epsilon s(1-s)$, ϵ and η are given positive parameters and (x_0, λ_0) is a known solution of (20) near the point (x^*, λ^*) that may be obtained by tracking the solution branch of (20). Assume that $\lambda_0 < \lambda^*$, we shall show that the system (21) owns the same property as (3) in the second section. For convenience, suppose that $\psi^T \phi_0 > 0$. Let

$$x = x^* + \xi\phi + y, \tag{22}$$

in which $\xi \in R, y \in R^n$ and $\phi^T y = 0$. It is easy to see that the first one of (21) is equivalent to the system of equations:

$$(I - \psi\psi^T)(F(x^* + \xi\phi + y, \lambda) - \alpha\phi_0) + \psi\phi^T y = 0, \tag{23}$$

and

$$\psi^T(F(x^* + \xi\phi + y, \lambda) - \alpha\phi_0) = 0. \tag{24}$$

In terms of the implicit function theorem, we may obtain unique smooth solution $y(\xi, \lambda; \alpha)$ of (23) with $y(0, \lambda^*, 0) = 0$. Moreover, it is not difficult to see that

$$y_\xi(0, \lambda^*; 0) = 0, \qquad y_{\xi\xi}(0, \lambda^*; 0) = u, \qquad y_\lambda(0, \lambda^*; 0) = v. \tag{25}$$

Substituting $y = y(\xi, \lambda; \alpha)$ into (24), it yields a scale equation

$$h(\xi, \lambda; \alpha) = \psi^T F(x^* + \xi\phi + y(\xi, \lambda; \alpha), \lambda) - \alpha\psi^T\phi_0 = 0. \tag{26}$$

From (25) we have

$$h(0, \lambda^*; 0) = h_\xi(0, \lambda^*; 0) = h_{\xi\xi}(0, \lambda^*; 0) = h_\lambda(0, \lambda^*; 0) = 0,$$

$$h_{\xi\xi\xi}(0, \lambda^*; 0) = b > 0, \qquad h_{\xi\lambda}(0, \lambda^*; 0) = a < 0.$$

Thus, following the recognition theorem of pitchfork in the theory of singularity points in [3], we assert that there exist smooth functions $\eta(\xi, \lambda)$ and $R(\xi, \lambda)$ in a neighborhood of $(\xi, \lambda) = (0, \lambda^*)$ such that

$$h(\xi, \lambda; 0) = R(\xi, \lambda)(\eta^3(\xi, \lambda) - (\lambda - \lambda^*)\eta(\xi, \lambda)), \tag{27}$$

and

$$\eta(0, \lambda^*) = 0, \qquad \eta_\xi(0, \lambda^*) > 0, \tag{28}$$

$$R(0, \lambda^*) > 0. \tag{29}$$

Expanding $h(\xi, \lambda; \alpha)$ at $\alpha = 0$, $h(\xi, \lambda; \alpha)$ may be rewritten as

$$h(\xi, \lambda; \alpha) = h(\xi, \lambda; 0) - \alpha\theta_1(\xi, \lambda; \alpha), \tag{30}$$

where $\theta_1(0, \lambda^*; 0) = \psi^T\phi_0 > 0$. Let the transformation $\gamma : (\xi, \lambda) \longrightarrow (\eta, \mu)$ be defined as follows:

$$\eta = \eta(\xi, \lambda), \qquad \mu = \lambda - \lambda^*. \tag{31}$$

Obviously, γ is a local diffehomorphism at $(0, \lambda^*)$ and the inverse transformation of γ denoted by γ^{-1} may be represented as

$$\gamma^{-1}(\eta, \mu) := (\xi(\eta, \mu), \lambda^* + \mu). \tag{32}$$

Substituting (27) into (30) and using (32) we get

$$h(\xi(\eta,\mu), \lambda^* + \mu; \alpha) = R(\xi(\eta,\mu), \lambda^* + \mu)(\eta^3 - \mu\eta) - \alpha\theta_1(\xi(\eta,\mu), \lambda^* + \mu; \alpha) = 0. \tag{33}$$

Dividing two sides of (33) by $R(\xi(\eta,\mu), \lambda^* + \mu)$, we obtain the equation be equivalent to (33):

$$\eta^3 - \mu\eta - \alpha\theta(\eta, \mu; \alpha) = 0, \tag{34}$$

in which, $\theta(\eta, \mu; \alpha)$ is smooth and satisfies that

$$\theta(0, 0; 0) = R^{-1}(0, \lambda^*)\theta_1(0, \lambda^*; 0) > 0. \tag{35}$$

Firstly, we investigate solutions of (20) that corresponds to the first equation of (21) with $\alpha = 0$. Clearly, the solution set of (34) with $\alpha = 0$ consists of $\eta = 0$ and $\eta^2 = \mu$. Hence, the solution set of (26) near $(0, \lambda^*)$ consists of the curve

$$\xi = \xi^{(1)}(\lambda) := \xi(0, \lambda - \lambda^*), \tag{36}$$

and the curve

$$\xi = \xi_{\pm}^{(2)}(\lambda) := \xi(\pm(\lambda - \lambda^*)^{\frac{1}{2}}, \lambda - \lambda^*), \qquad \lambda > \lambda^*. \tag{37}$$

Substituting (36) and (37) into (22), we conclude that the solution set of (20) in a neighborhood of (x^*, λ^*) consits of two smooth solution curve Γ_1 and Γ_2 , in which

$$\begin{cases} \Gamma_1 := \{(x,\lambda) : x = x^* + \xi^{(1)}(\lambda)\phi + y(\xi^{(1)}(\lambda), \lambda; 0)\}, \\ \\ \Gamma_2 := \{(x,\lambda) : x = x^* + \xi_{\pm}^{(2)}(\lambda)\phi + y(\xi_{\pm}^{(2)}(\lambda), \lambda; 0)\}. \end{cases} \tag{38}$$

Because (x_0, λ_0) is a solution of (20) near (x^*, λ^*) and $\lambda_0 < \lambda^*$, we have

$$x_0 = x^* + \xi^{(1)}(\lambda_0)\phi + y(\xi^{(1)}(\lambda_0), \lambda_0; 0). \tag{39}$$

Next,we investigate solutions of (21). Let us consider the system of equations

$$G(\eta, \mu; s) = \begin{pmatrix} \eta^3 - \mu\eta - \alpha(s)\theta(\eta, \mu; \alpha(s)) \\ \mu - \mu_0 - \eta s \end{pmatrix} = 0, \tag{40}$$

in which, $\mu_0 := \lambda^* - \lambda_0 < 0$ and $\alpha(s) := \epsilon s(1 - s)$. According to the Theorem 2.3 , we conclude that if ϵ and η are chosen so that (6) (7) are hold, there exists unique smooth solution curve $\Gamma' : (\eta(s), \mu(s)), 0 \le s \le 1$ of (40), in which, $\eta(s)$ and $\mu(s)$ satisfy that

$$\begin{cases} \eta(0) = 0, \qquad \mu(0) = \mu_0, \\ \bar{\eta} := \eta(1) > 0, \qquad \bar{\mu} := \mu(1) = \bar{\eta}^2, \end{cases} \tag{41}$$

and the Jacobian matrices $\frac{\partial G}{\partial(\eta,\mu)}$ at any points on Γ' are all nonsingular. By using the inverse transformation γ^{-1}, we conclude immediately that there exists unique smooth solution Γ : $(x(s), \lambda(s))$ $(0 \le s \le 1)$ of the system (21) in the neighborhood of $(x, \lambda) = (x^*, \lambda^*)$, in which

$$\begin{cases} x(s) := x^* + \xi(\eta(s), \mu(s))\phi + y(\xi(\eta(s), \mu(s)), \lambda^* + \mu(s), \alpha(s)), \\ \lambda(s) := \lambda^* + \mu(s). \end{cases} \tag{42}$$

Moreover, the Jacobian matrices $\frac{\partial H}{\partial(x,\lambda)}$ at any points on Γ are all nonsingular. Finally, it is not difficult to check that

$$\begin{cases} x(0)=x_0, & \lambda(0)=\lambda_0, \\ (\bar{x},\bar{\lambda})\in\Gamma_2, & \bar{\lambda}>\lambda^*. \end{cases} \tag{43}$$

Summarizing above discussions, we obtain the main conclusion of this paper.

Theorem 3.1. *Suppose that (x^*,λ^*) is a pitchfork bifurcation point of (20),Ω is a neighborhood of (x^*,λ^*) , $(x_0,\lambda_0)\in\Omega$ is a known solution of (20) near (x^*,λ^*) and $\lambda_0<\lambda^*$. If ϵ and η are chosen such that*

$$-\eta_0<\lambda_0-\lambda^*<0, \quad 0<\epsilon<\epsilon_0, \quad 0<\eta-(\lambda^*-\lambda_0)<\eta_0 \tag{44}$$

are hold, there exists unique smooth curve $\Gamma:(x(s),\lambda(s))\in\Omega\ \ (0\le s\le 1)$ of (21) such that (43) is true and the Jacobian matrices $\frac{\partial H}{\partial(x,\lambda)}$ at any points on Γ are all nonsingular.

From the Theorem 3.1, the solution $(\bar{x},\bar{\lambda}):=(x(1),\lambda(1))$ of (20) can be obtained by tracking solution curve Γ of (21) in terms of the standard continuation program with initial value $(x_0,\lambda_0,0)$. Thus, we realize the switching solution branches of (20) from $(x_0,\lambda_0)\in\Gamma_1$ to $(\bar{x},\bar{\lambda})\in\Gamma_2$.

4. Some Numerical Examples

In this section, we give some numerical results to illustrate the efficiency of the switching method proposed in this paper. A finite element analogue of the Euler's rod is solved in the example 1, it is that there are no trivial solution in the example 2.

Example 1. We discuss the finite element analogue of Euler's rod in this example. This system consists of three rigid rods of unit length connected to one another by torsional springs. An endpoint of this system is simply-supported and the another sliding simply-supported and subjected to a thrust μ. The deformed configuration of the system can be described by the angles $\theta_i, i=1,2,3$. The equilibrium equation of the system is given as follows:

$$F(\theta,\lambda):=\begin{pmatrix} \theta_1(\cos\theta_1+\cos\theta_3)-\theta_2\cos\theta_1-\theta_3(2\cos\theta_1+\cos\theta_3)-\lambda\sin(\theta_1-\theta_3) \\ -\theta_1\cos\theta_2+\theta_2(\cos\theta_2+\cos\theta_3)+\theta_3(2\cos\theta_2+\cos\theta_3)-\lambda\sin(\theta_2+\theta_3) \\ \sin\theta_1+\sin\theta_3-\sin\theta_2 \end{pmatrix}=0, \tag{45}$$

where $\theta:=(\theta_1,\theta_2,\theta_3),\lambda=\frac{\mu}{A}$ and A is the constant of the trosional springs. Obviously, $F(0,\lambda)=0$ for any λ. It is not difficult to see that $(\theta,\lambda)=(0,1)$ and $(\theta,\lambda)=(0,3)$ are two pitchfork bifurcation points of (45). Now we switch the solution branches of (45) at these points by using the method of this paper. By taking $\eta=0.4,\epsilon=0.1$ and $\phi_0=(1,2,1)^T$ in (21), the numerical results are entered in table 1 and table 2 for $\lambda_0=0.8$ and $\lambda_0=2.8$, respectively.

Table 1 The switching results of (45) for $\lambda_0=0.8,\eta=0.4,\epsilon=0.1$

s	λ	θ_1	θ_2	θ_3
0.2	0.88	0.1955451	0.1873185	0.007923755
0.4	0.96	0.4816525	0.4688640	0.01262921
0.6	1.04	0.7347430	0.7210976	0.01381339
0.8	1.12	0.9091566	0.8994816	0.01001892
1.0	1.20	1.026738	1.026738	-3.24E(-9)

Table 2 The switching results of (45) for $\lambda_0 = 2.8, \eta = 0.4, \epsilon = 0.1$

s	λ	θ_1	θ_2	θ_3
0.2	2.88	-0.02954560	0.003973478	0.04953503
0.4	2.96	-0.09140737	0.005396048	0.1700329
0.6	3.04	-0.2081435	0.1685575	0.4097765
0.8	3.12	-0.2869976	0.2581846	0.5876415
1.0	3.20	-0.3339928	0.3339928	0.7150242

In the above computing, if we replace $\epsilon = 0.1$ by $\epsilon = -0.1$ (this is equivalent to taking $\phi_0 = -(1,2,1)^T$ and $\epsilon = 0.1$ in (21)), the numerical results of θ_i are the same as those entered in Table 1 and Table 2 except their signs. Thus we obtain two non-zero solutions of (45) as $\lambda = 1.2$ and $\lambda = 3.2$,respectively.

$$\theta = \pm(1.026738, 1.026738, -3.24E(-9)), \qquad as \qquad \lambda = 1.2,$$

$$\theta = \pm(-0.3339928, 0.3339928, 0.7150242), \qquad as \qquad \lambda = 3.2.$$

Example 2. Let us consider the nonlinear system of equation

$$F(x,\lambda) := A(x-x^*)+(\lambda-\lambda^*)c+(\lambda-\lambda^*)B(x-x^*)+((x-x^*)^T D(x-x^*))E(x-x^*) = 0, \quad (46)$$

where

$$A = \begin{pmatrix} 0 & 6 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 8 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & -3 & 1 & 6 \\ -5 & 4 & 1 & 8 \\ 1 & 0 & 1 & 2 \\ 0 & 3 & 6 & 2 \end{pmatrix},$$

$$D = \begin{pmatrix} 4 & 0 & 0 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & 0 \\ -1 & -2 & 0 & 3 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 2 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix},$$

$$c = (1,0,1,1)^T, \qquad x^* = (1,2,3,-1)^T, \qquad \lambda^* = 8.$$

It is easy to check that (x^*, λ^*) is a pitchfork point of (46). At first, in terms of the continuation method proposed in [1] by H.B. Keller, some points on a solution branch of (46) through (x^*, λ^*) are obtained and are entered in Table 3. To obtain a point on the another solution branch,we take $\lambda_0 = 7.70, \quad \phi_0 = (1,1,1,1)^T, \quad \eta = 0.6$ in (21). For different ϵ, the numerical results are entered in Table 4, 5 and 6,respectively.

Table 3 The points on a solution branch of (46) near (x^*, λ^*)

λ	x_1	x_2	x_3	x_4
7.70	1.179158	2.080276	3.008017	-0.9148825
7.90	1.045698	2.013849	3.005753	-0.9783836
8.10	0.9659757	1.995194	2.992482	-1.017642
8.30	0.9256955	2.005614	2.973317	-1.044974

Table 4 The switching results of (46) for $\epsilon = 0.1$

s	λ	x_1	x_2	x_3	x_4
0.2	7.82	1.114435	2.038133	3.008666	-0.9533656
0.6	8.06	1.224977	1.991156	2.991533	-1.002029
1.0	8.30	1.434742	1.909760	2.933908	-1.003131

Table 5 The switching results of (46) for $\epsilon = 0.4$

s	λ	x_1	x_2	x_3	x_4
0.2	7.82	1.171332	2.052098	3.008342	-0.9419821
0.6	8.06	1.277802	1.999387	2.990074	-0.9873479
1.0	8.30	1.434742	1.909760	2.933908	-1.003131

Table 6 The switching results of (46) for $\epsilon = -0.1$

s	λ	x_1	x_2	x_3	x_4
0.2	7.82	1.067333	2.028072	3.008428	-0.9607119
0.6	8.06	0.7809138	1.998880	2.999144	-1.018056
1.0	8.30	0.5893231	2.052729	3.003435	-1.058332

The last row in Table 4 is same as one in Table 5. They give a solution of (46) which is different to the solution entered in Table 3. Analogously, the last row in Table 6 give another solution of (46) that differs from two solution mentioned above. Thus we get all three solution of (46) in a neighborhood of (x^*, λ^*) as $\lambda = 8.3$ which are entered in last rows of Table 3, 4 (5) and 6,respectively.

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